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EXISTENCE AND APPROXIMATE SOLUTIONS FOR NONLINEAR HYBRID FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

B.C. DHAGE[1,*], G.T. KHURPE[1], A.Y. SHETE[2] AND J.N. SALUNKE[2]

Abstract. In this paper we prove existence and approximation of the solutions for initial value problems of nonlinear hybrid fractional differential equations with maxima and with a linear as well as quadratic perturbation of second type. The main results rely on Dhage iteration method embodied in the recent hybrid fixed point theorem of Dhage (2014) in a partially ordered normed linear space. The approximation of the solutions of the considered nonlinear fractional differential equations are obtained under weaker mixed partial continuity and Lipschitz conditions. Our hypotheses and the main results are also illustrated by a numerical example.

1. Introduction

In this paper we prove existence and approximations of the solutions for initial value problems of nonlinear hybrid fractional differential equations. Consider the following initial value problem of fractional differential equations,

$$\left\{\begin{array}{rcl} {}^cD^\alpha\left(\dfrac{x(t)-I^\beta h(t,x(t))}{f(t,x(t))}\right) & = & g\left(t,x(t),\displaystyle\int_0^t k(s,x(s))\,ds\right),\ t\in J:=[0,T], \\ x(0) & = & x_0\in\mathbb{R}_+, \end{array}\right. \tag{1.1}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, $0<\alpha<1$, I^β is the Riemann-Liouville fractional integral of order β, and $f: J\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$, $h,k: J\times\mathbb{R}\to\mathbb{R}$ and $g: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

By a *solution* of the problem (1.1) we mean a function $x\in C^1(J,\mathbb{R})$ if

(i) the function $t\mapsto \dfrac{x(t)-I^\beta h(t,x(t))}{f(t,x(t))}$ is Caputo differentiable, and
(ii) x satisfies the relations in ((1.1) on J,

where $C^1(J,\mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

Fractional differential equations have aroused great interest, which is caused by both the intensive development of the theory of fractional calculus and the applications to rheology, physics, mechanics and chemistry engineering [16, 17]. For some recent development on the topic see [1] and the references cited therein. For some recent results on hybrid fractional differential equations we refer to [1], [2], [14], [18], [19] and the references cited therein.

The origin of the problem (1.1) lies in the initial value problems of first order quadratic differential equations with ordinary derivative wherein only existence of the solutions is proved using classical hybrid fixed point theorem of Dhage [3]. The problem (1.1) considered here is general in the sense that it includes the following three well-known classes of initial value problems of fractional differential equations.

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Case I: Let $f(t,x)=1$ and $h(t,x)=0$ for all $t\in J$ and $x\in\mathbb{R}$. Then the problem (1.1) reduces to standard initial value problem of fractional differential equation

$$\left\{\begin{array}{rcl} {}^cD^{\alpha}x(t) &=& g\left(t,x(t),\displaystyle\int_0^t k(s,x(s))\,ds\right),\ t\in J:=[0,T],\\ xx(0) &=& x_0\in\mathbb{R}. \end{array}\right. \tag{1.2}$$

Case II: If $h(t,x)=0$ for all $t\in J$ and $x\in\mathbb{R}$ in (1.1), we obtain the following quadratic fractional differential equation,

$$\left\{\begin{array}{rcl} {}^cD^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) &=& g\left(t,x(t),\displaystyle\int_0^t k(s,x(s))\,ds\right),\ t\in J:=[0,T],\\ x(0) &=& x_0\in\mathbb{R}. \end{array}\right. \tag{1.3}$$

Case III: If $f(t,x)=1$ for all $t\in J$ and $x\in\mathbb{R}$ in (1.1), we obtain the following interesting fractional differential equation,

$$\left\{\begin{array}{rcl} {}^cD^{\alpha}\big[x(t)-I^{\beta}h(t,x(t))\big] &=& g\left(t,x(t),\displaystyle\int_0^t k(s,x(s))\,ds\right),\ t\in J:=[0,T],\\ x(0) &=& x_0\in\mathbb{R}. \end{array}\right. \tag{1.4}$$

Therefore, the main result of this paper also includes the existence as well as approximation results for the solutions of above mentioned initial value problems of fractional differential equations as special cases. Again our approach here in this paper is different than that employed in the related paper of Dhage [3].

In the present paper we prove the existence and approximations of the solutions of problem (1.1) under weaker partial compactness and partial Lipschitz type conditions via Dhage iteration method [7]. Very recently, Dhage iteration method has been applied in [7, 8, 9, 11, 12, 13] to nonlinear ordinary differential equations for proving the existence and algorithms of the solutions.

