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COMMON BEST PROXIMITY POINTS FOR CYCLIC $\varphi-$CONTRACTION MAPS

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Abstract. The purpose of this paper is to introduce new types of contraction condition for a pair of maps (S,T) in metric spaces. We give convergence and existence results of best proximity points of such maps in the setting of uniformly convex Banach spaces. Moreover, we obtain existence theorems of best proximity points for such contraction pairs in reflexive Banach spaces. Our results generalize, extend and improve results on the topic in the literature.

1. Introduction and preliminaries

Let A and B be nonempty subsets of a metric space $X := (X, d)$ and T a cyclic map on $A \cup B$ i.e. $T(A) \subseteq B$, $T(B) \subseteq A$. An element $x \in A \cup B$ is called a best proximity point of the mapping T if $d(x, Tx) = d(A, B)$, where $d(A, B)$ is distance of A and B i.e.

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

The map T is called a cyclic contraction [2] if

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B),$$

for some $k \in (0, 1)$ and for all $x \in A$ and $y \in B$. If $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map then the cyclic map T is called cyclic $\varphi-$contraction map [1] if

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)),$$

for every $x \in A$ and $y \in B$.

Given two self maps S and T on $A \cup B$, a common best proximity point of the pair (S,T) is a point $x \in A \cup B$ satisfying $d(x, Sx) = d(x, Tx) = d(A, B)$.
The pair (S, T) is called a semi-cyclic contraction [3] if:
(i) $S(A) \subseteq B,\ T(B) \subseteq A$
(ii) $\exists \alpha \in (0, 1)$, such that $d(Sx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B)$, for every $x \in A$ *and* $y \in B$.
Let $S(A) \subseteq B,\ T(B) \subseteq A$. If $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map and

$$d(Sx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)),$$

for every $x \in A$ and $y \in B$, then the pair (S, T) is called semi-cyclic $\varphi-$contraction [8].

Clearly, if $S = T$ then a semi-cyclic $\varphi-$contraction pair reduces to a cyclic $\varphi-$ contraction map.
The best proximity point theorems emerge as a natural generalization of fixed point theorems, because a best proximity point reduces to a fixed point if $A \cap B \neq \emptyset$. A fundamental result in fixed point theory is the Banach contraction principle. One of the interesting extentions of this result was given by Kirk, Srinivasan and Veermani [5]. Eldred and Veeramani [2] gave existence and convergence results of best proximity point for cyclic contraction maps in uniformly convex Banach spaces and metric spaces, to include the case $A \cap B = \emptyset$. Al-Thagafi and Shahzad [1] obtained some such results for cyclic φ-contraction maps . Also Rezapour, Derafshpour and Shahzad [7] gave best proximity point of cyclic φ- contraction map on reflexive Banach spaces.

In 2011, Gabeleh and Abkar [3] proved theorems on the existence and convergence of best proximity point for semi-cyclic contraction pair in Banach spaces. In 2014, Thakur and Sharma [8] obtaind best

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proximity point for semi-cyclic $\varphi-$contraction pair in uniformly convex Banach spaces. Inspired by these results, we introduce a generalized semi-cyclic φ-contraction pair in metric spaces, which contain the general contractive pair of maps, and prove the existence and convergence of best proximity point for such pair of maps in metric spaces, uniformly convex Banach spaces and in reflexive Banach spaces. The following, definitions are needed for our resuls.

Definition 1.1. *[1] Let A and B be nonempty subsets of a normed linear space X, $T : A\cup B \to A\cup B$, $T(A) \subseteq B$ and $T(B) \subseteq A$. We say that T satisfies the proximal property if*

$$x_n \overset{w}{\to} x \in A \cup B,\ \|x_n - Tx_n\| \to d(A, B) \Longrightarrow \|x - Tx\| = d(A, B),$$

for $\{x_n\}_{n\geq 0} \in A \cup B$.

