

ON CHLODOWSKY VARIANT OF (p, q) KANTOROVICH-STANCU-SCHURER OPERATORS

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ABSTRACT. In the present paper, we introduce the Chlodowsky variant of (p, q) Kantorovich-Stancu-Schurer operators on the unbounded domain which is a generalization of (p, q) Bernstein-Stancu-Kantorovich operators. We have also derived its Korovkin type approximation properties and rate of convergence.

1. INTRODUCTION AND PRELIMINARIES

Approximation theory has an important role in mathematical research because of its great potential for applications. Korovkin gave his famous approximation theorem in 1950, since then the study of the linear methods of approximation given by sequences of positive and linear operators became a deep-rooted part of approximation theory. Considering it, various operators as Bernstein, Szász, Baskakov etc. and their generalizations are being studied. In recent years, many results about the generalization of positive linear operators have been obtained by several mathematicians ([6], [8]-[12], [21]). In last two decades, the applications of q -calculus has played an important role in the area of approximation theory, number theory and theoretical physics. In 1987, Lupaş and in 1997, Phillips introduced a sequence of Bernstein polynomials based on q -integers and investigated its approximation properties. Several researchers obtained various other generalizations of operators based on q -calculus (See [3], [17]-[19]).

Recently, Mursaleen et al. applied (p, q) -calculus in approximation theory and introduced first (p, q) -analogue of Bernstein operators. They investigated uniform convergence of the operators and order of convergence, obtained Voronovskaja theorem as well. Also, (p, q) -analogue of Bernstein operators, Bernstein-Stancu operators and Bernstein-Schurer operators, Kantorovich Bernstein Schurer, and Bleimann-Butzer-Hahn operators were introduced in ([13]-[16]), respectively. Further, Acar [1] have studied recently, (p, q) -Generalization of SzászMirakyan operators.

In the present paper, we introduce the Chlodowsky variant of (p, q) Kantorovich-Stancu-Schurer operators on the unbounded domain. We begin by recalling certain notations of (p, q) -calculus.

For $0 < q < p \leq 1$, the (p, q) integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}.$$

(p, q) factorial is expressed as

$$[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}[n-2]_{p,q} \dots 1.$$

(p, q) binomial coefficient is expressed as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

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(p, q) binomial expansion as

$$(ax + by)_{p,q}^n := \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k.$$

$$(x + y)_{p,q}^n := (x + y)(px + qy)(p^2x + q^2y) \dots (p^{n-1}x + q^{n-1}y).$$

The definite integrals of the function f are given by

$$\int_0^a f(x) d_{p,q}x = (q - p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right), \quad \left|\frac{p}{q}\right| < 1,$$

and

$$\int_0^a f(x) d_{p,q}x = (p - q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right), \quad \left|\frac{p}{q}\right| > 1.$$

Further (p, q) analysis can be found in [2].

In 1932, Chlodowsky [7] presented a generalization of Bernstein polynomials on an unbounded set, known as Bernstein-Chlodowsky polynomials given by,

$$(1.1) \quad B_n(f, x) = \sum_{k=0}^n f\left(\frac{k}{n}b_n\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}, \quad 0 \leq x \leq b_n,$$

where b_n is an increasing sequence of positive terms with the properties $b_n \rightarrow \infty$ and $\frac{b_n}{n} \rightarrow 0$ as $n \rightarrow \infty$.

In 2015, Vedi and Özarşlan [20] investigated Chlodowsky-type q -Bernstein-Stancu-Kantorovich operators, and Wafi and Rao investigated (p, q) form of Kantorovich type Bernstein-Stancu-Schurer operator. Mursaleen and Khan [15] defined the Kantorovich type (p, q) -Bernstein-Schurer Operators, given by

$$T_{n,m}(f; x, p, q) = \sum_{k=0}^{n+m} \begin{bmatrix} n+m \\ k \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n+m-k-1} (p^s - q^s x) \int_0^1 f\left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t}{[n+1]_{p,q}}\right) d_{p,q}t$$

$$k = 0, 1, 2, \dots, n = 1, 2, 3, \dots$$

where

Lemma 1. (See [15]) For the Operators $T_{n,m}^{(\alpha,\beta)}$, we have

$$T_{n,m}(1; x, p, q) = 1,$$

$$T_{n,m}(t; x, p, q) = \frac{(px + 1 - x)_{p,q}^{n+m}}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p + 2q - 1)[n+m]_{p,q}}{[2]_{p,q}[n+1]_{p,q}}x,$$

$$T_{n,m}(t^2; x, p, q) = \frac{(p^2x + 1 - x)_{p,q}^{n+m}}{[3]_{p,q}[n+1]_{p,q}^2} + \left\{1 + \frac{2q}{[2]_{p,q}} + \frac{q^2 - 1}{[3]_{p,q}}\right\} \frac{[n+m]_{p,q}}{[n+1]_{p,q}^2} (px + 1 - x)_{p,q}^{n+m-1}x$$

$$+ \left\{1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}}\right\} \frac{[n+m]_{p,q}[n+m-1]_{p,q}}{[n+1]_{p,q}^2} x^2.$$

2. CONSTRUCTION OF THE OPERATORS

We construct the Chlodowsky variant of (p, q) Kantorovich-Stancu-Schurer operators as

$$(2.1) \quad K_{n,m}^{(\alpha,\beta)}(f; x, p, q) = \sum_{k=0}^{n+m} \begin{bmatrix} n+m \\ k \end{bmatrix}_{p,q} \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{x}{b_n}\right)^k \int_0^1 f\left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n\right) d_{p,q}t,$$

where $n \in \mathbb{N}$, $m, \alpha, \beta \in \mathbb{N}_0$ with $0 \leq \alpha \leq \beta$, $0 \leq x \leq b_n$, $0 < q < p \leq 1$.

