International Journal of Analysis and Applications
ISSN 2291-8639
Volume 3, Number 1 (2013), 47-52
http://www.etamaths.com

EXISTENCE OF HETEROCLINIC SOLUTIONS TO FOURTH ORDER $\Phi-$LAPLACIAN DYNAMICAL EQUATIONS

K. R. PRASAD[1], P. MURALI[1] AND N.V.V.S. SURYANARAYANA[2,*]

Abstract. In this paper, we derive sufficient conditions for the existence of heteroclinic solutions to fourth order $\Phi-$Laplacian dynamical equation,

$$\left[\Phi\left(y^{\Delta^2}(t)\right)\right]^{\Delta^2} = f(y(t)),\ t \in \mathbb{T},$$

on infinite time scales by using variational approach as minimizers of an action functional on special functional space. And also, as an application we demonstrate our result with an example.

1. Introduction

The study of heteroclinic solutions for p and Φ-Laplacian operators on infinite time scales have a certain impulse in recent years, which are motivated by applications in various Biological, Physical, Mechanical and Chemical models, such as phase transition, physical processes in which the variable transits from an unstable equilibrium to a stable one, or front propagation in reaction diffusion equation. These solutions provide an important information on the dynamics of the system. Due to the importance in both theory and applications, the study of heteroclinic solutions gained momentum on real intervals, we list a few; Avrameseu and Vladimirecu [1], Cabada and Cid [3, 4], Cabada and Tersion [5], and Marcelli and Papalini [7, 8].

The history of p and Φ-Laplacian operators for boundary value problems also enjoys a good history, first for differential equations, then finite difference equations, and recently, unifying results for dynamic equations. These operators have been widely studied by many researchers. In this theory, the most investigated operator is the classical p-Laplacian, generally $\Phi_p(y) := y|y|^{p-2}$ with $p > 1$, which, in recent years, has been generalized to other types of differential operators that preserve the monotonicity of the p-Laplacian, but are not homogeneous. These more general operators, which are usually referred to as Φ-Laplacian, are involved in the modeling of non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. The related nonlinear differential equation has the form

$$[\Phi(y')]' = f(t, y, y').$$

In this paper, we are dealing with some special class of time scales namely infinite and semi infinite time scales because continuous time orbits and discrete time orbits

---

2010 *Mathematics Subject Classification.* 34A40, 34B15, 34L30.

Key words and phrases. Heteroclinic solution, Φ-Laplacian, variational approach, infinite time scales.

are topologically different. A time scale is an arbitrary nonempty closed subset of the real numbers and we denote the time scale by the symbol $\mathbb{T}$. A time scale $\mathbb{T}$ is said to be infinite time scale, we mean, if it has no infimum and no supremum. (i.e., $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$.) A time scale $\mathbb{T}$ is said to be semi infinite time scale, we mean, if it has either infimum but no supremum or supremum but no infimum. (i.e., $\inf \mathbb{T} = a$ and $\sup \mathbb{T} = +\infty$ or $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = b$)and we denote as $\mathbb{T}_a^+ = [a, +\infty)$ and $\mathbb{T}_b^- = (-\infty, b]$ respectively. For example if we consider time scale $\mathbb{T} = \{-(a)^n\}_{n\in\mathbb{N}} \cup \{0\} \cup \{b^n\}_{n\in\mathbb{N}}, (a, b > 1)$ then it is an infinite time scale and if we remove either negative terms or positive terms then it is an example of semi infinite time scale. By an interval $[a, b]_{\mathbb{T}}$ means the intersection of the real interval with a given time scale.

$$i.e\ [a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}.$$

Most of the definitions and results on time scales are from the text book by Bohner and Peterson [2] and Lakshmikantham, Sivasundaram and Kaymakcalan [6]. Heteroclinic solutions to time scale dynamical systems are not available in the literature which will unify both the continuous and discrete dynamical systems.

Now, we consider fourth order $\Phi-$Laplacian dynamical equation on infinite time scales,

$$\left[\Phi\left(y^{\Delta^2}(t)\right)\right]^{\Delta^2} = f(y(t)),\ t \in \mathbb{T}, \tag{1}$$

where $f : \mathbb{T} \to \mathbb{R}$ is continuous. By using variational approach, we establish sufficient conditions for the existence of heteroclinic solution to the dynamical equation (1) under certain assumptions.