Definition 1.2. *A Banach space X is said to be*
(i) uniformly convex if there exists a stricly increasing function $\delta : (0, 2] \to [0, 1]$ such that the following implication holds for all $x_1, x_2, p \in X, R > 0$ and $r \in [0, 2R]$:

$$\|x_i - p\| \leq R,\ i = 1, 2 \text{ and } \|x_1 - x_2\| \geq r \Rightarrow \|(x_1 + x_2)/2 - p\| \leq (1 - \delta(r/R))R$$

(ii) strictly convex if the following impilication holds for all $x_1, x_2, p \in X$ and $R > 0$

$$\|x_i - p\| \leq R,\ i = 1, 2 \text{ and } x_1 \neq x_2 \Rightarrow \|(x_1 + x_2)/2 - p\| < R.$$

2. Main results

We introduce generalized semi-cyclic $\varphi-$contraction pair in metric spaces as under:

Definition 2.1. *Let A and B be nonempty subsets of a metric space X and $S, T : A \cup B \to A \cup B$ be such that $S(A) \subseteq B$ and $T(B) \subseteq A$. Then the pair (S, T) is said to be generalized semi-cyclic φ-contraction if $\varphi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing map and*

$$\begin{aligned} d(Sx, Ty) &\leq (1/3)\{d(x, y) + d(Sx, x) + d(Ty, y)\} \\ &- \varphi(d(x, y) + d(Sx, x) + d(Ty, y)) + \varphi(3d(A, B)), \end{aligned}$$

for all $x \in A$ and $y \in B$.

Example 2.1. *Take $\varphi(t) = (1 - k)(t/3)$ for $t \geq 0$ and $0 < k < 1$, we obtain*

$$d(Sx, Ty) \leq (k/3)\{d(x, y) + d(x, Sx) + d(y, Ty)\} + (1 - k)d(A, B),$$

for all $x \in A$ and $y \in B$, which is generalization of semi-cyclic contraction.
Note that, $S = T$ then we obtain generalized cyclic contraction map [4].

Example 2.2. *Let $X = \mathbb{R}$ with the usual metric. For $A = [0, 1],\ B = [-1, 0]$, define $S, T : A \cup B \to A \cup B$ by*

$$S(x) = \begin{cases} \frac{-x}{2} & x \in A \\ \frac{x}{2} & x \in B, \end{cases} \qquad T(x) = \begin{cases} \frac{x}{2} & x \in A \\ \frac{-x}{2} & x \in B. \end{cases}$$

Clearly $S(A) \subseteq B$ and $T(B) \subseteq A$. With $a \in A,\ b \in B$ and $\varphi(t) = \frac{t^2}{1+8t}$ for $t \geq 0$, (S, T) is a generalized semi-cyclic φ-contraction.

Let (S, T) be a generalized semi-cyclic φ-contraction. Consider $x_0 \in A$, then $Sx_0 \in B$, so there exists $y_0 \in B$ such that $y_0 = Sx_0$. Now $Ty_0 \in A$, so there exists $x_1 \in A$ such that $x_1 = Ty_0$. Inductively, we define sequences $\{x_n\}$ and $\{y_n\}$ in A and B, respectively by

$$x_{n+1} = Ty_n,\ y_n = Sx_n. \tag{1}$$

Lemma 2.1. *Let A and B be nonempty subsets of a metric space X and $S, T : A \cup B \to A \cup B$ be such that the pair (S, T) is generalized semi-cyclic φ-contraction . For $x_0 \in A \cup B$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (1) then for all $x \in A,\ y \in B$, and $n \geq 1$, we have*
(a) $-\varphi(d(x, y) + d(x, Sx) + d(y, Ty)) + \varphi(3d(A, B)) \leq 0$,
(b) $d(Sx, Ty) \leq (1/3)\{d(x, y) + d(x, Sx) + d(y, Ty)\}$,
(c) $d(x_n, Sx_n) \leq d(x_{n-1}, Sx_{n-1})$,
(d) $d(x_{n+1}, y_n) \leq d(y_n, Ty_{n-1})$,

(e) $d(y_{n+1}, Ty_n) \leq d(y_n, Ty_{n-1})$.

Proof. We have $3d(A,B) \leq d(x,y) + d(Tx,x) + d(Ty,y)$. Hence φ is a strictly increasing map, (a) and (b) are obtained. Since

$$d(x_n, Sx_n) \leq (1/3)\{d(y_{n-1}, x_n) + d(x_n, Sx_n) + d(y_{n-1}, x_n)\},$$

so

$$d(x_n, Sx_n) \leq d(y_{n-1}, x_n). \tag{2}$$

Also, since

$$d(y_{n-1}, x_n) \leq (1/3)\{d(y_{n-1}, x_{n-1}) + d(x_{n-1}, Sx_{n-1}) + d(y_{n-1}, x_n)\}, \text{ we have}$$

