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# ON QUASI-POWER INCREASING SEQUENCES AND THEIR SOME APPLICATIONS

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ABSTRACT. In [6], we proved a main theorem dealing with $\mid \bar{N}, p_n, \theta_n \mid_k$ summability factors using a new general class of power increasing sequences instead of a quasi-$\sigma$-power increasing sequence. In this paper, we prove that theorem under weaker conditions. This theorem also includes some new results.

## 1. Introduction

A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $K f_n X_n \geq f_m X_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{n^\sigma (\log n)^\eta,\ \eta \geq 0,\ 0 < \sigma < 1\}$ (see [13]). If we set $\eta$=0, then we get a quasi-$\sigma$-power increasing sequence (see [10]). We write $\mathcal{B}V_O = \mathcal{B}V \cap C_O$, where $\mathcal{C}_O = \{\ x = (x_k) \in \Omega\ :\ \lim_k |x_k| = 0\ \}$, $\mathcal{B}V$=$\{\ x = (x_k) \in \Omega\ :\ \sum_k |x_k - x_{k+1}| < \infty\ \}$ and $\Omega$ being the space of all real-valued sequences. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $(s_n)$. We denote by $u_n^\alpha$ the $n$th Cesàro mean of order $\alpha$, with $\alpha > -1$, of the sequence $(s_n)$, that is (see [7]),

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0. \tag{2}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k < \infty. \tag{3}$$

If we take $\alpha = 1$, then we get the $\mid C, 1 \mid_k$ summability. Let $(p_n)$ be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{4}$$

The sequence-to-sequence transformation

$$v_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence $(v_n)$ of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence $(s_n)$, generated by the sequence of coefficients $(p_n)$ (see [9]). Let $(\theta_n)$ be any sequence of positive constants. The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$, if (see [12])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid v_n - v_{n-1} \mid^k < \infty. \tag{6}$$

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If we take $\theta_n = \frac{P_n}{p_n}$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid \bar{N}, p_n \mid_k$ summability (see [1]). Also, if we take $\theta_n = \frac{P_n}{p_n}$ and $p_n = 1$ for all values of $n$, then we get $\mid C, 1 \mid_k$ summability. Furthermore, if we take $\theta_n = n$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid R, p_n \mid_k$ summability (see [2]).

**2. Known Results.** The following theorems are known:

**Theorem A ([4]).** Let $\left(\frac{\theta_n p_n}{P_n}\right)$ be a non-increasing sequence. Let $(\lambda_n) \in \mathcal{B}V_O$ and let $(X_n)$ be a quasi- $\sigma$-power increasing sequence for some $\sigma$ $(0 < \sigma < 1)$ . Suppose also that there exist sequences $(\beta_n)$ and $(\lambda_n)$ such that

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{7}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{8}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{9}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{10}$$

If

$$\sum_{v=1}^{n} \theta_v^{k-1} \frac{\mid s_v \mid^k}{v^k} = O(X_n) \quad as \quad n \to \infty, \tag{11}$$

and $(p_n)$ is a sequence such that

$$P_n = O(np_n), \tag{12}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{13}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n, \theta_n \mid_k$, $k \geq 1$.

**Remark.** We can take $(\lambda_n) \in \mathcal{B}V$ instead of $(\lambda_n) \in \mathcal{B}V_O$ and it is sufficient to prove Theorem A.

**Theorem B ([6]).** Let $\left(\frac{\theta_n p_n}{P_n}\right)$ be a non-increasing sequence. Let $(\lambda_n) \in \mathcal{B}V$ and let $(X_n)$ be a quasi-f-power increasing sequence for some $\sigma$ $(0 < \sigma < 1)$ and $\eta \geq 0$. If the conditions (7)-(13) are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n, \theta_n \mid_k$, $k \geq 1$.

It should be noted that if we take $\eta$=0, then we obtain Theorem A.

**3. The Main result.** The purpose of this paper is to prove Theorem B under weaker conditions. Now, we shall prove the following general theorem.

**Theorem.** Let $\left(\frac{\theta_n p_n}{P_n}\right)$ be a non-increasing sequence. Let $(X_n)$ be a quasi-f-power increasing sequence for some $\sigma$ $(0 < \sigma < 1)$ and $\eta \geq 0$. If the conditions (7)-(10), (12)-(13), and

$$\sum_{v=1}^{n} \theta_v^{k-1} \frac{\mid s_v \mid^k}{v^k {X_v}^{k-1}} = O(X_n) \quad as \quad n \to \infty \tag{14}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n, \theta_n \mid_k$, $k \geq 1$.

**Remark.** It should be noted that condition (14) is reduced to the condition (11), when k=1. When $k > 1$, the condition (14) is weaker than the condition (11), but the converse is not true. As in [14] we can show that if (11) is satisfied, then we get that

$$\sum_{v=1}^{n} \theta_v^{k-1} \frac{\mid s_v \mid^k}{v^k {X_v}^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{v=1}^{n} \theta_v^{k-1} \frac{\mid s_v \mid^k}{v^k} = O(X_n).$$

If (14) is satisfied, then for $k > 1$ we obtain that

$$\sum_{v=1}^{n} \theta_v^{k-1} \frac{\mid s_v \mid^k}{v^k} = \sum_{v=1}^{n} \theta_v^{k-1} X_v^{k-1} \frac{\mid s_v \mid^k}{v^k {X_v}^{k-1}} = O(X_n^{k-1}) \sum_{v=1}^{n} \theta_v^{k-1} \frac{\mid s_v \mid^k}{v^k {X_v}^{k-1}} = O(X_n^k) \neq O(X_n).$$

Also, it should be noted that the condition "$(\lambda_n) \in \mathcal{B}V$ " has been removed.