Obviously, $K_{n,m}^{(\alpha,\beta)}$ is a linear and positive operator. To begin with, we obtain the following important lemma.

Lemma 2. Let $K_{n,m}^{(\alpha,\beta)}(f; x, p, q)$ be given by (2.1). The first few moments of the operators are

$$\begin{aligned}
(i) \quad & K_{n,m}^{(\alpha,\beta)}(1; x, p, q) = 1, \\
(ii) \quad & K_{n,m}^{(\alpha,\beta)}(t; x, p, q) = \left(\frac{1}{[n+1]_{p,q} + \beta} \right) \left(\alpha b_n + \frac{(p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}} b_n + \frac{(p+2q-1)[n+m]_{p,q} x}{[2]_{p,q}} \right), \\
(iii) \quad & K_{n,m}^{(\alpha,\beta)}(t^2; x, p, q) \\
&= \left(\frac{1}{[n+1]_{p,q} + \beta} \right)^2 \left[\left(\alpha^2 + \frac{2\alpha}{[2]_{p,q}} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q} + \frac{(p^2 \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[3]_{p,q}} \right) b_n^2 \right. \\
&\quad \left. + \left(\frac{2\alpha}{[2]_{p,q}} (p+2q-1) + \left\{ 1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}} \right\} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m-1} \right) [n+m]_{p,q} b_n x \right. \\
&\quad \left. + \left\{ 1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}} \right\} [n+m]_{p,q} [n+m-1]_{p,q} x^2 \right], \\
(iv) \quad & K_{n,m}^{(\alpha,\beta)}((t-x); x, p, q) = \frac{[2]_{p,q} \alpha + (p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q} ([n+1]_{p,q} + \beta)} b_n + \left(\frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q} ([n+1]_{p,q} + \beta)} - 1 \right) x \\
(v) \quad & K_{n,m}^{(\alpha,\beta)}((t-x)^2; x, p, q) \\
&= \left[\frac{\alpha^2}{([n+1]_{p,q} + \beta)^2} + \frac{2\alpha}{[2]_{p,q} ([n+1]_{p,q} + \beta)^2} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q} + \frac{(p^2 \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[3]_{p,q} ([n+1]_{p,q} + \beta)^2} \right] b_n^2 \\
&\quad + \left[\frac{2\alpha(p+2q-1)[n+m]_{p,q}}{[2]_{p,q} ([n+1]_{p,q} + \beta)^2} + \left\{ 1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}}{([n+1]_{p,q} + \beta)^2} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m-1} \right. \\
&\quad \left. - \frac{2\alpha}{([n+1]_{p,q} + \beta)} - \frac{2(p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q} ([n+1]_{p,q} + \beta)} \right] b_n x \\
&\quad + \left[\left\{ 1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q} [n+m-1]_{p,q}}{([n+1]_{p,q} + \beta)^2} - 2 \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q} ([n+1]_{p,q} + \beta)} + 1 \right] x^2.
\end{aligned}$$

Proof. From operator 2.1,

$$\begin{aligned}
K_{n,m}^{(\alpha,\beta)}(t^u; x, p, q) &= \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{x}{b_n} \right)^k \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q} t + \alpha}{[n+1]_{p,q} + \beta} b_n \right)^u d_{p,q} t \\
&= \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{x}{b_n} \right)^k \frac{[n+1]_{p,q}^u b_n^u}{([n+1]_{p,q} + \beta)^u} \\
&\quad \times \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q} t + \alpha}{[n+1]_{p,q}} \right)^u d_{p,q} t \\
&= \frac{[n+1]_{p,q}^u b_n^u}{([n+1]_{p,q} + \beta)^u} \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^k \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \sum_{i=0}^u \binom{u}{i} \left(\frac{\alpha}{[n+1]_{p,q}} \right)^{u-i} \\
&\quad \times \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q} t}{[n+1]_{p,q}} \right)^u d_{p,q} t \\
&= \frac{[n+1]_{p,q}^u b_n^u}{([n+1]_{p,q} + \beta)^u} \sum_{i=0}^u \binom{u}{i} \left(\frac{\alpha}{[n+1]_{p,q}} \right)^{u-i} \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^k \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \\
&\quad \times \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q} t}{[n+1]_{p,q}} \right)^i d_{p,q} t \\
(2.2) \quad & K_{n,m}^{(\alpha,\beta)}(t^u; x, p, q) = \frac{[n+1]_{p,q}^u b_n^u}{([n+1]_{p,q} + \beta)^u} \sum_{i=0}^u \binom{u}{i} \left(\frac{\alpha}{[n+1]_{p,q}} \right)^{u-i} T_{n,m}(t^i; \frac{x}{b_n}, p, q).
\end{aligned}$$