$$d(y_{n-1}, x_n) \leq d(x_{n-1}, Sx_{n-1}). \tag{3}$$

From (2) and (3), inequality (c) is obtained.
Since

$$d(x_{n+1}, y_n) \leq (1/3)\{d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_n, y_n)\},$$

so

$$d(x_{n+1}, y_n) \leq d(y_n, Ty_{n-1}),$$

that is inequlity (d).
Now, since

$$d(y_{n+1}, Ty_n) \leq (1/3)\{d(x_{n+1}, y_n) + d(y_{n+1}, Ty_n) + d(x_{n+1}, y_n)\}, \text{ we have}$$

$$d(y_{n+1}, Ty_n) \leq d(x_{n+1}, y_n).$$

By using (d), inequality (e) is obtained.

Following result will be needed in what follows.

Proposition 2.1. *Let A and B be nonempty subsets of a metric space X and $S, T : A \cup B \to A \cup B$ be such that the pair (S,T) is generalized semi-cyclic φ-contraction map. For $x_0 \in A \cup B$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (1). Then*
$d(x_n, Sx_n) \to d(A,B)$ and $d(y_n, Ty_{n-1}) \to d(A,B)$.

Proof. Let $d_n = d(x_n, Sx_n)$. It follows from Lemma 2.1(c), that $\{d_n\}$ is decreasing and bounded. So $\lim_{n\to\infty} d_n = t_0$. Since (S,T) is a generalized semi-cyclic φ-contraction pair, we obtain
$d_{n+1} \leq d(Sx_n, Ty_n)$
$\leq (1/3)\{2d_n + d(y_n, Ty_n)\} - \varphi(2d_n + d(y_n, Ty_n)) + \varphi(3d(A,B))$
$\leq d_n - \varphi(2d_n + d(y_n, Ty_n)) + \varphi(3d(A,B))$.
Hence,

$$\varphi(3d(A,B)) \leq \varphi(2d_n + d(y_n, Ty_n)) \leq d_n - d_{n+1} + \varphi(3d(A,B)).$$

Thus

$$\lim_{n\to\infty} \varphi(2d_n + d(y_n, Ty_n)) = \varphi(3d(A,B)). \tag{4}$$

Since φ is strictly increasing and $d_n \geq d(y_n, Ty_n) \geq d_{n+1} \geq t_0 \geq d(A,B)$, we have

$$\lim_{n\to\infty} \varphi(2d_n + d(y_n, Ty_n)) \geq \varphi(3t_0) \geq \varphi(3d(A,B)).$$

From (4),

$$\varphi(3t_0) = \varphi(3d(A,B)).$$

As φ is strictly increasing, we have $t_0 = d(A,B)$.

Theorem 2.1. *Let A and B be nonempty subsets of a metric space X and $S, T : A \cup B \to A \cup B$ be such that the pair (S,T) is a generalized semi-cyclic φ-contraction map. For $x_0 \in A \cup B$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (1). If $\{x_n\}$ and $\{y_n\}$ have convergent subsequences in A and B, then there exists $x \in A$ and $y \in B$ such that*

$$d(x, Sx) = d(A,B) = d(y, Ty).$$

Proof. Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that $y_{n_k} \to y$. The relation

$$d(A,B) \leq d(Ty_{n_k}, y) \leq d(y_{n_k}, y) + d(y_{n_k}, Ty_{n_k})$$

holds for each $k \geq 1$. Letting $k \to \infty$, by Proposion 2.1 and Lemma 2.1(d), we obtain

$$\lim_{k\to\infty} d(Ty_{n_k}, y) = d(A,B).$$

From Lemma2.1(b),
$d(Ty, y_{n_k}) \leq (1/3)\{d(y, x_{n_k}) + d(y, Ty) + d(x_{n_k}, Sx_{n_k})\}$
$\leq (1/3)\{d(y, y_{n_k}) + d(y_{n_k}, x_{n_k}) + d(y, y_{n_k}) + d(Ty, y_{n_k}) + d(x_{n_k}, Sx_{n_k})\}$.
Letting $k \to \infty$, by Proposion 2.1, we get

$$(2/3)d(A,B) \leq (2/3)\lim_{k\to\infty} d(Ty, y_{n_k}) \leq (2/3)d(A,B).$$

So $d(Ty, y) = d(A,B)$. Similary, it can be proved that $d(x, Sx) = d(A,B)$.