We require the following lemmas for the proof of the theorem.

**Lemma 1 ([5]).** Under the conditions on $(X_n)$, $(\beta_n)$ and $(\lambda_n)$ as expressed in the statement of the theorem, we have the following;

$$nX_n\beta_n = O(1), \tag{15}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{16}$$

**Lemma 2 ([11]).** If the conditions (12) and (13) are satisfied, then we have that

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{17}$$

**4. Proof of the theorem.** Let $(T_n)$ be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{rp_r} = \frac{1}{P_n}\sum_{v=1}^{n}(P_n - P_{v-1})\frac{a_v P_v \lambda_v}{vp_v}.$$

Then, for $n \geq 1$ we obtain that

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{vp_v}.$$

Using Abel's transformation, we get

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \Delta\left(\frac{P_{v-1} P_v \lambda_v}{vp_v}\right) + \frac{\lambda_n s_n}{n} \\
&= \frac{s_n \lambda_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \frac{P_{v+1} P_v \Delta\lambda_v}{(v+1)p_{v+1}} \\
&+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta\left(\frac{P_v}{vp_v}\right) - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v \lambda_v \frac{1}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1,2,3,4. \tag{18}$$

Firstly, by using Abel's transformation, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \theta_n^{k-1} \mid T_{n,1} \mid^k &= \sum_{n=1}^{m} \theta_n^{k-1} n^{-k} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid\mid s_n \mid^k \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid \left(\frac{1}{X_n}\right)^{k-1} \theta_n^{k-1} n^{-k} \mid s_n \mid^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \theta_v^{k-1} \frac{\mid s_v \mid^k}{{X_v}^{k-1} v^k} \\
&+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \theta_n^{k-1} \frac{\mid s_n \mid^k}{{X_n}^{k-1} n^k} \\
&= O(1) \sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 1. Now, using (12) and applying Hölder's inequality, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1} \mid T_{n,2}\mid^k &= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k}\mid \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v \mid^k \\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}\mid s_v \mid p_v \mid \Delta\lambda_v \mid\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k \mid s_v\mid^k p_v\ (\beta_v)^k \\
&\times \left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k \mid s_v\mid^k p_v\ (\beta_v)^k \sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_n P_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k \mid s_v\mid^k p_v\ (\beta_v)^k \left(\frac{\theta_v p_v}{P_v}\right)^{k-1}\sum_{n=v+1}^{m+1}\frac{p_n}{P_n P_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k \mid s_v\mid^k\ (\beta_v)^k \left(\frac{p_v}{P_v}\right)\theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}(v\beta_v)^{k-1} v\beta_v \frac{1}{v^k}\theta_v^{k-1}\mid s_v\mid^k \\
&= O(1)\sum_{v=1}^{m}\left(\frac{1}{X_v}\right)^{k-1} v\beta_v \frac{1}{v^k}\theta_v^{k-1}\mid s_v\mid^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\theta_r^{k-1}\frac{\mid s_r\mid^k}{r^k {X_r}^{k-1}} + O(1) m\beta_m \sum_{v=1}^{m}\theta_v^{k-1}\frac{\mid s_v\mid^k}{v^k {X_v}^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}\mid \Delta(v\beta_v)\mid X_v + O(1) m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1}\mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1) m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1} v\mid \Delta\beta_v\mid X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1) m\beta_m X_m = O(1)
\end{aligned}
$$

as $m\to\infty$, in view of the hypotheses of the theorem and Lemma 1. Again, as in $T_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1} \mid T_{n,3}\mid^k &= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1} P_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v}\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k v^{-k} p_v \mid s_v\mid^k\mid \lambda_v\mid^k \\
&\times \left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right\}^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k v^{-k} \mid s_v \mid^k p_v \mid \lambda_v \mid^k \sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} v^{-k}\theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1} \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid\mid s_v \mid^k\\
&= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \left(\frac{1}{X_v}\right)^{k-1} \theta_v^{k-1} v^{-k} \mid s_v \mid^k\\
&= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \theta_v^{k-1}\frac{\mid s_v \mid^k}{v^k {X_v}^{k-1}} = O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

in view of the hypotheses of the theorem, Lemma 1 and Lemma 2. Finally, using Hölder's inequality, as in $T_{n,1}$ we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1} \mid T_{n,4} \mid^k &= \sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} s_v \frac{P_v}{v}\lambda_v \mid^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} s_v \frac{P_v}{v p_v} p_v \lambda \mid^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \mid s_v \mid^k \left(\frac{P_v}{p_v}\right)^k v^{-k} p_v \mid \lambda_v \mid^k\\
&\times \left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k v^{-k} \mid s_v \mid^k p_v \mid \lambda_v \mid^k \frac{1}{P_v}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} v^{-k}\left(\frac{p_v}{P_v}\right)^{k-1}\theta_v^{k-1} \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid\mid s_v \mid^k\\
&= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \theta_v^{k-1}\frac{\mid s_v \mid^k}{v^k {X_v}^{k-1}} = O(1) \quad as \quad m\to\infty.
\end{aligned}
$$

This completes the proof of the theorem. If we set $\eta \geq 0$, then we obtain Theorem B under weaker conditions. If we take $p_n = 1$ for all values of $n$, then we have a new result for $\mid C, 1, \theta_n \mid_k$ summability. Furthermore, if we take $\theta_n = n$, then we have another new result for $\mid R, p_n \mid_k$ summability. Finally, if we take $p_n = 1$ for all values of n and $\theta_n = n$, then we get a new result dealing with $\mid C, 1 \mid_k$ summability factors.

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