Thus for $u=0,1,2$ we get

$$K_{n,m}^{(\alpha,\beta)}(1; x, p, q) = T_{n,m}(1; \frac{x}{b_n}, p, q),$$

$$\begin{aligned}
K_{n,m}^{(\alpha,\beta)}(t; x, p, q) &= \frac{[n+1]_{p,q} b_n}{[n+1]_{p,q} + \beta} \sum_{i=0}^1 \binom{1}{i} \left(\frac{\alpha}{[n+1]_{p,q}} \right)^{1-i} T_{n,m}(t^i; \frac{x}{b_n}, p, q) \\
&= \left(\frac{[n+1]_{p,q}}{[n+1]_{p,q} + \beta} \right) b_n \left\{ \frac{\alpha}{[n+1]_{p,q}} + T_{n,m}(t; \frac{x}{b_n}, p, q) \right\}, \\
K_{n,m}^{(\alpha,\beta)}(t^2; x, p, q) &= \frac{[n+1]_{p,q}^2 b_n^2}{([n+1]_{p,q} + \beta)^2} \sum_{i=0}^2 \binom{2}{i} \left(\frac{\alpha}{[n+1]_{p,q}} \right)^{2-i} T_{n,m}(t^i; \frac{x}{b_n}, p, q) \\
&= \frac{[n+1]_{p,q}^2 b_n^2}{([n+1]_{p,q} + \beta)^2} \left[\left(\frac{\alpha}{[n+1]_{p,q}} \right)^2 + 2 \left(\frac{\alpha}{[n+1]_{p,q}} \right) T_{n,m}(t; \frac{x}{b_n}, p, q) + T_{n,m}(t^2; \frac{x}{b_n}, p, q) \right].
\end{aligned}$$

Using Lemma 1 and in view of the above relations we get the statements (i), (ii) and (iii).

Using linear property of operators, we have

$$\begin{aligned}
K_{n,m}^{(\alpha,\beta)}((t-x); x, p, q) &= K_{n,m}^{(\alpha,\beta)}(t; x, p, q) - x K_{n,m}^{(\alpha,\beta)}(1; x, p, q) \\
&= \frac{[2]_{p,q} \alpha + (p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q} ([n+1]_{p,q} + \beta)} b_n + \left(\frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q} ([n+1]_{p,q} + \beta)} - 1 \right) x.
\end{aligned}$$

Hence, we get (iv).

Similar calculations give

$$K_{n,m}^{(\alpha,\beta)}((t-x)^2; x, p, q) = K_{n,m}^{(\alpha,\beta)}(t^2; x, p, q) - 2x K_{n,m}^{(\alpha,\beta)}(t; x, p, q) + x^2 K_{n,m}^{(\alpha,\beta)}(1; x, p, q).$$

Then we obtain,

$$\begin{aligned}
&K_{n,m}^{(\alpha,\beta)}((t-x)^2; x, p, q) \\
&= \left[\frac{\alpha^2}{([n+1]_{p,q} + \beta)^2} + \frac{2\alpha}{[2]_{p,q} ([n+1]_{p,q} + \beta)^2} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m} + \frac{(p^2 \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[3]_{p,q} ([n+1]_{p,q} + \beta)^2} \right] b_n^2 \\
&+ \left[\frac{2\alpha(p+2q-1)[n+m]_{p,q}}{[2]_{p,q} ([n+1]_{p,q} + \beta)^2} + \left\{ 1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}}{([n+1]_{p,q} + \beta)^2} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m-1} \right. \\
&- \left. \frac{2\alpha}{([n+1]_{p,q} + \beta)} - \frac{2(p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q} ([n+1]_{p,q} + \beta)} \right] b_n x \\
&+ \left[\left\{ 1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q} [n+m-1]_{p,q}}{([n+1]_{p,q} + \beta)^2} - 2 \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q} ([n+1]_{p,q} + \beta)} + 1 \right] x^2.
\end{aligned}$$

This proves (v). \square

3. KOROVKIN-TYPE APPROXIMATION THEOREM

Assume C_ρ is the space of all continuous functions f such that

$$|f(x)| \leq M\rho(x), \quad -\infty < x < \infty.$$

Then C_ρ is a Banach space with the norm

$$\|f\|_\rho = \sup_{-\infty < x < \infty} \frac{|f(x)|}{\rho(x)}.$$

The subsequent results are used for proving Korovkin approximation theorem on unbounded sets.