Proposition 2.2. *Let A and B be nonempty subsets of a metric space X and $S, T : A \cup B \to A \cup B$ be such that the pair (S,T) is generalized semi-cyclic φ-contraction.Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1) are bounded.*

Proof. By Proposition 2.1, we have $d(x_n, Sx_n) \to d(A,B)$ as $n \to \infty$. It is sufficient to show that $\{Sx_n\}$ is bounded. For the unbounded map φ, take $M > 0$ such that

$$\varphi(M) > (4/3)d(x_0, Sx_0) + \varphi(3d(A,B)).$$

If $\{Sx_n\}$ is not bounded, then there exists a natural number $N \in \mathbb{N}$, such that

$$d(x_1, Sx_N) > M,\ d(x_1, Sx_{N-1}) \leq M.$$

Then

$M < d(x_1, Sx_N)$
$\leq d(y_0, x_N)$
$\leq (1/3)\{d(x_0, y_{N-1}) + d(x_0, Sx_0) + d(y_{N-1}, Ty_{N-1})\}$
$-\varphi(d(x_0, y_{N-1}) + d(x_0, Sx_0) + d(y_{N-1}, Ty_{N-1})) + \varphi(3d(A,B))$
$\leq (1/3)\{d(x_0, x_1) + d(x_1, y_{N-1}) + d(x_0, Sx_0) + d(x_{N-1}, y_{N-1})\}$
$-\varphi(d(x_0, y_{N-1})) + \varphi(3d(A,B))$
$\leq (1/3)\{d(x_0, y_0) + d(y_0, x_1) + M + d(x_0, Sx_0) + d(x_{N-1}, y_{N-2})\}$
$-\varphi(d(x_0, y_{N-1})) + \varphi(3d(A,B))$
$\leq (1/3)\{3d(x_0, Sx_0) + M + d(x_{N-2}, y_{N-2})\}$
$-\varphi(d(x_0, y_{N-1})) + \varphi(3d(A,B)) + \varphi(3d(A,B))$
$< (4/3)d(x_0, Sx_0) + M - \varphi(d(x_0, y_{N-1})) + \varphi(3d(A,B))$.
Hence

$$\varphi(d(x_0, y_{N-1})) < (4/3)d(x_0, Sx_0) + \varphi(3d(A,B)).$$

Therefore

$\varphi(M) < \varphi(d(x_1, Sx_N)) \leq \varphi(d(y_0, x_N)) \leq \varphi(d(x_0, y_{N-1}))$
$< (4/3)d(x_0, Sx_0) + \varphi(3d(A,B))$,
which is a contradiction. Hence $\{Sx_n\}$ is bounded, therefore $\{x_n\}$ is bounded.

Corollary 2.1. *Let A and B be nonempty subsets of a metric space X and $S, T : A \cup B \to A \cup B$ be such that the pair (S,T) is generalized semi-cyclic φ-contraction. If A and B are boundedly compact then there exists $x \in A$ and $y \in B$ such that*

$$d(x, Sx) = d(A,B) = d(y, Ty).$$

Proof. The result is an immediately consequence of Proposition 2.2 and Theorem 2.1.

Now, define a sequence $\{z_n\}$ in $A \cup B$ as:

$$z_n = \begin{cases} Ty_k & n = 2k \\ Sx_k & n = 2k-1. \end{cases} \tag{5}$$

In the following, we consider a uniformly convex Banach space X and give best proximity point for generalized semi-cyclic φ-contraction pair (S,T).

Lemma 2.2. *Let A and B be nonempty convex subsets of a uniformly convex Banach space X and $S,T : A\cup B \to A\cup B$ be such that the pair (S,T) is generalized semi-cyclic φ-contraction. For $x_0 \in A\cup B$, if the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (1) and sequence $\{z_n\}$ is generated by (5), then $\|z_{2n+2}-z_{2n}\| \to 0$ and $\|z_{2n+3}-z_{2n+1}\| \to 0$ as $n\to\infty$.*

