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A NEW RESULT ON GENERALIZED ABSOLUTE CESÀRO SUMMABILITY

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Abstract. In [4], a main theorem dealing with an application of almost increasing sequences, has been proved. In this paper, we have extended that theorem by using a general class of quasi power increasing sequences, which is a wider class of sequences, instead of an almost increasing sequence. This theorem also includes some new and known results.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). A sequence (d_n) is said to be δ-quasi monotone, if $d_n \to 0$, $d_n > 0$ ultimately, and $\Delta d_n \geq -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$ and $\delta = (\delta_n)$ is a sequence of positive numbers (see [2]). A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = \{f_n(\sigma, \gamma)\} = \{n^\sigma (\log n)^\gamma,\ \gamma \geq 0,\ 0 < \sigma < 1\}$(see [11]). If we take γ=0, then we get a quasi-σ-power increasing sequence. Every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse is not true for $\sigma > 0$ (see [9]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α, β), with $\alpha + \beta > -1$, of the sequence (na_n), that is (see [6])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad \texttt{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \texttt{for} \quad n > 0. \tag{2}$$

Let $(\theta_n^{\alpha,\beta})$ be a sequence defined by (see [3])

$$\theta_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1 \\ \max_{1\leq v\leq n} \left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1. \end{cases} \tag{3}$$

The series $\sum a_n$ is said to be summable $\mid C, \alpha, \beta \mid_k$, $k \geq 1$, if (see [7])

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{4}$$

If we take $\beta = 0$, then $\mid C, \alpha, \beta \mid_k$ summability reduces to $\mid C, \alpha \mid_k$ summability (see [8]).
The first author has proved the following main theorem.
Theorem A ([4]). Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (3). Let (X_n) be an almost increasing sequence such that $\mid \Delta X_n \mid = O(X_n/n)$ and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent, and $\mid \Delta\lambda_n \mid \leq \mid A_n \mid$ for all n. If the condition

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n} = O(X_m) \quad as \quad m \to \infty \tag{5}$$

satisfies, then the series $\sum a_n\lambda_n$ is summable $\mid C, \alpha, \beta \mid_k$, $0 < \alpha \leq 1$, $\alpha + \beta > 0$, and $k \geq 1$.

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2. The main result.

The aim of this paper is to extent Theorem A by using a quasi-f-power increasing sequence, which is a general class of quasi power increasing sequences, instead of an almost increasing sequence. We shall prove the following theorem.

Theorem. Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (3). Let (X_n) be a quasi-f-power increasing sequence and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\Delta A_n \leq \delta_n$, $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent, and $\mid \Delta\lambda_n \mid \leq \mid A_n \mid$ for all n. If the condition (5) is satisfied, then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha,\beta \mid_k$, $0<\alpha\leq 1$, $\alpha+\beta>0$, and $k\geq 1$.

If we take (X_n) as an almost increasing sequence such that $\mid \Delta X_n \mid = O(X_n/n)$, then we get Theorem A, in this case condition '$\Delta A_n \leq \delta_n$' is not needed.

We need the following lemmas for the proof of our theorem.

Lemma 1 ([3]). If $0<\alpha\leq 1$, $\beta>-1$, and $1\leq v\leq n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1\leq m\leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{6}$$

Lemma 2 ([5]). Let (X_n) be a quasi-f-power increasing sequence. If (A_n) is a δ-quasi-monotone sequence with $\Delta A_n \leq \delta_n$ and $\sum n\delta_n X_n < \infty$, then

$$\sum_{n=1}^{\infty} nX_n \mid \Delta A_n \mid < \infty, \tag{7}$$

$$nA_nX_n = O(1) \quad as \quad n\to\infty. \tag{8}$$

3. Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v .$$

Applying Abel's transformation first and then using Lemma 1, we obtain that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v ,$$

$$\begin{aligned} \mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid \mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\ &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} . \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty, \quad for \quad r=1,2.$$

When $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n}\mid T_{n,1}^{\alpha,\beta}\mid^k &\leq \sum_{n=2}^{m+1}\frac{1}{n}\mid \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{(\alpha+\beta)}\theta_v^{\alpha,\beta}\Delta\lambda_v\mid^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\left\{\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}\mid A_v\mid(\theta_v^{\alpha,\beta})^k\right\}\times\left\{\sum_{v=1}^{n-1}\mid A_v\mid\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}\mid A_v\mid(\theta_v^{\alpha,\beta})^k\sum_{n=v+1}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}\mid A_v\mid(\theta_v^{\alpha,\beta})^k\int_v^\infty\frac{dx}{x^{1+(\alpha+\beta)k}}=O(1)\sum_{v=1}^{m}v\mid A_v\mid\frac{(\theta_v^{\alpha,\beta})^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\mid A_v\mid)\sum_{p=1}^{v}\frac{(\theta_p^{\alpha,\beta})^k}{p}+O(1)m\mid A_m\mid\sum_{v=1}^{m}\frac{(\theta_v^{\alpha,\beta})^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\mid(v+1)\Delta\mid A_v\mid-\mid A_v\mid\mid X_v+O(1)m\mid A_m\mid X_m\\
&= O(1)\sum_{v=1}^{m-1}v\mid\Delta A_v\mid X_v+O(1)\sum_{v=1}^{m-1}\mid A_v\mid X_v+O(1)m\mid A_m\mid X_m\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

in view of hypotheses of the theorem and Lemma 2. Similarly, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\frac{1}{n}\mid T_{n,2}^{\alpha,\beta}\mid^k &= O(1)\sum_{n=1}^{m}\frac{\mid\lambda_n\mid}{n}(\theta_n^{\alpha,\beta})^k=O(1)\sum_{n=1}^{m}\frac{(\theta_n^{\alpha,\beta})^k}{n}\sum_{v=n}^{\infty}\mid\Delta\lambda_v\mid\\
&= O(1)\sum_{v=1}^{\infty}\mid\Delta\lambda_v\mid\sum_{n=1}^{v}\frac{(\theta_n^{\alpha,\beta})^k}{n}=O(1)\sum_{v=1}^{\infty}\mid\Delta\lambda_v\mid X_v\\
&= O(1)\sum_{v=1}^{\infty}\mid A_v\mid X_v<\infty.
\end{aligned}$$

This completes the proof of the theorem. If we take $\beta=0$, then we get a new result concerning the $\mid C,\alpha\mid_k$ summability factors. If we set $\beta=0$, $\alpha=1$, and $X_n=$ logn, then we obtain the result of Mazhar dealing with $\mid C,1\mid_k$ summability factors (see [10]). Finally, if we take γ=0, then we get a new result dealing with an application of quasi-σ-power increasing sequences.

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