Theorem 1. (See [5]) *There exists a sequence of positive linear operators \mathbb{U}_n , acting from C_ρ to C_ρ , satisfying the conditions*

- (1) $\lim_{n \rightarrow \infty} \|\mathbb{U}_n(1; x) - 1\|_\rho = 0$,
- (2) $\lim_{n \rightarrow \infty} \|\mathbb{U}_n(\phi; x) - \phi\|_\rho = 0$,
- (3) $\lim_{n \rightarrow \infty} \|\mathbb{U}_n(\phi^2; x) - \phi^2\|_\rho = 0$,

where $\phi(x)$ is a continuous and increasing function on $(-\infty, \infty)$ such that $\lim_{x \rightarrow \pm\infty} \phi(x) = \pm\infty$ and $\rho(x) = 1 + \phi^2$, and there exists a function $f^* \in C_\rho$ for which $\lim_{n \rightarrow \infty} \|\mathbb{U}_n f^* - f^*\|_\rho > 0$.

Theorem 2. (See [5]) Conditions (1),(2),(3) of above theorem implies that

$$\lim_{n \rightarrow \infty} \|\mathbb{U}_n f - f\|_\rho = 0$$

for any function f belonging to the subset $C_\rho^0 := \{f \in C_\rho : \lim_{|x| \rightarrow \infty} \frac{|f(x)|}{\rho(x)} \text{ is finite}\}$.

Consider the weight function $\rho(x) = 1 + x^2$ and operators:

$$\mathbb{U}_{n,m}^{\alpha,\beta}(f; x, p, q) = \begin{cases} K_{n,m}^{\alpha,\beta}(f; x, p, q) & \text{if } x \in [0, b_n], \\ f(x) & \text{if } x \in [0, \infty)/[0, b_n]. \end{cases}$$

Thus for $f \in C_{1+x^2}$, we have

$$\begin{aligned} \|\mathbb{U}_{n,m}^{\alpha,\beta}(f; \cdot, p, q)\| &\leq \sup_{x \in [0, b_n]} \frac{|\mathbb{U}_{n,m}^{\alpha,\beta}(f; x, p, q)|}{1+x^2} + \sup_{b_n < x < \infty} \frac{|f(x)|}{1+x^2} \\ &\leq \|f\|_{1+x^2} \left[\sup_{x \in [0, \infty)} \frac{|\mathbb{U}_{n,m}^{\alpha,\beta}(1+t^2; x, p, q)|}{1+x^2} + 1 \right]. \end{aligned}$$

Now, using Lemma 2, we will obtain,

$$\|\mathbb{U}_{n,m}^{\alpha,\beta}(f; \cdot, p, q)\|_{1+x^2} \leq M \|f\|_{1+x^2}$$

if $p := (p)_n, q := (q)_n$ with $0 < q_n < p_n \leq 1, \lim_{n \rightarrow \infty} p_n = 1, \lim_{n \rightarrow \infty} q_n = 1, \lim_{n \rightarrow \infty} p_n^n = \lim_{n \rightarrow \infty} q_n^n = N, N < \infty$ and $\lim_{n \rightarrow \infty} \frac{b_n}{[n]} = 0$.

Theorem 3. For all $f \in C_{1+x^2}^0$, we have

$$\lim_{n \rightarrow \infty} \|\mathbb{U}_{n,m}^{\alpha,\beta}(f; \cdot, p_n, q_n) - f(\cdot)\|_{1+x^2} = 0$$

provided that $p := (p)_n, q := (q)_n$ with $0 < q_n < p_n \leq 1, \lim_{n \rightarrow \infty} p_n = 1, \lim_{n \rightarrow \infty} q_n = 1, \lim_{n \rightarrow \infty} p_n^n = \lim_{n \rightarrow \infty} q_n^n = N, N < \infty$ and $\lim_{n \rightarrow \infty} \frac{b_n}{[n]} = 0$.

Proof. Using the results of Theorem 1 and Lemma 2(i),(ii) and (iii), we will achieve the following assessments, respectively:

$$\begin{aligned} \sup_{x \in [0, \infty)} \frac{|\mathbb{U}_{n,m}^{\alpha,\beta}(1; x, p_n, q_n) - 1|}{1+x^2} &= \sup_{0 \leq x \leq b_n} \frac{|K_{n,m}^{\alpha,\beta}(1; x, p_n, q_n) - 1|}{1+x^2} = 0 \\ \sup_{x \in [0, \infty)} \frac{|\mathbb{U}_{n,m}^{\alpha,\beta}(t; x, p_n, q_n) - t|}{1+x^2} &= \sup_{0 \leq x \leq b_n} \frac{|K_{n,m}^{\alpha,\beta}(t; x, p_n, q_n) - x|}{1+x^2} \\ &\leq \sup_{0 \leq x \leq b_n} \frac{\frac{[2]_{p,q} \alpha + (p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \left(\frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} - 1 \right) x}{1+x^2} \\ &\leq \frac{[2]_{p,q} \alpha + (p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \left| \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} - 1 \right| \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \sup_{x \in [0, \infty)} \frac{|\mathbb{U}_{n,m}^{\alpha,\beta}(t^2; x, p_n, q_n) - t^2|}{1+x^2} &= \sup_{0 \leq x \leq b_n} \frac{|K_{n,m}^{\alpha,\beta}(t^2; x, p_n, q_n) - x^2|}{1+x^2} \\ &\leq \sup_{0 \leq x \leq b_n} \frac{1}{1+x^2} \left(\left[\frac{\alpha^2}{([n+1]_{p,q} + \beta)^2} + \frac{2\alpha}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m} + \frac{(p^2 \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[3]_{p,q}([n+1]_{p,q} + \beta)^2} \right] b_n^2 \right. \\ &\quad \left. + \left[\frac{2\alpha(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} + \left\{ 1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}}{([n+1]_{p,q} + \beta)^2} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m-1} \right] \right) \end{aligned}$$

