International Journal of Analysis and Applications
ISSN 2291-8639
Volume 10, Number 2 (2016), 101-111
http://www.etamaths.com

DHAGE ITERATION METHOD FOR APPROXIMATING POSITIVE SOLUTIONS OF PBVPS OF NONLINEAR QUADRATIC DIFFERENTIAL EQUATIONS WITH MAXIMA

SHYAM B. DHAGE, BAPURAO C. DHAGE*

Abstract. In this paper authors prove the existence as well as approximation of the positive solutions for a periodic boundary value problem of first order ordinary nonlinear quadratic differential equations with maxima. An algorithm for the solutions is developed and it is shown that certain sequence of successive approximations converges monotonically to the positive solution of considered quadratic differential equations under some suitable mixed hybrid conditions. Our results rely on the Dhage iteration principle embodied in a recent hybrid fixed point theorem of Dhage (2014). A numerical example is also provided to illustrate the hypotheses and abstract theory developed in this paper.

1. Introduction

Motivated by the linear differential equation with maxima that occurs in the automatic control of some technical system, several mathematicians started working in the field of differential equations with maxima for qualitative aspects of the solutions. See Magomedov [17, 18], Bainov and Hristova [1], Otrocol and Rus [15] and the references therein for the details. Again, Mishkis [19] pointed out in his survey the necessity of the study of differential equations with maxima and gave an impetus in the study of such equations. Similarly, the study of periodic boundary value problems of nonlinear quadratic differential equations is initiated in the works of Dhage *et.al.* [13] and Dhage [2]. Very recently, Dhage *et.al.* [13] initiated the study of initial value problems of nonlinear quadratic differential equations with maxima and proved the existence and approximation results for such equations. The purpose of the present paper is to blend these two ideas together and discuss the PBVPs of first order quadratic differential equations with maxima for existence and numerical aspects of the solutions. It is well-known that the hybrid differential equations can be tackled using the Dhage iteration method embodied in the hybrid fixed point theory initiated by Dhage [3, 5, 7]) which also yields the algorithms for the solutions. Therefore, it is of interest to establish algorithms for the quadratic differential equations with maxima (1.1) for existence and approximation of the solutions along similar lines. We claim that our method as well as our results of this paper are new to the literature in the theory of nonlinear differential equations with maxima.

Given a closed and bounded interval $J = [0, T]$ in the real line $\mathbb{R}$, consider the periodic boundary value problem (in short PBVP) of nonlinear first order ordinary quadratic differential equation (QDE) with maxima,

$$\left.\begin{aligned} &\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] + \lambda\left[\frac{x(t)}{f(t,x(t)))}\right] = g(t,x(t),X(t))),\ t \in J,\\ &x(0) = x(T), \end{aligned}\right\} \tag{1.1}$$

for $\lambda \in \mathbb{R}$, $\lambda > 0$, where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $X(t) = \max_{t_0 \le \xi \le t} x(\xi)$ for $t \in J$.

2010 *Mathematics Subject Classification.* 34A12, 34A38.

Key words and phrases. quadratic differential equation with maxima; periodic boundary value problem; Dhage iteration principle; approximate positive solution.

By a *solution* of the QDE (1.1) we mean a function $x \in C^1(J, \mathbb{R})$ that satisfies

(i) $t \mapsto \dfrac{x}{f(t,x)}$ is a continuously differentiable function for each $x, y \in \mathbb{R}$, and

(ii) x satisfies the equations in (1.1) on J,

where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The QDE (1.1) is a quadratic perturbation of second type of the nonlinear PBVP of first order differential equations,

$$\left.\begin{aligned} x'(t) &= f(t, x(t)), \ \ t \in J, \\ x(0) &= x(T), \end{aligned}\right\} \tag{1.2}$$

The details of different types perturbations is given in Dhage [3]. The QDE (1.1) includes some known classes of nonlinear differential equations studied earlier in the literature. In the following we list a few special cases of (1.1) defined on J.