Proof. To show $\|z_{2n+2}-z_{2n}\|\to 0$ as $n\to\infty$, assume the contrary. Then there exists $\epsilon_0>0$ such that for each $k\geq 1$, there exists $n_k \geq k$ such that

$$\|z_{2n_k+2}-z_{2n_k}\| \geq \epsilon_0. \tag{6}$$

Choose $\epsilon>0$ such that $\left(1-\delta\left(\frac{\epsilon_0}{d(A,B)+\epsilon}\right)\right)(d(A,B)+\epsilon) < d(A,B)$. By Proposition 2.1, there exists N_1 such that

$$\|z_{2n_k+2}-z_{2n_k+1}\| \leq d(A,B)+\epsilon, \tag{7}$$

for every $n_k \geq N_1$. Also,
$\|z_{2n_k}-z_{2n_k+1}\| \leq \|y_{n_k}-x_{n_k+1}\| \leq \|x_{n_k}-y_{n_k}\| \to d(A,B)$,
so, there exists N_2 such that

$$\|z_{2n_k}-z_{2n_k+1}\| \leq d(A,B)+\epsilon, \tag{8}$$

for all $n_k\geq N_2$. Let $N=\max\{N_1,N_2\}$. From (6)-(8) and the uniform convexity of X, we get

$$\left\|\frac{z_{2n_k+2}+z_{2n_k}}{2}-z_{2n_k+1}\right\| \leq \left(1-\delta\left(\frac{\epsilon_0}{d(A,B)+\epsilon}\right)\right)(d(A,B)+\epsilon),$$

for all $n_k\geq N$. As $(z_{2n_k+2}+z_{2n_k})/2\in A$, the choice of ϵ implies that

$$\left\|\frac{z_{2n_k+2}+z_{2n_k}}{2}-z_{2n_k+1}\right\| < d(A,B),$$

for all $n_k \geq N$, a contradiction. By a similar argument we can show that $\|z_{2n+3}-z_{2n+1}\|\to 0$ as $n\to\infty$.

Proposition 2.3. *Let A and B be nonempty convex subsets of a uniformly convex Banach space X and $S,T : A\cup B\to A\cup B$ be such that the pair (S,T) is generalized semi-cyclic φ-contraction. For $x_0\in A\cup B$, if the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (1) and sequence $\{z_n\}$ is generated by (5), then for each $\epsilon>0$, there exists a positive integer N_0 such that for all $m>n\geq N_0$,*

$$\|z_{2m}-z_{2n+1}\| < d(A,B)+\epsilon.$$

Proof. Suppose the contrary. So there exists $\epsilon_0>0$ such that for each $k\geq 1$, there is $m_k>n_k\geq k$ satisfying

$$\|z_{2m_k}-z_{2n_k+1}\| \geq d(A,B)+\epsilon_0 \tag{9}$$

and

$$\|z_{2(m_k-1)}-z_{2n_k+1}\| < d(A,B)+\epsilon_0. \tag{10}$$

Now from (9) and (10), we get

$$\begin{aligned} d(A,B)+\epsilon_0 \leq \|z_{2m_k}-z_{2n_k+1}\| &\leq \|z_{2m_k}-z_{2(m_k-1)}\| + \|z_{2(m_k-1)}-z_{2n_k+1}\| \\ &< \|z_{2m_k}-z_{2(m_k-1)}\| + d(A,B)+\epsilon_0. \end{aligned}$$

Letting $k\to\infty$, Lemma 2.2 implies

$$\lim_{k\to\infty}\|z_{2m_k}-z_{2n_k+1}\| = d(A,B)+\epsilon_0. \tag{11}$$

By Lemma 2.1(b) and (d),

$$\|z_{2m_k} - z_{2n_k+1}\| \leq \|z_{2m_k} - z_{2m_k+2}\| + \|z_{2m_k+2} - z_{2n_k+3}\| + \|z_{2n_k+3} - z_{2n_k+1}\|$$
$$\leq \|z_{2m_k} - z_{2m_k+2}\| + (1/3)\{\|y_{m_k+1} - x_{n_k+2}\|$$
$$+\|y_{m_k+1} - x_{m_k+2}\| + \|x_{n_k+2} - y_{n_k+2}\|\} + \|z_{2n_k+3} - z_{2n_k+1}\|$$
$$\leq \|z_{2m_k} - z_{2m_k+2}\| + (1/9)\{\|x_{m_k+1} - y_{n_k+1}\|$$
$$+\|x_{m_k+1} - y_{m_k+1}\| + \|y_{n_k+1} - x_{n_k+1}\|\}$$
$$+(1/3)\{\|y_{m_k+1} - x_{m_k+1}\| + \|x_{n_k+2} - y_{n_k+2}\|\} + \|z_{2n_k+3} - z_{2n_k+1}\|.$$

Letting $k \to \infty$, from (11), Lemma 2.2 and Proposition 2.1 we get

$$d(A,B) + \epsilon_0 \leq (1/9)(d(A,B) + \epsilon_0) + (2/9)d(A,B) + (2/3)d(A,B),$$

so

$$d(A,B) + \epsilon_0 \leq d(A,B) + (1/9)\epsilon_0,$$

this is a contradiction.