$$\begin{aligned}
& - \frac{2\alpha}{([n+1]_{p,q} + \beta)} - \frac{2(p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} \Big] b_n x \\
& + \left[\left\{ 1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}[n+m-1]_{p,q}}{([n+1]_{p,q} + \beta)^2} - 2 \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} + 1 \right] x^2 \\
& \leq \left(\left[\frac{\alpha^2}{([n+1]_{p,q} + \beta)^2} + \frac{2\alpha}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} \left(p\frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m} + \frac{(p^2\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[3]_{p,q}([n+1]_{p,q} + \beta)^2} \right] b_n^2 \right. \\
& + \left. \left| \frac{2\alpha(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} + \left\{ 1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}}{([n+1]_{p,q} + \beta)^2} \left(p\frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m-1} \right. \right. \\
& - \left. \left. \frac{2\alpha}{([n+1]_{p,q} + \beta)} - \frac{2(p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} \right| b_n \right. \\
& + \left. \left[\left\{ 1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}[n+m-1]_{p,q}}{([n+1]_{p,q} + \beta)^2} - 2 \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} + 1 \right] \right) \rightarrow 0
\end{aligned}$$

whenever $n \rightarrow \infty$, because we have $\lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} q_n = 1$ and $\frac{b_n}{[n]_{p,q}} = 0$ as $n \rightarrow \infty$. \square

Theorem 4. *Assuming C as a positive and real number independent of n and f as a continuous function which vanishes on $[C, \infty)$. Let $p := (p_n), q := (q_n)$ with $0 < q_n < p_n \leq 1, \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} q_n = 1, \lim_{n \rightarrow \infty} p_n^n = \lim_{n \rightarrow \infty} q_n^n = N < \infty$ and $\lim_{n \rightarrow \infty} \frac{b_n}{[n]_{p,q}} = 0$. Then we have*

$$\lim_{n \rightarrow \infty} \sup_{0 \leq x \leq b_n} |K_{n,m}^{\alpha,\beta}(f; x, p_n, q_n) - f(x)| = 0.$$

Proof. From the hypothesis on f , it is bounded i.e. $|f(x)| \leq M$ ($M > 0$). For any $\epsilon > 0$, we have

$$\left| f \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n \right) - f(x) \right| < \epsilon + \frac{2M}{\delta^2} \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right)^2,$$

where $x \in [0, b_n]$ and $\delta = \delta(\epsilon)$ are independent of n . Now since we know,

$$K_{n,m}^{\alpha,\beta}((t-x)^2; x, p_n, q_n) = \sum_{k=0}^{n+m} \binom{n+m}{k}_{p,q} \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{x}{b_n} \right)^k \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n \right)^2 dt.$$

we can conclude by Theorem 3,

$$\begin{aligned}
& \sup_{0 \leq x \leq b_n} \left| K_{n,m}^{\alpha,\beta}(f; x, p_n, q_n) - f(x) \right| \\
& \leq \epsilon + \frac{2M}{\delta^2} b_n^2 \left(\left[\frac{\alpha^2}{([n+1]_{p,q} + \beta)^2} + \frac{2\alpha}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} \left(p\frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m} + \frac{(p^2\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[3]_{p,q}([n+1]_{p,q} + \beta)^2} \right] \right. \\
& + \left| \frac{2\alpha(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} + \left\{ 1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}}{([n+1]_{p,q} + \beta)^2} \left(p\frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m-1} \right. \\
& - \left. \left. \frac{2\alpha}{([n+1]_{p,q} + \beta)} - \frac{2(p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} \right| \right. \\
& + \left. \left[\left\{ 1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}[n+m-1]_{p,q}}{([n+1]_{p,q} + \beta)^2} - 2 \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} + 1 \right] \right).
\end{aligned}$$

Since $\frac{b_n}{[n]_{p,q}} = 0$ as $n \rightarrow \infty$, we have the desired result. \square

4. RATE OF CONVERGENCE

We will find the rate of convergence in terms of the Lipschitz class $Lip_M(\gamma)$, for $0 < \gamma \leq 1$. Assume that $C_B[0, \infty)$ denote the space of bounded continuous functions on $[0, \infty)$. A function $f \in C_B[0, \infty)$ belongs to $Lip_M(\gamma)$ if

$$|f(t) - f(x)| \leq M|t - x|^\gamma, \quad t, x \in [0, \infty)$$

is satisfied.