1. When $f(t, x) = 1$ for all $t \in J$ and $x, y \in \mathbb{R}$, the QDE (1.1) reduces to the following known nonlinear differential equation with maxima

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= g(t, x(t), X(t)), \ \ t \in J, \\ x(0) &= x(T). \end{aligned}\right\} \tag{1.3}$$

The above nonlinear differential equation with maxima (3.10) has already been discussed in the literature for existence and uniqueness of the solutions via classical methods of Schauder and Banach fixed point principles. See Bainov and Hristova [1] and the references therein. Here our method is different and constructive. Therefore, Theorem 3.1 includes the existence and approximation theorem for the differential equation with maxima (3.10) as a special case under weak partial compactness type conditions.

2. Again, when $g(t, x, y) = g(t, x)$ for all $t \in J$ and $x, y \in \mathbb{R}$, the QDE (1.1) reduces to the following QDE without maxima,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f\,(t, x(t))}\right] + \lambda \left[\frac{x(t)}{f\,(t, x(t))}\right] &= g\,(t, x(t))\,, \ t \in J, \\ x(0) &= x(T)., \end{aligned}\right\} \tag{1.4}$$

which has been discussed in Dhage and Dhage [11] via Dhage iteration method and established the existence and approximation result for positive solutions.

3. If we take $g(t, x, y) = p\,y + F(t)$ for all $t \in J$ and $x, y \in \mathbb{R}$ in (1.3), then it reduces to the standard linear differential equation of automatic regulation,

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= pX(t) + F(t) \\ x(0) &= x(T), \end{aligned}\right\} \tag{1.5}$$

for all $t \in J$, where $\lambda > 0$, $p > 0$ are constants and $F \ : \ J \to \mathbb{R}$ is a continuous perturbation function. The differential equation with maxima (1.5) is the motivation for development of the subject of differential equations with maxima. Therefore, our QDE (1.1) is more general and the existence and approximation result of this problem includes the existence and approximation results for all the above differential equations with maxima as special cases.

2. Auxiliary Results

In this section we give all the preliminaries and key tool that is used in subsequent part of the paper. Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is **regular** if $\{x_n\}_{n\in\mathbb{N}}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Clearly, the partially ordered Banach space $C(J, \mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space E may be found in Heikkilä and Lakshmikantham [16] and the references therein.

We need the following preliminary definitions given in Dhage [4, 5, 6] in what follows.

A mapping $\mathcal{T} : E \to E$ is called **isotone** or **nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E. A non-empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded. An operator $\mathcal{T} : E \to E$ is called **partially bounded** if every chain C in $T(E)$ is bounded. $\mathcal{T}$ is called **uniformly partially bounded** if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant. $\mathcal{T}$ is called **bounded** if $T(E)$ is a bounded subset of E. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is compact. An operator $\mathcal{T} : E \to E$ is called **partially compact** if every chain or totally ordered set C in $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}(E)$ is a uniformly partially bounded and partially compact on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.1. *Suppose that $\mathcal{T}$ is a nondecreasing operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact if $\mathcal{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.*

Definition 2.1 (Dhage [4])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called* **Janhavi** *if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a* **Janhavi metric** *or* **Janhavi Banach space**.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

Definition 2.2 (Dhage [5])**.** *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called **partially nonlinear $\mathcal{D}$-Lipschitz** if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathcal{T}$ is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \big\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\big\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}. \tag{2.2}$$

The elements of the set $\mathcal{K}$ are called the positive vectors in E. The following lemma follows immediately from the definition of the set $\mathcal{K}$ which is often times used in the hybrid fixed point theory of Banach algebras and applications to nonlinear differential and integral equations.

