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ON $\mid C,1\mid_k$ INTEGRABILITY OF IMPROPER INTEGRALS

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Abstract. In this paper, we introduce the concept of $\mid C,1\mid_k, k\geq 1$, integrability of improper integrals and we prove a known theorem of Mazhar [3] by using this definition.

1. Introduction

Throughout this paper we assume that f is a real valued function which is continuous on $[0,\infty)$ and $s(x)=\int_0^x f(t)dt$. The Cesàro mean of $s(x)$ is defined by

$$\sigma(x)=\frac{1}{x}\int_0^x s(t)dt.$$

The integral $\int_0^\infty f(t)dt$ is said to be integrable $\mid C,1\mid_k, k\geq 1$, in the sense of Flett [2], if

$$\int_0^\infty x^{k-1}\mid \sigma'(x)\mid^k dx \tag{1.1}$$

is convergent. The Kronecker identity (see [1]): $s(x)-\sigma(x)=v(x)$, where $v(x)=\frac{1}{x}\int_0^x tf(t)dt$ is well-known and will be used in the various steps of proofs.
Since $\sigma'(x)=\frac{1}{x}v(x)$, condition (1.1) can also be written as

$$\int_0^\infty \frac{1}{x}\mid v(x)\mid^k dx \tag{1.2}$$

is convergent.

We note that for infinite series, an analogous definition was introduced by Flett [2]. Using this definition, Mazhar [3] established the following theorem for $\mid C,1\mid_k$ summability factors of infinite series.

Given any functions f,g, it is customary to write $g(x)=O(f(x))$, if there exist η and N, for every $x>N, \mid \frac{g(x)}{f(x)}\mid\leq\eta$.

Theorem 1.1. *If (X_n) is a positive monotonic non-decreasing sequence such that*

$$\lambda_m X_m = O(1) \text{ as } m\to\infty, \tag{1.3}$$

$$\sum_{n=1}^m nX_n\mid \Delta^2\lambda_n\mid = O(1), \tag{1.4}$$

$$\sum_{n=1}^m \frac{1}{n}\mid t_n\mid^k = O(X_m) \text{ as } m\to\infty, \tag{1.5}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,1\mid_k, k\geq 1$.

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2. The Main Result

The aim of this paper is to prove Mazhar's theorem for $\mid C,1\mid_k$ integrability of improper integrals. Now, we shall state the following theorem.

Theorem 2.1. *If $\gamma(x)$ is a positive monotonic non-decreasing function such that*

$$\lambda(x)\gamma(x) = O(1) \text{ as } x\to\infty, \tag{2.1}$$

$$\int_0^x u\mid \lambda''(u)\mid \gamma(u)du = O(1), \tag{2.2}$$

$$\int_0^x \frac{\mid v(u)\mid^k}{u}du = O(\gamma(x)) \text{ as } x\to\infty, \tag{2.3}$$

then the integral $\int_0^\infty f(t)dt$ is integrable $\mid C,1\mid_k, k\geq 1$.

We need the following lemma for the proof of our theorem.

Lemma 2.2. *Under the conditions of the theorem we have that*

$$\int_0^\infty \gamma(t)\mid \lambda'(t)\mid dt \text{ is convergent}, \tag{2.4}$$

$$x\gamma(x)\mid \lambda'(x)\mid = O(1) \text{ as } x\to\infty. \tag{2.5}$$

Proof. Since $\lambda'(t)=\int_0^t \lambda''(u)$, we have

$$\begin{aligned}
\int_0^x \gamma(t)\mid \lambda'(t)\mid dt &= \int_0^x \gamma(t)\mid \int_0^t \lambda''(u)du\mid dt \\
&\leq \int_0^x \gamma(t)\int_0^t \mid \lambda''(u)\mid dudt \\
&= \int_0^x \mid \lambda''(u)\mid du \int_u^x \gamma(t)dt \\
&\leq \int_0^x u\gamma(u)\mid \lambda''(u)\mid du = O(1) \text{ as } x\to\infty
\end{aligned}$$

by (2.2).
Since $x\gamma(x)$ is a non decreasing function, we get

$$\begin{aligned}
x\gamma(x)\mid \lambda'(x)\mid &= x\gamma(x)\mid \int_0^x \lambda''(u)du\mid \\
&\leq x\gamma(x)\int_0^x \mid \lambda''(u)\mid du \\
&= \int_0^x u\gamma(u)\mid \lambda''(u)\mid du = O(1) \\
&\leq \int_0^x u\gamma(u)\mid \lambda''(u)\mid du = O(1) \text{ as } x\to\infty
\end{aligned}$$