Theorem 5. Let $f \in Lip_M(\gamma)$, then

$$K_{n,m}^{\alpha,\beta}(f; x, p, q) \leq M(\mu_{n,p,q}(x))^{\gamma/2},$$

where $\mu_{n,p,q}(x) = K_{n,m}^{\alpha,\beta}((t-x)^2; x, p, q)$.

Proof. Since $f \in Lip_M(\gamma)$, and the operator $K_{n,m}^{\alpha,\beta}(f; x, p, q)$ is linear and monotone,

$$\begin{aligned} & |K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| \\ &= \left| \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{x}{b_n} \right)^k \right. \\ &\quad \left. \times \int_0^1 \left(f \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n \right) - f(x) \right) d_{p,q}t \right| \\ &\leq \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{x}{b_n} \right)^k \\ &\quad \times \int_0^1 \left| f \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n \right) - f(x) \right| d_{p,q}t \\ &\leq M \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{x}{b_n} \right)^k \\ &\quad \times \int_0^1 \left| \frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right|^\gamma d_{p,q}t. \end{aligned}$$

Applying Hölder's inequality with the values $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get following inequality,

$$\begin{aligned} & \int_0^1 \left| \frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right|^\gamma d_{p,q}t \\ & \leq \left\{ \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right)^2 d_{p,q}t \right\}^{\frac{\gamma}{2}} \left\{ \int_0^1 d_{p,q}t \right\}^{\frac{2-\gamma}{2}} \\ & = \left\{ \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right)^2 d_{p,q}t \right\}^{\frac{\gamma}{2}} \end{aligned}$$

Using this, we get

$$\begin{aligned} & |K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| \\ & \leq M \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{x}{b_n} \right)^k \\ & \quad \times \left\{ \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right)^2 d_{p,q}t \right\}^{\frac{\gamma}{2}} \\ & = M \sum_{k=0}^{n+m} \left\{ \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right)^2 d_{p,q}t \right\}^{\frac{\gamma}{2}} w_{n,k}(p, q; x), \end{aligned}$$

where $w_{n,k}(p, q; x) = \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{x}{b_n} \right)^k$. Again using Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we have

$$\begin{aligned} & |K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| \\ & \leq M \left\{ \sum_{k=0}^{n+m} \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right)^2 d_{p,q}t w_{n,k}(p, q; x) \right\}^{\frac{\gamma}{2}} \left\{ \sum_{k=0}^{n+m} w_{n,k}(p, q; x) \right\}^{\frac{2-\gamma}{2}} \end{aligned}$$

$$\begin{aligned}
&= M \left\{ \sum_{k=0}^{n+m} w_{n,k}(p, q; x) \int_0^1 \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right)^2 d_{p,q}t \right\}^{\frac{\gamma}{2}} \\
&= M(\mu_{n,p,q}(x))^{\gamma/2},
\end{aligned}$$

where $(\mu_{n,p,q}(x))^{\gamma/2} = K_{n,m}^{\alpha,\beta}((t-x)^2; x, p, q)$. \square

In order to obtain rate of convergence in terms of modulus of continuity $\omega(f; \delta)$, we assume that for any uniformly continuous $f \in C_B[0, \infty)$ and $x \geq 0$, modulus of continuity of f is given by

$$(4.1) \quad \omega(f; \delta) = \max_{\substack{|t-x| \leq \delta \\ t, x \in [0, \infty)}} |f(t) - f(x)|.$$

Thus it implies for any $\delta > 0$

$$(4.2) \quad |f(x) - f(y)| \leq \omega(f; \delta) \left(\frac{|x-y|}{\delta} + 1 \right),$$

is satisfied.

Theorem 6. *If $f \in C_B[0, \infty)$, we have*

$$|K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| \leq 2\omega(f; \sqrt{\mu_{n,p,q}(x)}),$$

where $\omega(f; \cdot)$ is modulus of continuity of f and $\lambda_{n,p,q}(x)$ be the same as in Theorem 5.

Proof. Using triangular inequality, we get

$$\begin{aligned}
&|K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| \\
&= \left| \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^k \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(f \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n \right) - f(x) \right) \right| \\
&\leq \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^k \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left| f \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n \right) - f(x) \right|,
\end{aligned}$$

Now using (4.2) and Hölder's inequality, we get

$$\begin{aligned}
&|K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| \\
&= \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^k \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left(\frac{|\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x|}{\delta} + 1 \right) \omega(f; \delta) \\
&\leq \omega(f; \delta) \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^k \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \\
&\quad + \frac{\omega(f; \delta)}{\delta} \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^k \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \left| \frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right| \\
&= \omega(f; \delta) + \frac{\omega(f; \delta)}{\delta} \left\{ \sum_{k=0}^{n+m} \left[\begin{matrix} n+m \\ k \end{matrix} \right]_{p,q} \left(\frac{x}{b_n} \right)^k \prod_{s=0}^{n+m-k-1} (p^s - q^s \frac{x}{b_n}) \right. \\
&\quad \left. \times \left(\frac{(1-t)[k]_{p,q} + [k+1]_{p,q}t + \alpha}{[n+1]_{p,q} + \beta} b_n - x \right)^2 \right\}^{\frac{1}{2}} \\
&= \omega(f; \delta) + \frac{\omega(f; \delta)}{\delta} \{ K_{n,m}^{\alpha,\beta}((t-x)^2; x, p, q) \}^{1/2}.
\end{aligned}$$