Lemma 2.1 (Dhage [4])**.** *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.3. *An operator $\mathcal{T} : E \to E$ is said to be positive if the range $R(\mathcal{T})$ of $\mathcal{T}$ is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.*

The Dhage iteration principle (in short DIP) developed in Dhage [5, 7] may be described as "***the monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation.***" The aforesaid convergence principle is a useful tool in nonlinear analysis and is called Dhage iteration method (in short DIM) for nonlinear equations. Dhage iteration method embodied in the following applicable hybrid fixed point theorem of Dhage [7] is used as a key tool for our work contained in the present paper. A few other hybrid fixed point theorems containing the Dhage iteration method along with applications appear in Dhage [5, 7].

Theorem 2.1 (Dhage [5, 7])**.** *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that every compact chain of E is Janhavi. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\psi_{\mathcal{A}}$,*
(b) *$\mathcal{B}$ is partially continuous and uniformly partially compact,*
(c) *$0 < M\psi_{\mathcal{A}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C$ is a chain in $E\}$, and*
(d) *there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0$.*

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x = x \tag{2.3}$$

has a positive solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n$, $n = 0, 1, \ldots$; converges monotonically to x^*.

Remark 2.2. The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it.

3. Main Results

The QDE (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{3.1}$$

and

$$x \le y \iff x(t) \le y(t) \tag{3.2}$$

for all $t \in J$ respectively. Clearly, $C(J, \mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\le$. It is known that the partially ordered Banach algebra $C(J, \mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzellá-Ascolli theorem.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), \le, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\le$ defined by (3.1) and (3.2) respectively. Then every partially compact subset S of $C(J, \mathbb{R})$ is Janhavi.*

Proof. The proof of the lemma is given in Dhage and Dhage [9, 10, 11] and so we omit the details of it. □

We need the following definition in what follows.

Definition 3.1. *A function $u \in C(J, \mathbb{R})$ is said to be a lower solution of the QDE (1.1) if the function $t \mapsto \dfrac{u(t)}{f(t, u(t))}$ is differentiable and satisfies*

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{u(t)}{f(t, u(t))}\right] + \lambda\left[\frac{u(t)}{f(t, u(t))}\right] &\le g(t, u(t), U(t)), \\ u(0) &\le u(T), \end{aligned}\right\}$$

for all $t \in J$*, where* $U(t) = \max_{0\le\xi\le t} x(\xi)$*,* $t \in J$*. Similarly, a function* $v \in C(J,\mathbb{R})$ *is said to be an upper solution of the QDE (1.1) if the function* $t \mapsto \dfrac{v(t)}{f(t,v(t))}$ *is differentiable and satisfies the above inequalities with reverse sign.*

We consider the following set of assumptions in what follows:

- (A_0) The map $x \mapsto \dfrac{x}{f(t,x)}$ is increasing for each $t \in J$.
- (A_1) f defines a function $f : J \times \mathbb{R} \to \mathbb{R}_+$.
- (A_2) There exists a constant $M_f > 0$ such that $0 < f(t,x) \le M_f$ for all $t \in J$ and $x \in \mathbb{R}$.
- (A_3) There exists a $\mathcal{D}$-function φ such that
$$0 \le f(t,x) - f(t,y) \le \varphi\big(x-y\big),$$
for all $t \in J$ and $x, y \in \mathbb{R}$, $x_1 \ge y_1$.
- (A_4) The function $f(t,x)$ is periodic in t with period T for all $x \in \mathbb{R}$, i.e., $f(t,x) = f(t+T,x)$ for each $x \in \mathbb{R}$.
- (B_1) g defines a function $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$.
- (B_2) There exists a constant $M_g > 0$ such that $g(t,x,y) \le M_g$ for all $t \in J$ and $x, y \in \mathbb{R}$.
- (B_3) $g(t,x,y)$ is nondecreasing in x and y for all $t \in J$.
- (B_4) The QDE (1.1) has a lower solution $u \in C^1(J,\mathbb{R})$.

Remark 3.1. *Notice that if hypothesis (A_0) holds, then the function* $x \mapsto \dfrac{x}{f(t,x)}$ *is injective for each* $t \in J$*.*

The following useful lemma is obvious and may be found in Nieto and Lopez [20].