Theorem 2.2. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and $S,T : A\cup B \to A\cup B$ be such that the pair (S,T) is generalized semi-cyclic φ-contraction. For $x_0 \in A \cup B$, if the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (1) and sequence $\{z_n\}$ is generated by (5), then there exist unique $x \in A$ and $y \in B$ such that $z_{2n} \to x$, $z_{2n+1} \to y$ and $\|x - Sx\| = d(A,B) = \|y - Ty\|$.*

Proof. First, we show that $\{z_{2n}\}$ is a Cauchy sequence in A. If $d(A,B) = 0$, then let $\epsilon_0 > 0$ be given. By Lemma 2.1(d) and Proposition 2.1,

$$\|z_{2n} - z_{2n+1}\| = \|Ty_n - Sx_{n+1}\| \leq \|x_n - Sx_n\| \to d(A,B) = 0.$$

So, there exists a positive integer N_1 such that

$$\|z_{2n} - z_{2n+1}\| < \epsilon,$$

for every $n \geq N_1$. By Proposition 2.2, there exists a positive integer N_2 such that

$$\|z_{2m} - z_{2n+1}\| < \epsilon,$$

for every $m > n \geq N_2$. Let $N = \max\{N_1, N_2\}$. It follows that

$$\|z_{2m} - z_{2n}\| \leq \|z_{2m} - z_{2n+1}\| + \|z_{2n} - z_{2n+1}\| < 2\epsilon,$$

for all $m > n \geq N$. Therefore $\{z_{2n}\}$ is a Cauchy sequence in A. Now, we assume that $d(A,B) > 0$. To show that $\{z_{2n}\}$ is a Cauchy sequence in A, we assume the contrary. Then there exists $\epsilon_0 > 0$ such that for each $k \geq 1$ there exists $m_k > n_k \geq k$ so that

$$\|z_{2m_k} - z_{2n_k}\| \geq \epsilon_0. \tag{12}$$

Choose $\epsilon > 0$ such that

$$\left(1 - \delta\left(\frac{\epsilon_0}{d(A,B)+\epsilon}\right)\right)(d(A,B)+\epsilon) < d(A,B).$$

By Lemma 2.1(d) and Proposition 2.1,

$$\|z_{2n_k} - z_{2n_k+1}\| = \|Ty_{n_k} - Sx_{n_k+1}\| \leq \|x_{n_k} - Sx_{n_k}\| \to d(A,B).$$

Hence, there exists a positive integer N_1 such that

$$\|z_{2n_k} - z_{2n_k+1}\| \leq d(A,B) + \epsilon, \tag{13}$$

for all $n_k \geq N_1$. By Proposition 2.2 there exists a positive integer N_2 such that

$$\|z_{2m_k} - z_{2n_k+1}\| \leq d(A,B) + \epsilon, \tag{14}$$

for all $m_k > n_k \geq N_2$. Let $N = \max\{N_1, N_2\}$. From (12)-(14) and the uniform convexity of X, we get

$$\left\|\frac{z_{2m_k} + z_{2n_k}}{2} - z_{2n_k+1}\right\| \leq \left(1 - \delta\left(\frac{\epsilon_0}{d(A,B)+\epsilon}\right)\right)(d(A,B)+\epsilon),$$

for all $m_k > n_k \geq N$. As $(z_{2m_k} + z_{2n_k})/2 \in A$, the choice of ϵ implies that

$$\left\|\frac{z_{2m_k} + z_{2n_k}}{2} - z_{2n_k+1}\right\| < d(A,B),$$

for all $m_k > n_k \geq N$, a contradiction. Thus $\{z_{2n}\}$ Cauchy sequence in A. By a similar argument, we can show that $\{z_{2n+1}\}$ is a Cauchy sequence in B. The completeness of X and the closedness of A implies that $z_{2n} \to x$ as $n \to \infty$. By Theorem 2.1, $\|x - Sx\| = d(A, B)$. Also, it follows from closedness of B and Theorem 2.1 that $\|y - Ty\| = d(A, B)$. To prove uniqueness, assume that there is $a \in A$ such that $a \neq x$ and $\|a - Sa\| = d(A, B)$. By Lemma 2.1 (b),