Now choosing $\delta = \mu_{n,p,q}(x)$ as in Theorem 5, we have

$$|K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| \leq 2\omega(f; \sqrt{\mu_{n,p,q}(x)}).$$

\square

Now let us denote by $C_B^2[0, \infty)$ the space of all functions $f \in C_B[0, \infty)$ such that $f', f'' \in C_B[0, \infty)$. Let $\|f\|$ denote the usual supremum norm of f . The classical Peetre's K-functional and the second modulus of smoothness of the function $f \in C_B[0, \infty)$ are defined respectively as

$$K(f, \delta) := \inf_{g \in C_B^2[0, \infty)} [\|f - g\| + \delta \|g''\|]$$

and

$$\omega_2(f, \delta) = \sup_{\substack{0 < h < \delta, \\ x, x+h \in I}} |f(x+2h) - 2f(x+h) + f(x)|,$$

where $\delta > 0$. It is known that [see [4], p. 177] there exists a constant $A > 0$ such that

$$(4.3) \quad K(f, \delta) \leq A\omega_2(f, \delta).$$

Theorem 7. *Let $x \in [0, b_n]$ and $f \in C_B[0, \infty)$. Then, for fixed $p \in \mathbb{N}_0$, we have*

$$|K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| \leq C\omega_2(f, \sqrt{\alpha_{n,p,q}(x)}) + \omega(f, \beta_{n,p,q}(x))$$

for some positive constant C , where

$$(4.4) \quad \begin{aligned} \alpha_{n,p,q}(x) = & \left[\left\{ 1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}} + \frac{(p+2q-1)^2}{[2]_{p,q}^2} \right\} \frac{[n+m]_{p,q}^2}{([n+1]_{p,q} + \beta)^2} - 4 \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} + 2 \right] x^2 \\ & + \left[\left\{ 1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}} + 2 \frac{(p+2q-1)}{[2]_{p,q}^2} \right\} \frac{[n+m]_{p,q}}{([n+1]_{p,q} + \beta)^2} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m} \right. \\ & \quad \left. + 4 \frac{\alpha(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} - 4 \frac{(p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} - 4 \frac{\alpha}{([n+1]_{p,q} + \beta)} \right] b_n x \\ & + \left[\frac{(p^2 \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[3]_{p,q}} + \frac{(p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{2n+2m}}{[2]_{p,q}^2} + 4 \frac{\alpha}{[2]_{p,q}} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m} + 2\alpha^2 \right] \frac{b_n^2}{([n+1]_{p,q} + \beta)^2}, \end{aligned}$$

and

$$(4.5) \quad \beta_{n,p,q}(x) = \left(\frac{[2]_{p,q}\alpha + (p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \left(\frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} - 1 \right) x \right).$$

Proof. Consider an auxiliary operator $K_{n,m}^*(f; x, p, q) : C_B[0, \infty) \rightarrow C_B[0, \infty)$ by

$$(4.6) \quad K_{n,m}^*(f; x, p, q) := K_{n,m}^{\alpha,\beta}(f; x, p, q) - f \left(\frac{[2]_{p,q}\alpha + (p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} x \right) + f(x).$$

Then by Lemma 2 we get

$$(4.7) \quad \begin{aligned} K_{n,m}^*(1; x, p, q) &= 1, \\ K_{n,m}^*((t-x); x, p, q) &= 0. \end{aligned}$$

For given $g \in C_B[0, \infty)$, it follows by the Taylor formula that

$$g(y) - g(x) = (y-x)g'(x) + \int_x^y (y-u)g''(u) du.$$

Taking into account 4.6 and using 4.7, we get

$$\begin{aligned} |K_{n,m}^*(g; x, p, q) - g(x)| &= |K_{n,m}^*(g(y) - g(x); x, p, q)| \\ &= \left| g'(x)K_{n,m}^*((y-x); x, p, q) + K_{n,m}^* \left(\int_x^y (y-u)g''(u) du; x, p, q \right) \right| \\ &= \left| K_{n,m}^* \left(\int_x^y (y-u)g''(u) du; x, p, q \right) \right| \end{aligned}$$

Then by 4.6

$$\begin{aligned} & |K_{n,m}^*(g; x, p, q) - g(x)| \\ &= \left| K_{n,m}^* \left(\int_x^y (y-u)g''(u) du; x, p, q \right) - \int_x^y \left(\frac{[2]_{p,q}\alpha + (p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} x \right) g''(u) du \right| \end{aligned}$$

$$\begin{aligned}
& \left| \left(\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} x - u \right) g''(u) du \right| \\
& \leq \left| K_{n,m}^* \left(\int_x^y (y-u)g''(u) du; x, p, q \right) \right| + \left| \int_x^{\left(\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} x \right)} \right. \\
& \left. \left(\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} x - u \right) g''(u) du \right|.
\end{aligned}$$