This completes the proof of Lemma 2.2. □

3. Proof of the Theorem

Let $A(x)$ be the function of $(C,1)$ means of the integral $\int_0^\infty f(t)dt$. Then, by definition, we have

$$\begin{aligned} A(x) &= \frac{1}{x}\int_0^x \int_0^t \lambda(u)f(u)dudt \\ &= \frac{1}{x}\int_0^x \lambda(u)f(u)du \int_u^x dt \\ &= \frac{1}{x}\int_0^x (x-u)\lambda(u)f(u)du \\ &= \int_0^x \left(1-\frac{u}{x}\right)\lambda(u)f(u)du \end{aligned}$$

Differentiating the function $A(x)$ and later integrating by parts, we obtain

$$\begin{aligned} A'(x) &= \frac{1}{x^2}\int_0^x u\lambda(u)f(u)du \\ &= \frac{v(x)\lambda(x)}{x} - \frac{1}{x^2}\int_0^x \lambda'(u)uv(u)du \\ &= A_1(x)+A_2(x),\ say. \end{aligned}$$

To complete the proof of the theorem, it is sufficient to show that

$$\int_0^x t^{k-1} \mid A_r(t) \mid^k dt = O(1)\ as\ x\to\infty,\ for\ r=1,2. \tag{3.1}$$

First, applying Hölder's inequality, we have

$$\begin{aligned} \int_0^x t^{k-1} \mid A_1(t) \mid^k dt &= \int_0^x t^{k-1}\frac{\mid v(t)\mid^k \mid \lambda(t)\mid^k}{t^k}dt \\ &= \int_0^x \frac{1}{t}\mid v(t)\mid^k \mid \lambda(t)\mid^{k-1} \mid \lambda(t)\mid dt \\ &\le \int_0^x \frac{\mid v(t)\mid^k}{t}\mid \lambda(t)\mid dt \\ &= \mid \lambda(x)\mid \int_0^x \frac{\mid v(t)\mid^k}{t}dt - \int_0^x \mid \lambda'(t)\mid \int_0^t \frac{\mid v(u)\mid^k}{u}dudt \\ &= \mid \lambda(x)\mid \gamma(x) - \int_0^x \mid \lambda'(t)\mid \gamma(t)dt \\ &= O(1)\ as\ x\to\infty \end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.
Now, as in $A_1(x)$, we have that

$$\begin{aligned} \int_0^x t^{k-1} \mid A_2(t) \mid^k dt &= \int_0^x t^{k-1}\frac{1}{t^{2k}} \mid \int_0^t u\lambda'(u)v(u)du \mid^k dt \\ &\le \int_0^x \frac{1}{t^2}\left\{\int_0^t \mid \lambda'(u)\mid^k u^k \mid v(u)\mid^k du\right\} x \left\{\frac{1}{t}\int_0^t du\right\}^{k-1} dt \\ &= \int_0^x \mid u\lambda'(u)\mid^{k-1} \mid u\lambda'(u)\mid\mid v(u)\mid^k du \int_u^x \frac{dt}{t^2} \\ &= \int_0^x \mid u\lambda'(u)\mid\mid v(u)\mid^k \left(\frac{1}{u}-\frac{1}{x}\right)du \\ &\le \int_0^x \mid u\lambda'(u)\mid \frac{\mid v(u)\mid^k}{u}du \end{aligned}$$

Integrating by parts,we get

$$\begin{aligned}\int_0^x t^{k-1} \mid A_2(t) \mid^k dt &= x \mid \lambda'(x) \mid \int_0^x \frac{\mid v(u) \mid^k}{u} du + \int_0^x (u \mid \lambda'(u) \mid)' \int_0^u \frac{\mid v(t) \mid^k}{t} dt du \\ &= x \mid \lambda'(x) \mid \gamma(x) - \int_0^x (u \mid \lambda'(u) \mid)' \gamma(u) du \\ &= x \mid \lambda'(x) \mid \gamma(x) - \int_0^x \mid \lambda'(u) \mid \gamma(u) du - \int_0^x u \mid \lambda''(u) \mid \gamma(u) du \\ &= O(1) \; as \; x \to \infty\end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.
Thus, we obtain

$$\int_0^x t^{k-1} \mid A'(t) \mid^k dt = O(1) \; as \; x \to \infty.$$

This completes the proof of the theorem.

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