$$\|TSx - Sx\| \leq (1/3)\{2\|Sx - x\| + \|TSx - Sx\|\},$$

hence

$$(2/3)d(A, B) \leq (2/3)\|TSx - Sx\| \leq (2/3)\|Sx - x\| = (2/3)d(A, B).$$

Therefore, $\|TSx - Sx\| = d(A, B)$, it follows that $TSx = x$. Now

$$\begin{aligned}&\|Sx - a\| = \|Sx - TSa\| \leq (1/3)\{\|Sa - x\| + \|x - Sx\| + \|Sa - a\|\}\\ &\leq (1/9)\{\|Sx - a\| + \|Sa - a\| + \|x - Sx\|\}\\ &+(1/3)\{\|x - Sx\| + \|Sa - a\|\}.\end{aligned}$$

Hence

$$(8/9)d(A, B) \leq (8/9)\|Sx - a\| \leq (8/9)d(A, B),$$

which implies that, $\|Sx - a\| = d(A, B)$. From convexity of A and strict convexity of X, we get

$$\left\|\frac{x + a}{2} - Sx\right\| = \left\|\frac{x - Sx}{2} + \frac{a - Sx}{2}\right\| < d(A, B),$$

a contradiction. Thus $x = a$. Similarly, we show the uniqueness of $y \in B$.

Now, we show the existence of a best proximity point for generalized semi-cyclic φ-contraction pair (S, T) in reflexive Banach spaces.
First, we prove the following theorem.

Theorem 2.3. *Let A and B be nonempty weakly closed subsets of a reflexive Banach space X and $S, T : A \cup B \to A \cup B$ be such that the pair (S, T) is generalized semi-cyclic φ-contraction. Then there exists $(x, y) \in A \times B$ such that*

$$\|x - y\| = d(A, B).$$

Proof. If $d(A, B) = 0$, the result follows by Theorem 3.1(i) of [6]. So we assume that $d(A, B) > 0$. For $x_0 \in A$, if the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (1) and sequence $\{z_n\}$ is generated by (5) then from Proposition 2.3, the sequences $\{z_{2n}\}$ and $\{z_{2n+1}\}$ are bounded. Since X is reflexive and A is weakly closed, the sequence $\{z_{2n}\}$ has a subsequence $\{z_{2n_k}\}$ such that $z_{2n_k} \overset{w}{\to} x \in A$. Also B is weakly closed, hence $z_{2n_k+1} \overset{w}{\to} y \in B$ as $k \to \infty$. Since $z_{2n_k} - z_{2n_k+1} \overset{w}{\to} x - y \neq 0$ as $k \to \infty$, there exists a bounded liner functional $f : X \to [0, \infty)$ such that $\|f\| = 1$ and $f(x - y) = \|x - y\|$. For all $k \geq 1$, we have

$$|f(z_{2n_k} - z_{2n_k+1})| \leq \|f\|\|z_{2n_k} - z_{2n_k+1}\| = \|z_{2n_k} - z_{2n_k+1}\|.$$

Since $\lim_{k\to\infty} |f(z_{2n_k} - z_{2n_k+1})| = \|x - y\|$, by Lemma 2.1(d) and Proposition 2.1, we get

$$\begin{aligned}&\|x - y\| = \lim_{k\to\infty} |f(z_{2n_k} - z_{2n_k+1})| \leq \lim_{k\to\infty} \|z_{2n_k} - z_{2n_k+1}\|\\ &\leq \lim_{k\to\infty} \|x_{n_k} - Sx_{n_k}\|\\ &= d(A, B).\end{aligned}$$

So, $\|x - y\| = d(A, B)$.

Theorem 2.4. *Let A and B be nonempty weakly closed subsets of a reflexive Banach space X and $S, T : A \cup B \to A \cup B$ be such that the pair (S, T) is generalized semi-cyclic φ-contraction. Then there exists $x \in A$ and $y \in B$ such that*

$$\|x - Sx\| = d(A, B) = \|Ty - y\|,$$

provided that one of the following conditions is satisfied
(i) S is weakly continuous on A and T is weakly continuous on B.
(ii) T, S satisfy the proximal property.