Lemma 3.2. *For any* $h \in L^1(J,\mathbb{R}_+)$ *and* $\sigma \in L^1(J,\mathbb{R})$*,* x *is a solution to the differential equation*

$$\left.\begin{aligned} x'(t) + h(t)x(t) = \sigma(t),\ \ t \in J,\\ x(0) = x(T), \end{aligned}\right\} \tag{3.3}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_h(t,s)\,\sigma(s)\,ds \tag{3.4}$$

where,

$$G_h(t,s) = \begin{cases} \dfrac{e^{H(s)-H(t)+H(T)}}{e^{H(T)}-1}, & \textit{if} \quad 0 \le s \le t \le T,\\[2ex] \dfrac{e^{H(s)-H(t)}}{e^{H(T)}-1}, & \textit{if} \quad 0 \le t < s \le T. \end{cases} \tag{3.5}$$

and $H(t) = \displaystyle\int_0^t h(s)\,ds$*.*

Notice that the Green's function G_h is continuous and nonnegative on $J \times J$ and therefore, the number

$$K_h := \max\{\,|G_h(t,s)| \,:\, t,s \in [0,T]\,\}$$

exists for all $h \in L^1(J,\mathbb{R}^+)$. In particular, if $h(t) = \lambda$ for all $t \in J$, then for the sake of convenience we write $G_\lambda(t,s) = G(t,s)$ and $K_\lambda = K$.

An application of above Lemma 3.2 we obtain

Lemma 3.3. *Suppose that hypothesis (A_1) holds. Then a function* $u \in C(J,\mathbb{R})$ *is a solution of the PBVP* (1.1) *if and only if it is a solution of the nonlinear integral equation,*

$$x(t) = \big[f(t,x(t))\big]\left(\int_0^T G(t,s)g(s,x(s),X(s))\,ds\right) \tag{3.6}$$

for all $t \in J$*, where*

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda s-\lambda t+\lambda T}}{e^{\lambda T}-1}, & \text{if} \quad 0 \le s \le t \le T, \\ \dfrac{e^{\lambda s-\lambda t}}{e^{\lambda T}-1}, & \text{if} \quad 0 \le t < s \le T. \end{cases} \tag{3.7}$$

Theorem 3.1. *Assume that hypotheses* (A_0)*-*(A_4) *and* (B_1)*-*(B_3) *hold. Furthermore, assume that*

$$K\,M_g\,T\,\varphi(r) < r, \;\; r > 0, \tag{3.8}$$

then the QDE (1.1) has a positive solution x^* *defined on* J *and the sequence* $\{x_n\}_{n=1}^{\infty}$ *of successive approximations defined by*

$$x_{n+1}(t) = \big[f(t,x_n(t))\big]\left(\int_0^T G(t,s)\,g(s,x_n(s),X_n(s))\,ds\right), \;\; t \in J, \tag{3.9}$$

where $x_1 = u$*, converges monotonically to* x^**.*

Proof. Set $E = C(J,\mathbb{R})$. Then, by Lemma 3.1, every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ in E.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = f(t,x(t)), \;\; t \in J, \tag{3.10}$$

and

$$\mathcal{B}x(t) = \int_0^T G(t,s)g(s,x(s),X(t))\,ds, \;\; t \in J. \tag{3.11}$$

From the continuity of the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the maps $\mathcal{A},\mathcal{B} : E \to E$. Now by Lemma 3.3, the QDE (1.1) is equivalent to the operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t) = x(t), \;\; t \in J. \tag{3.12}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: $\mathcal{A}$ *and* $\mathcal{B}$ *are nondecreasing on* E*.*