Since,

$$\left| K_{n,m}^{\alpha,\beta} \left(\int_x^y (y-u)g''(u) du; x, p, q \right) \right| \leq \|g''(x)\| K_{n,m}^{\alpha,\beta}((y-x)^2; x, p, q)$$

and

$$\begin{aligned}
& \left| \int_x^{\left(\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} x \right)} \right. \\
& \left. \left(\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} x - u \right) g''(u) du \right| \\
& \leq \|g''\| \left[\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \left(\frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} - 1 \right) x \right]^2,
\end{aligned}$$

we get

$$\begin{aligned}
|K_{n,m}^*(g; x, p, q) - g(x)| & \leq \|g''\| K_{n,m}^{\alpha,\beta}((y-x)^2; x, p, q) + \|g''\| \left[\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n \right. \\
& \quad \left. + \left(\frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} - 1 \right) x \right]^2.
\end{aligned}$$

Hence Lemma 2 implies that

$$\begin{aligned}
(4.8) \quad & |K_{n,m}^*(g; x, p, q) - g(x)| \\
& \leq \|g''\| \left[\left(\frac{\alpha^2}{([n+1]_{p,q} + \beta)^2} + \frac{2\alpha}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} \left(p\frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m} + \frac{(p^2\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[3]_{p,q}([n+1]_{p,q} + \beta)^2} \right) b_n^2 \right. \\
& \quad + \left(\frac{2\alpha(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} + \left\{ 1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}}{([n+1]_{p,q} + \beta)^2} \left(p\frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m-1} \right. \\
& \quad \left. - \frac{2\alpha}{([n+1]_{p,q} + \beta)} - \frac{2(p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} \right) b_n x \\
& \quad + \left(\left\{ 1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}} \right\} \frac{[n+m]_{p,q}[n+m-1]_{p,q}}{([n+1]_{p,q} + \beta)^2} - 2\frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} + 1 \right) x^2 \\
(4.9) \quad & \left. + \left(\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \left(\frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} - 1 \right) x \right)^2 \right].
\end{aligned}$$

Since $K_{n,m}^*(f; x, p, q) \leq 3\|f\|$, considering 4.4 and 4.5, for all $f \in C_B[0, \infty)$ and $g \in C_B^2[0, \infty)$, we may write from 4.8 that

$$\begin{aligned}
& |K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| \leq |K_{n,m}^*(f-g; x, p, q) - (f-g)(x)| + |K_{n,m}^*(g; x, p, q) - g(x)| \\
& + \left| f \left(\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} x \right) - f(x) \right| \\
& \leq 4\|f-g\| + \alpha_{n,p,q}(x)\|g'\| + \left| f \left(\frac{[2]_{p,q}\alpha + (p\frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} x \right) - f(x) \right| \\
& \leq 4\|f-g\| + \alpha_{n,p,q}(x)\|g'\| + \omega(f, \beta_{n,p,q}(x)),
\end{aligned}$$

which yields that

$$\begin{aligned} |K_{n,m}^{\alpha,\beta}(f; x, p, q) - f(x)| &\leq 4K(f, \alpha_{n,p,q}(x)) + \omega(f, \beta_{n,p,q}(x)) \\ &\leq C\omega_2(f, \sqrt{\alpha_{n,p,q}(x)}) + \omega(f, \beta_{n,p,q}(x)), \end{aligned}$$

where

$$\begin{aligned} \alpha_{n,p,q}(x) &= \left[\left\{ 1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}} + \frac{(p+2q-1)^2}{[2]_{p,q}^2} \right\} \frac{[n+m]_{p,q}^2}{([n+1]_{p,q} + \beta)^2} - 4 \frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} + 2 \right] x^2 \\ &+ \left[\left\{ 1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}} + 2 \frac{(p+2q-1)}{[2]_{p,q}^2} \right\} \frac{[n+m]_{p,q}}{([n+1]_{p,q} + \beta)^2} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m} \right. \\ &+ 4 \frac{\alpha(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)^2} - 4 \frac{(p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} - 4 \frac{\alpha}{([n+1]_{p,q} + \beta)} \left. \right] b_n x \\ &+ \left[\frac{(p^2 \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[3]_{p,q}} + \frac{(p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{2n+2m}}{[2]_{p,q}^2} + 4 \frac{\alpha}{[2]_{p,q}} \left(p \frac{x}{b_n} + 1 - \frac{x}{b_n} \right)_{p,q}^{n+m} + 2\alpha^2 \right] \frac{b_n^2}{([n+1]_{p,q} + \beta)^2}, \end{aligned}$$

and

$$\beta_{n,p,q}(x) = \left(\frac{[2]_{p,q}\alpha + (p \frac{x}{b_n} + 1 - \frac{x}{b_n})_{p,q}^{n+m}}{[2]_{p,q}([n+1]_{p,q} + \beta)} b_n + \left(\frac{(p+2q-1)[n+m]_{p,q}}{[2]_{p,q}([n+1]_{p,q} + \beta)} - 1 \right) x \right).$$

Hence we get the result. \square

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