A heteroclinic solution of equation (1) connecting -1 to $+1$ in the phase-space, is a function $y^{\Delta^2} \in C_{rd}^2(\mathbb{T})$ such that $y^{\Delta^2} \in (-a, a)$, $\Phi \circ y^{\Delta^2} \in C_{rd}^2(\mathbb{T})$ and y satisfies the dynamical equation (1) and with the property

$$\lim_{t\to\pm\infty} (y(t), y^{\Delta}(t), y^{\Delta^2}(t), y^{\Delta^3}(t))) = (\pm 1, 0, 0, 0).$$

Through out the paper we assume the following:

- (A1) $f(y) = 0$ if and only if $y = \pm 1$.
- (A2) there exists a primitive F of f such that $F(-1) = F(+1)$ and $F(y) \geq 0$ for all $y \in \mathbb{R}$ and
$$\lim_{|y|\to+\infty} \inf f(y) > 0.$$
- (A3) $\Phi : (-a, a) \to \mathbb{R}$ is a positive increasing homeomorphism with $\Phi(0) = 0$ and $0 < a < +\infty$.
- (A4) Φ and F satisfy symmetric conditions $\Phi(y) = \Phi(-y)$ and $F(y) = F(-y)$.
- (A5) the energy is conserved. ($i.e.,\ \widetilde{\Phi}(y_2) + F(y) = K,\ K \in \mathbb{R}^+$)
- (A6) if $\{y_n(t)\}$ is a sequence of solutions of (1) for which for each $n \in \mathbb{T}_0^+$ an arbitrary compact interval $[0, \sigma^2(n)]_{\mathbb{T}}$ and there exists an $M > 0$ such that $y_n(t) \leq M$ for all $t \in [0, \sigma^2(n)]_{\mathbb{T}}$ and for all $t \in \mathbb{N}$, then there exists a subsequence $\{y_{n_j}(t)\}$, such that $\{y_{n_j}^{\Delta^i}\}$ converges uniformly on $[0, \sigma^2(n)]_{\mathbb{T}},\ i = 0, 1.$

The rest of the paper is organized as follows, in Section 2, by using variational approach, we establish sufficient conditions for the existence of heteroclinic solution to the differential equation (1). As an application, we give an example to demonstrate our result.

2. Existence of Heteroclinic Solutions

In this section, by using variational approach, we establish sufficient conditions for the existence of heteroclinic solution to the differential equation (1).

The equation (1) is the Euler-Lagrange equation corresponding to the action functional

$$\mathcal{F}(y) = \int_{\mathbb{T}} \left(\int \Phi(y_2)\Delta y_2 + F(y) \right) \Delta t, \tag{2}$$

where y_2 is the second delta derivative of y with respect to t and $F(y)$ is the primitive of $f(y)$. The action functional is defined in functional space

$$\mathcal{E} = \left\{ y : \mathbb{T} \to \mathbb{R} |\ y(0) = 0,\ y+1 \in H^2(\mathbb{T}^-),\ y-1 \in H^2(\mathbb{T}^+) \right\}. \tag{3}$$

Theorem 2.1. *The functional $\mathcal{F} : \mathcal{E} \to \mathbb{R}$ defined by* (2) *on* (3) *is of class C^1 and any critical point in C^∞ is a heteroclinic solution of* (1) *connecting -1 to $+1$.*

Proof. For any function $\eta \in C_c^2(\mathbb{T})$, for all $\tau \in \mathbb{R}$, $y+\tau\eta \in \mathcal{F}$ and $\mathcal{F}(y+\tau\eta)$ is delta differentiable as a real function of the parameter τ. Since y minimizes $\mathcal{F}$ in the space $\mathcal{E}$, the function $\mathcal{F}(y+\tau\eta)$ archives a minimum at $\tau = 0$. Let $\widetilde{\Phi}(y_2) = \int \Phi(y_2)\Delta y_2$,

$$\mathcal{F}^{\Delta_\tau}(y+\tau\eta)|_{\tau=0} = \left[\int_{\mathbb{T}} \left(\widetilde{\Phi}(y_2) + F(y) \right) \Delta t \right]^{\Delta_\tau} \text{ at } \tau = 0$$

$$\mathcal{F}^{\Delta_\tau}\eta = \int_{\mathbb{T}} \left(\widetilde{\Phi}^{\Delta_\tau}(y_2)\eta^{\Delta_\tau^2} + f(y)\eta \right) \Delta t.$$