Let $x,y \in E$ be such that $x \ge y$. Then $x(t) \ge y(t)$ for all $t \in J$. Since y is continuous on $[a,t]$, there exists a $\xi^* \in [a,t]$ such that $y(\xi^*) = \max\limits_{a\le\xi\le t} y(\xi)$. By definition of $\le$, one has $x(\xi^*) \ge y(\xi^*)$. Consequently, we obtain

$$X(t) = \max_{a\le\xi\le t} x(\xi) \ge x(\xi^*) \ge y(\xi^*) = \max_{a\le\xi\le t} y(\xi) = Y(t)$$

for each $t \in J$. Then by hypothesis (A_3), we obtain

$$\mathcal{A}x(t) = f(t,x(t)) \ge f(t,x(t)) = \mathcal{A}y(t),$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E. Similarly using hypothesis (B_3), it is shown that the operator $\mathcal{B}$ is also nondecreasing on E into itself. Thus, $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing positive operators on E into itself.

Step II: $\mathcal{A}$ *is partially bounded and partially* $\mathcal{D}$*-Lipschitz on* E*.*

Let $x \in E$ be arbitrary. Then by (A_2),

$$|\mathcal{A}x(t)| \le \big|f(t,x(t))\big| \le M_f,$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x\| \le M_f$ and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x,y \in E$ be such that $x \ge y$. Then,

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= \big|f(t,x(t)) - f(t,y(t))\big| \\ &\le \varphi(|x(t) - y(t)) \end{aligned}$$

$$\leq \varphi(\|x-y\|),$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \varphi(\|x-y\|),$$

for all $x, y \in E$ with $x \geq y$. Hence, $\mathcal{A}$ is a partial nonlinear $\mathcal{D}$-Lipschitz on E which further also implies that $\mathcal{A}$ is a partially continuous on operator on E.

Step III: *$\mathcal{B}$ is a partially continuous operator on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \int_0^T G(t,s) \left[\lim_{n\to\infty} g(s, x_n(s), X_n(s))\right] ds \\
&= \int_0^T G(t,s) g(s, x(s), X_n(s))\, ds \\
&= \mathcal{B}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary. Then, by hypothesis (B$_2$),

$$\begin{aligned}
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &\leq \bigg| \int_0^T G(t_2,s) g(s, x_n(s), X_n(s))\, ds \\
&\qquad - \int_0^T G(t_1,s) g(s, x_n(s), X_n(s))\, ds \bigg| \\
&\leq \int_0^T \big|G(t_2,s) - G(t_1,s)\big| \, \big|g(s, x_n(s), X_n(s))\big|\, ds \\
&\leq M_g \int_0^T \big|G(t_2,s) - G(t_1,s)\big|\, ds \\
&\to 0 \quad as \quad t_2 - t_1 \to 0
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and hence $\mathcal{B}$ is a partially continuous on E.

Step IV: *$\mathcal{B}$ is a uniformly partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let$y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ be such that $y = \mathcal{B}x$. Now, by hypothesis (B$_2$),

$$\begin{aligned}
|y(t)| &\leq \bigg| \int_0^T G(t,s) g(s, x(s), X(t))\, ds \bigg| \\
&\leq K \int_0^T |g(s, x(s), X(t))|\, ds \\
&\leq K\, M_g\, T = M,
\end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|y\| = \|\mathcal{B}x\| \leq M$ for all $y \in \mathcal{B}(C)$. Hence, $\mathcal{B}(C)$ is a uniformly bounded subset of E. Moreover, $\|\mathcal{B}(C)\| \leq M$ for all chains C in E. Hence, $\mathcal{B}$ is a uniformly partially bounded operator on E.

Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary. Then, for any $y \in \mathcal{B}(C)$, there is a $x \in C$ such that $y(t) = \mathcal{B}x(t)$. Hence, proceeding with arguments as in Step

III,

$$\begin{aligned}|y(t_2)-y(t_1)| &\le \Big|\int_0^T G(t_2,s)g(s,x(s),X(s))\,ds - \int_0^T G(t_1,s)g(s,x(s),X(s))\,ds\Big| \\ &\le M_g \int_0^T |G(t_2,s)-G(t_1,s)|\,ds \\ &\to 0 \quad as \quad t_2-t_1\to 0\end{aligned}$$

uniformly for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathcal{B}$ is a uniformly partially compact operator on E into itself.