Proof. If $d(A,B)=0$, the result follows from Theorem 3.1(i) of [6]. So we assume that $d(A,B)>0$. For $x_0 \in A$, if the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (1) and sequence $\{z_n\}$ is generated by (5) then by Proposition 2.3 the sequences $\{z_{2n}\}$ and $\{z_{2n+1}\}$ are bounded. Since A is weakly closed, the sequence $\{z_{2n}\}$ has a subsequence $\{z_{2n_k}\}$ such that $z_{2n_k} \stackrel{w}{\to} x \in A$.
From (i), $z_{2n_k+1} \stackrel{w}{\to} Sx \in B$ as $k \to \infty$. So $z_{2n_k} - z_{2n_k+1} \stackrel{w}{\to} x - Sx \neq 0$ as $k \to \infty$. Now the proof continues similar to that of Theorem 2.3. Also B is weakly closed, so $z_{2n_k+1} \stackrel{w}{\to} y \in B$ as $k \to \infty$. Since, (i) holds, $z_{2n_k+2} \stackrel{w}{\to} Ty$, as $k \to \infty$. Next the proof continues similar to that of Theorem 2.3.
From (ii), by Lemma 2.1(d) and Proposition 2.1, $\|z_{2n_k+1} - Tz_{2n_k+1}\| \to d(A,B)$ as $k \to \infty$. So $\|y - Ty\| = d(A,B)$. Also, $\|z_{2n_k} - Sz_{2n_k}\| \to d(A,B)$ as $k \to \infty$. Thus $\|x - Sx\| = d(A,B)$.

Next, we consider reflexive and strictly convex Banach spaces and give best proximity point for generalized semi-cyclic φ-contraction pair.

Theorem 2.5. *Let A and B be nonempty closed and convex subsets of a reflexive and strictly convex Banach space X and $S,T : A \cup B \to A \cup B$ be such that the pair (S,T) is generalized semi-cyclic φ-contraction. If $(A-A) \cap (B-B) = \{0\}$, then there exists a unique $x \in A$ and a unique $y \in B$ such that*

$$\|x - Sx\| = d(A,B) = \|Ty - y\|.$$

Proof. If $d(A,B)=0$, the result follows from Theorem 3.1(i) of [6]. So we assume that $d(A,B)>0$. Since A and B are closed and convex, they are weakly closed. By Theorem 2.3, there exists $(x,y) \in A \times B$ with $\|x-y\| = d(A,B)$. Suppose that there exists $(a,b) \in A \times B$ with $\|a-b\| = d(A,B)$. Since $(A-A) \cap (B-B) = \{0\}$, $x - y \neq a - b$. By the strict convexity of X, and convexity of A and B, we have

$$\|(x+a)/2 - (y+b)/2\| = \|(x-y)/2 + (a-b)/2\| < d(A,B),$$

which is a contraction. This shows that (x,y) is unique.

Theorem 2.6. *Let A and B be nonempty closed and convex subsets of a reflexive and strictly convex Banach space X and $S,T : A \cup B \to A \cup B$ be such that the pair (S,T) is generalized semi-cyclic φ-contraction. Then there exist unique $x \in A$ and $y \in B$ such that*

$$\|x - Sx\| = d(A,B) = \|Ty - y\|,$$

provided that one of the following conditions is satisfied
(i) S is weakly continuous on A and T is weakly continuous on B.
(ii) T, S satisfy the proximal property.

Proof. If $d(A,B)=0$, the result follows from Theorem 3.1(i) of [6]. So we assume that $d(A,B)>0$. Since A and B are closed and convex, they are weakly closed. By Theorem 2.4, there exists $x \in A$ and $y \in B$ such that

$$\|x - Sx\| = d(A,B) = \|Ty - y\|.$$

For the uniqueness of x, suppose that there exists $a \in A$ such that $\|a - Sa\| = d(A,B)$. By the strict convexity of X, and convexity of A and B, we have

$$\|(x+a)/2 - (Sx+Sa)/2\| = \|(x-Sx)/2 + (a-Sa)/2\| < d(A,B),$$

which is a contraction.
Now, for uniqueness of y, suppose that there exists $b \in B$ such that $\|Tb - b\| = d(A,B)$. Since

$$\|(y+b)/2 - (Ty+Tb)/2\| = \|(y-Ty)/2 + (b-Tb)/2\| < d(A,B),$$

which is a contraction.

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