Now let y be a critical point of $\mathcal{F}$. We have $\mathcal{F}^{\Delta_\tau}(y)\eta = 0$ for every $\eta \in H^2(\mathbb{T})$ satisfies $\eta(0) = 0$. Starting with $\eta \in C_c^2(\mathbb{T})$ and Du Bois-Reymond Lemma, we have $\Phi \circ y^{\Delta^2} \in C^2(\mathbb{T})$ and y satisfies (1). Hence $y - 1 \in H^4(\mathbb{T}^+)$. Similarly we have $y + 1 \in H^4(\mathbb{T}^-)$. From the L^2-Integrability of the delta derivative implies that

$$\lim_{t\to\pm\infty} y(t) = \pm 1 \text{ and } \lim_{t\to\pm\infty} y^{\Delta^n}(t) = 0 \text{ for } n = 1, 2, 3,$$

so that y is a heteroclinic solution of (1) connecting -1 to $+1$ and a straightforward argument given as y is of class C^∞. □

Now, we prove main theorem which confirms the efficiency of a minimization approach.

Theorem 2.2. *The functional $\mathcal{F} : \mathcal{E} \to \mathbb{R}$ defined by* (2) *on* (3) *has a minimizer which is a heteroclinic solution of* (1) *connecting -1 to $+1$. Furthermore, any minimizer is odd and positive in $(0, +\infty)$.*

Proof. For convenience, we introduce the following spaces

$$\mathcal{E}^+ = \left\{ y : \mathbb{T}^+ \to \mathbb{R} : y(0) = 0,\ y - 1 \in H^2(\mathbb{T}^+) \right\}$$

$$\mathcal{E}^- = \left\{ y : \mathbb{T}^- \to \mathbb{R} : y(0) = 0,\ y + 1 \in H^2(\mathbb{T}^-) \right\}$$

and consider the action functionals $\mathcal{F}^{\pm} : \mathcal{E}^{\pm} \to \mathbb{R}$ by

$$\mathcal{F}^{\pm}(y) = \int_{\mathbb{T}^{\pm}} L(y, y^{\Delta}, y^{\Delta^2})\Delta t,$$

where $L(y, y^{\Delta}, y^{\Delta^2})$ is the Lagrangian given by

$$L(y, y^{\Delta}, y^{\Delta^2}) = \widetilde{\Phi}(y_2) + F(y).$$

Let us denote the values

$$c = \inf_{\mathcal{E}} \mathcal{F} \text{ and } c^{\pm} = \inf_{\mathcal{E}^{\pm}} \mathcal{F}^{\pm}.$$

Since $\mathcal{F}$ is symmetric, we have, for all $y^+ \in \mathcal{E}^+$,

$$\mathcal{F}^+(y^+) = \mathcal{F}^-(y^-),$$

where $y^- \in \mathcal{E}^-$ is defined by $y^-(t) = -y^+(-t)$. Therefore, we have

$$c^+ = c^- = \frac{c}{2}.$$

First we prove that the variational problem $\inf\{\mathcal{F}^+(y) : y \in \mathcal{E}^+\}$ has a positive solution.

Let $(v_n)_n \subset \mathcal{E}^+$ be a minimizing sequence for $\mathcal{F}^+$, i.e., $v_n \in \mathcal{E}^+$ for all $n \in \mathbb{N}$ and $\mathcal{F}^+(v_n) \to c^+$. For each $n \geq 0$, we define

$$t_n = \sup\{t \geq 0 : v_n(t) = 0\}.$$

Since $\lim_{t\to+\infty} v_n(t) = 1,\ t_n < +\infty$ for all $n \geq 0$. We now consider the positive sequence $(v_n^+)_n \subset \mathcal{E}^+$, where $v_n^+(t) = v_n(t + t_n)$ for $t \geq 0$. We observe that

$$\int_0^{t_n} L(v_n, v_n^{\Delta}, v_n^{\Delta^2})dt \geq 0$$

so that $\mathcal{F}^+(v_n^+) \leq \mathcal{F}^+(v_n)$ which implies that $(v_n^+)_n$ is also a minimizing sequence for $\mathcal{F}^+$.