Step V: *u satisfies the operator inequality $u \le \mathcal{A}u\,\mathcal{B}u$.*

By hypothesis (B_3), the QDE (1.1) has a lower solution u defined on J. Then, we have

$$\left.\begin{aligned}\frac{d}{dt}\left[\frac{u(t)}{f(t,u(t))}\right] + \lambda\left[\frac{u(t)}{f(t,u(t))}\right] &\le g(t,u(t),U(t)),\\ u(0)) &\le x(T),\end{aligned}\right\} \tag{3.13}$$

for all $t\in J$. Multiplying the first inequality in (3.13) by the integrating factor $e^{\lambda t}$, we obtain

$$\left(e^{\lambda t}\frac{u(t)}{f(t,u(t))}\right)' \le e^{\lambda t} g(t,u(t),U(t)), \tag{3.14}$$

for all $t \in J$. A direct integration of (3.14) from 0 to t yields

$$e^{\lambda t}\frac{u(t)}{f(t,u(t))} \le \frac{u(0)}{f(0,u(0))} + \int_0^t e^{\lambda s} g(s,u(s))\,ds, \tag{3.15}$$

for all $t \in J$. Therefore, in particular,

$$e^{\lambda T}\frac{u(T)}{f(T,u(T))} \le \frac{u(0)}{f(0,u(0))} + \int_0^T e^{\lambda s} g(s,u(s),U(s))\,ds. \tag{3.16}$$

Now $u(0)\le u(T)$, so by hypothesis (A_0) one has

$$\frac{u(0)}{f(0,u(0))}e^{\lambda T} \le \frac{u(T)}{f(T,u(T))}e^{\lambda T}. \tag{3.17}$$

From (3.15) and (3.17) it follows that

$$e^{\lambda T}\frac{u(0)}{f(0,u(0))} \le \frac{u(0)}{f(0,u(0))} + \int_0^T e^{\lambda s} g(s,u(s),U(s))\,ds \tag{3.18}$$

which further yields

$$\frac{u(0)}{f(0,u(0))} \le \int_0^T \frac{e^{\lambda s}}{(e^{\lambda T}-1)} g(s,u(s),U(s))\,ds \tag{3.19}$$

Substituting (3.19) in (3.15) we obtain

$$u(t) \le \big[f(t,u(t))\big]\left(\int_0^T G(t,s)g(s,u(s),U(s))\,ds\right),\ t\in J.$$

From the definitions of the operators $\mathcal{A}$ and $\mathcal{B}$ it follows that $u(t) \le \mathcal{A}u(t)\,\mathcal{B}u(t)$ for all $t \in J$. Hence $u \le \mathcal{A}u\,\mathcal{B}u$.

Step VI: *$\mathcal{D}$-function φ satisfies the growth condition $M\psi_{\mathcal{A}}(r) < r$, $r > 0$.*

Finally, the $\mathcal{D}$-function φ of the operator $\mathcal{A}$ satisfies the inequality given in hypothesis (d) of Theorem 2.1. Now from the estimate given in Step IV, it follows that $M\psi_{\mathcal{A}}(r) \le K\,M_g\,T\,\varphi(r) < r$ for all $r > 0$.

Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1 and we apply it to conclude that the operator equation $\mathcal{A}x\,\mathcal{B}x = x$ has a positive solution x^*. Therefore, the integral equation (3.6) and consequently the QDE (1.1) has a positive solution x^* defined on J. Furthermore, the sequence

$\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by (3.9) converges monotonically to x^*. This completes the proof. □

Remark 3.2. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (B_3) with the following:

(B_3') The QDE (1.1) has an upper solution $v \in C^1(J, \mathbb{R})$.

The proof under this new hypothesis is similar to the proof of Theorem 3.1 with appropriate modifications.