As the sequence $\mathcal{F}^+(v_n^+)$ is uniformly bounded, we deduce a uniform estimate for $\| v_n^+ - 1 \|_{H^2(\mathbb{T}^+)}$. From the positivity of v_n^+ that

$$\int_{\mathbb{T}^+} F(v_n^+)\Delta t \leq \mathcal{F}^+(v_n^+)$$

and there exists

$$v^+ \in H^2(\mathbb{T}^+) + 1$$

such that

$$v_n^+ - 1 \xrightarrow{H^2(\mathbb{T}^+)} v^+ - 1$$

and

$$v_n^+ \xrightarrow{C^1_{loc}(\mathbb{T}^+)} v^+.$$

As the first two terms in $\mathcal{F}^+$ are the sequence of seminorms and Fatou's Lemma is applicable to the last one, we have

$$\mathcal{F}^+(v^+) \leq \lim_{n\to+\infty} \inf \mathcal{F}(v_n^+) = c^+.$$

The converges being uniform an compact interval, we conclude that $v^+(0) = 0$ so that $v^+ \in \mathcal{E}^+$ and $\mathcal{F}^+(v^+) = c^+$. Observe that v^+ is positive on $(0, +\infty)$ otherwise we could proceed as above to construct a positive function having smaller action.

Secondly, we show that, if $v \in \mathcal{E}^+$ is such that $(\mathcal{F}^+)^{\Delta_\tau}(v) = 0$, then $v^{\Delta^2}(0) = 0$, $v^* \in \mathcal{E}$ defined by

$$v^*(t) = \begin{cases} v(t), & \text{if } t \geq 0 \\ -v(-t) & \text{if } t < 0 \end{cases} \tag{4}$$

is a minimizer of $\mathcal{F}$ in $\mathcal{E}$ and v^* is a heteroclinic solution of (1). We compute

$$(\mathcal{F}^+)^{\Delta_\tau}(v)(\eta) = \int_{\mathbb{T}^+} \left(\widetilde{\Phi}^{\Delta_\tau}(v_2)\eta^{\Delta_\tau^2} + f(v)\eta\right) \Delta t \tag{5}$$

for all $\eta \in H^2(\mathbb{T}^+) \cap H_0^1(\mathbb{T}^+)$.

From Theorem 2.1, we deduce that v is a solution of equation (1), $\Phi \circ v^{\Delta^2} \in C^2(\mathbb{T}^+)$ and

$$\lim_{t\to+\infty} v(t) = +1 \text{ and } \lim_{t\to+\infty} v^{\Delta^n}(t) = 0 \text{ for } n = 1, 2, 3, \tag{6}$$

Integrating (5), we obtain for all $n \in H^2(\mathbb{T}^+) \cap H_0^1(\mathbb{T}^+)$, as

$$\int_{\mathbb{T}^+} \left(\int [\Phi(y^{\Delta^2})]^{\Delta_\tau^2} + f(y) \right) \eta(t)\Delta t = 0.$$

Take (6) into account, we now deduce that

$$v^{\Delta^2}(0)\eta^{\Delta}(0) = 0$$

for all $\eta \in H^2(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+)$ which implies that $v^{\Delta^2}(0) = 0$. The function $v^* : \mathbb{T} \to \mathbb{R}$ defined by (4) is of class C^4, and solution of (1) and also as

$$\mathcal{F}(v^*) = 2\mathcal{F}^+(v) = 2c^+ = c.$$

We conclude that v^* is a minimization of $\mathcal{F}$ in $\mathcal{E}$.

Finally, we prove that if $y \in \mathcal{E}$ minimizes $\mathcal{F}$, then y is odd.

Let us define $y^\pm = y|_{\mathbb{R}^\pm}$. As $\mathcal{F}(y) = c$, we obviously have

$$\mathcal{F}^+(y^+) = \mathcal{F}^-(y^-) = \frac{c}{2}$$

otherwise the odd extension of y^+ or y^- would have a lower action than c. Define $v^+ \in \mathcal{E}^+$ by $v^+(t) = -y^-(-t)$. Then v^+ satisfies $\mathcal{F}^+(v^+) = c^+$ and therefore minimizer $\mathcal{F}^+$ in $\mathcal{E}^+$. It follows that both y^+ and v^+ are minimizers of $\mathcal{F}^+$ in $\mathcal{E}^+$. From claim 2, we have that $y^{\Delta^2}(0) = (v^+)^{\Delta^2}(0) = 0$ and as $(v^+)^{\Delta}(0) = (-y^-)^{\Delta}(0) = (y^+)^{\Delta}(0)$ and $(v^+)^{\Delta^3}(0) = (-y^-)^{\Delta^3}(0) = (y^+)^{\Delta^3}(0)$, the functions v^+ and y^+ are the solutions of the Cauchy problem,

$$\left[\Phi\left(y^{\Delta^2}(t)\right)\right]^{\Delta^2} = f(y(t)),\ t \in \mathbb{T},$$

$$y(0) = 0,\ y^{\Delta}(0) = (y^+)^{\Delta}(0);\ y^{\Delta}(0) = 0, y^{\Delta^3}(0) = (y^+)^{\Delta^3}(0).$$

By uniqueness, this implies $y^+(t) = v^+(t)$ for $t \in \mathbb{R}^+$, that is $y^+(t) = -y^-(-t)$ for all $t \in \mathbb{T}^+$. □

Example

In this section, as an application, we give an example to demonstrate our result. We consider the following fourth order Φ-Laplacian differential equation on infinite time scale

$$\mathbb{T} = \left\{ \{-2^n\}_{n\in\mathbb{N}\cup\{0\}} \cup [-1,1] \cup \{2^n\}_{n\in\mathbb{N}\cup\{0\}} \right\}$$

$$\left[\Phi(y^{\Delta^2})\right]^{\Delta^2} = f(y),\ t \in \mathbb{T}, \tag{7}$$

where $\Phi(y_2) = \frac{y_2^2}{\sqrt{1+y_2^2}}$ for $y_2 \in \mathbb{T}$ and $f(y) = y^2 - 1$. Then Φ and f satisfy the conditions (A1)-(A6) . Therefore, it follows from Theorem 2.2 that the fourth order Φ-Laplacian differential (7) has a heteroclinic solution.

Acknowledgement. One of the authors (Dr. P. Murali) is thankful to C.S.I.R. of India for awarding him an RA.

References

[1] C. Avrameseu and C. Vladimirecu, Existence of solution to second order differential equations having finite limits at $\pm\infty$, *Elec. J. Diff. Equa.,* Vol.**18**(2004), pp. 1-12.

[2] M.Bohner and A.C.Peterson, *Dynamic Equations on Time scales, An Introduction with Applications*, Birkhauser, Boston,MA,(2001).

[3] A. Cabada and J. A. Cid, Solvability of some Φ-Laplacian singular difference equation defined on integers, *The Arab. J. Scie. Engi.,* Vol **34**(2009), No.ID, pp.75-81.

[4] A. Cabada and J. A. Cid, Heteroclinic solution for non-autonomous boundary value problem with singular Φ-Laplacian operator, *Discr. Cont. Dyna. Syst.,* Vol **18**(2009), pp. 118-112.

[5] A. Cabada and S. Tersion, Existence of heteroclinic solutions for discrete P-Laplacian problems with a parameter, *Nonlinear; Real world appl.,* doi:10.1016/j.nonwrg.2011.02.022.

[6] V. Lakshmikantham, S. Sivasundaram and B. Kaymakcalan, *Dynanic Systems on Measure Chains*, Mathematics and its Applications, **370**, Kluwer, Dordrecht, 1996.

[7] C. Marcelli and F. Papalini, Heteroclinic solution for second order non-autonoms boundary value problem on real line, *Diff. Equa. Dyna. Syst.,* Vol**11**(2003), pp.333-353.

[8] C. Marcelli and F. Papalini, Heteroclinic connection for fully nonlinear non-autonums second order differential equations, *J. Diff. Equa.,* **241**(2007), pp.160-183.

[1]Department of Applied Mathematics, Andhra University, Visakhapatnam, 530003, India

[2]Department of Mathematics, VITAM College of Engineering, Visakhapatnam, 531173, India

*Corresponding author