Remark 3.3. We note that if the QDE (1.1) has a lower solution u as well as an upper solution v such that $u \leq v$, then under the given conditions of Theorem 3.1 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \leq x^*$. Hence they are the minimal and maximal solutions of the PBVP (1.1) in the vector segment $[u, v]$ of the Banach space $E = C^1(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set in $C^1(J, \mathbb{R})$ defined by

$$[u, v] = \{x \in C^1(J, \mathbb{R}) \mid u \leq x \leq v\}.$$

This is because the order relation $\leq$ defined by (3.2) is equivalent to the order relation defined by the order cone $\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x \geq \theta\}$ which is a closed set in $C(J, \mathbb{R})$.

Finally, we give an example to illustrate our hypotheses (A_0)-(A_4) and (B_1)-(B_3) and the abstract result formulated in Theorem 3.1.

Example 3.1. Given a closed and bounded interval $J = [0, 1]$, consider the PBVP of QDEs,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t, x(t))}\right] + \left[\frac{x(t)}{f(t, x(t))}\right] &= \frac{1}{8}\Big[2 + \tanh x(t) + \tanh X(t)\Big], \\ x(0) &= x(1), \end{aligned}\right\} \tag{3.20}$$

for all $t \in J$, where the functions $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined as

$$f(t, x, y) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 + x, & \text{if } 0 < x < 3, \\ 4, & \text{if } x \geq 3. \end{cases}$$

and

$$g(t, x, y) = \frac{1}{20}\big[3 + \tanh x\big].$$

Clearly, the functions f and g are continuous nonnegative real-valued functions on $J \times \mathbb{R}$ and $J \times \mathbb{R} \times \mathbb{R}$ respectively. As $\dfrac{\partial}{\partial x}\left(\dfrac{x}{f(t, x)}\right) > 0$ for all $t \in J$ and $x \in \mathbb{R}$, the function $x \mapsto \dfrac{x}{f(t, x)}$ is increasing for each $t \in J$ and so, the hypothesis (A_0) is satisfied. Next, f satisfies the hypothesis (A_3) with $\varphi(r) = r$. To see this, we have

$$0 \leq f(t, x) - f(t, y) \leq x - y$$

for all $x, y \in \mathbb{R}$, $x \geq y$. Therefore, $\varphi(r) = r$. Moreover, the function $f(t, x)$ is periodic in t for each $x \in \mathbb{R}$ and is also bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 4$ and so, the hypotheses (A_2) and(A_3) are satisfied. Again, since g is positive and bounded on $J \times \mathbb{R} \times \mathbb{R}$ by $M_g = \frac{1}{4}$, the hypothesis (B_2) holds. Furthermore, $g(t, x, y)$ is nondecreasing in x and y for each $t \in J$, and thus hypothesis (B_3) is satisfied. Here, the Green's function $G(t, s)$ associated with the homogeneous PBVP

$$\left.\begin{aligned} x'(t) + x(t) = 0, \ \ t \in [0, 1], \\ x(0) = x(T), \end{aligned}\right\} \tag{3.21}$$

is given by

$$0 \leq G(t,s) = \begin{cases} \dfrac{e^{s-t+1}}{e-1}, & \text{if } 0 \leq s \leq t \leq 1, \\ \\ \dfrac{e^{s-t}}{e-1}, & \text{if } 0 \leq t \leq s \leq 1 \end{cases} \leq \frac{e}{e-1} \leq 2 = K. \tag{3.22}$$

Also the condition (3.8) of Theorem 3.1 holds. Finally, after a simple computation, it is shown that the function $u(t) = \dfrac{1}{20} \displaystyle\int_0^1 G(t,s)\, ds$ is a lower solution of the QDE (3.20) defined on J. Thus, all the hypotheses of Theorem 3.1 are satisfied. Hence we apply Theorem 3.1 and conclude that the QDE (3.20) has a positive solution x^* defined on J and the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by

$$x_1(t) = \frac{1}{20} \int_0^1 G(t,s)\, ds,$$

$$x_{n+1}(t) = \frac{1}{20} \big[f(t, x_n(t))\big] \left(\int_0^1 G(t,s) \big[2 + \tanh x_n(s) + \tanh X_n(s)\big]\, ds \right),$$

for all $t \in J$, converges monotonically to x^*.

References

[1] D.D. Bainov, S. Hristova, *Differential equations with maxima*, Chapman & Hall/CRC Pure and Applied Mathematics, 2011.

[2] B.C. Dhage, *Periodic boundary value problems of first order Carathéodory and discontinuous differential equations*, Nonlinear Funct. Anal. & Appl. **13**(2) (2008), 323-352.

[3] B.C. Dhage, *Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations*, Differ. Equ. Appl. **2** (2010), 465–486.

[4] B.C. Dhage, *Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations*, Differ. Equ. Appl. **5** (2013), 155-184.

[5] B.C. Dhage, *Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations*, Tamkang J. Math. **45** (4) (2014), 397-426.

[6] B.C. Dhage, *Nonlinear $\mathcal{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations*, Malaya J. Mat. **3**(1) (2015), 62-85.

[7] B.C. Dhage, *Operator theoretic techniques in the theory of nonlinear hybrid differential equations*, Nonlinear Anal. Forum **20** (2015), 15-31.

[8] B.C. Dhage, *A new monotone iteration principle in the theory of nonlinear first order integro-differential equations*, Nonlinear Studies **22** (3) (2015), 397-417.

[9] B.C. Dhage, S.B. Dhage, *Approximating solutions of nonlinear first order ordinary differential equations*, GJMS Special issue for Recent Advances in Mathematical Sciences and Applications-13, Global Journal of Mathematical Sciences, **2** (2014), 25-35.

[10] B.C. Dhage, S.B. Dhage, *Approximating solutions of nonlinear pbvps of hybrid differential equations via hybrid fixed point theory*, Indian J. Math. **57**(1) (2015), 103-119.

[11] B.C. Dhage, S.B. Dhage, *Approximating positive solutions of PBVPs of nonlinear first order ordinary quadratic differential equations*, Appl. Math. Lett. **46** (2015), 133-142.

[12] S.B. Dhage, B.C. Dhage, D. Octrocol, *Dhage iteration method for approximating positive solutions of nonlinear first order ordinary quadratic differential equations with maxima*, Fixed Point Theory, in press.

[13] B.C. Dhage, J. Henderson, S.K. Ntouyas, *Periodic boundary value problems of first order differential equations in Banach algebras*, J. Nonlinear Funct. Anal. & Diff. Equ. **1** (2007), 103-120.

[14] B.C. Dhage, V. Lakshmikantham, *Basic results on hybrid differential equations*, Nonlinear Analysis: Hybrid Systems **4** (2010), 414-424.

[15] D. Otrocol, I.A. Rus, *Functional-differential equations with maxima of mixed type argument*, Fixed Point Theory, **9**(2008), no. 1, pp. 207–220.

[16] S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.

[17] A.R. Magomedov, *On some questions about differential equations with maxima*, Izv. Akad. NaukAzerbaidzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk, **1** (1977), 104–108 (in Russian).

[18] A.R. Magomedov, *Theorem of existence and uniqueness of solutions of linear differential equations with maxima*, Izv. Akad. NaukAzerbaidzhan. SSR Ser. Fiz. Tekhn. Mat. Nauk, **5** (1979), 116-118 (in Russian).

[19] A.D. Myshkis, *on some problems of the theory of differential equations with deviating argument*, Russian Math. Surveys **32** (1977), 181-210.

[20] J.J. Nieto, R. Rodriguez-Lopez, *Existence and approximation of solution for nonlinear differential equations with periodic bounday conditions*, Compt. Math. Appl. **40** (2000), 435-442.

Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur Maharashtra, India

*Corresponding author: bcdhage@gmail.com