We recall the basic definitions of fractional calculus [16, 17] which are useful in what follows.

Definition 1.1. *The Riemann-Liouville fractional integral of order q with the lower limit zero for a function f is defined as*

$$I^qf(t)=\frac{1}{\Gamma(q)}\int_0^t\frac{f(s)}{(t-s)^{1-q}}ds,\quad t>0,\quad q>0,$$

provided the right hand-side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(q)=\displaystyle\int_0^{\infty}t^{q-1}e^{-t}dt$.

Definition 1.2. *The Riemann-Liouville fractional derivative of order $q>0,\ n-1<q<n,\ n\in\mathbb{N}$, is defined as*

$$D^q_{0+}f(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-q-1}f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 1.3. *The Caputo derivative of order q for a function $f\in C^n(J,\mathbb{R})$ can be written as*

$${}^cD^qf(t)=D^q\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right),\quad t>0,\quad n-1<q<n.$$

Remark 1.4. *If $f\in C^n(J,\mathbb{R})$, then*

$${}^cD^qf(t)=\frac{1}{\Gamma(n-q)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{q+1-n}}ds=I^{n-q}f^{(n)}(t),\ t>0,\ n-1<q<n.$$

Lemma 1.5. *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

Remark 1.6. In view of Lemma 1.5, it follows that

$$I^q\ {}^cD^q x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1}, \tag{1.5}$$

for some $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

The rest of the paper will be organized as follows. In Section 2 we give some preliminaries and key fixed point theorems that will be used in subsequent part of the paper. In Section 3 we discuss the main existence and approximation result for initial value problems of fractional differential equations (1.1). An illustrative example is also discussed.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper we let E denote a partially ordered real normed linear space with the order relation $\preceq$ and the norm $\|\cdot\|$ in which addition and scalar multiplication by positive real numbers are preserved by $\preceq$. A few details on such partially ordered normed linear spaces appear in Dhage [5] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all elements of C are comparable. We say that E is *regular* if for any nondecreasing (resp. nonincreasing) sequence $\{x_n\}$ in E such that $x_n \to x^*$ as $n \to \infty$, we have that $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [15] and the references therein.

We need the following definitions (see Dhage [5, 6, 6] and the references therein) in what follows.

Definition 2.1. *A mapping $\mathcal{B} : E \to E$ is called* **isotone** *or* **nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{B}x \preceq \mathcal{B}y$ for all $x, y \in E$.*

Definition 2.2. *A mapping $\mathcal{B} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{B}x - \mathcal{B}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{B}$ called a partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{B}$ is a partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3. *A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. A nondecreasing mapping $\mathcal{B} : E \to E$ is called* **partially bounded** *if $\mathcal{B}(C)$ is bounded for every chain C in E. $\mathcal{B}$ is called* **uniformly partially bounded** *if all chains $\mathcal{B}(C)$ in E are bounded by a unique constant. $\mathcal{B}$ is called* **bounded** *if $\mathcal{B}(E)$ is a bounded subset of E.*

Definition 2.4. *A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is compact. A nondecreasing mapping $\mathcal{B} : E \to E$ is called* **partially compact** *if $\mathcal{B}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{B}$ is called* **uniformly partially compact** *if $\mathcal{B}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{B}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{B}(C)$ is a relatively compact subset of E. If $\mathcal{B}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Definition 2.5. *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the whole sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called* **Janhavi** *if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a* **Janhavi metric** *or* **Janhavi Banach space**.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.6. *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called **partially nonlinear $\mathcal{D}$-Lipschitz** if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathcal{T}$ is called a partially Lipschitz with a Lipschitz constant k. Furthermore, if $\psi(r) < r$, $r > 0$, $\mathcal{T}$ is called a partially nonlinear $\mathcal{D}$-contraction on E.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \big\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\big\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } \ u, v \in E^+\}. \tag{2.2}$$

The elements of the set $\mathcal{K}$ are called the positive vectors in E. Then following lemma is immediate.

Lemma 2.7 (Dhage [3])**.** *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.8. *An operator $\mathcal{B} : E \to E$ is said to be positive if the range $R(\mathcal{B})$ of $\mathcal{B}$ is such that $R(\mathcal{B}) \subseteq \mathcal{K}$.*

The Dhage iteration method is embodied in the following hybrid fixed point theorem proved in Dhage [6] which are useful tools in what follows. A few other such hybrid fixed point theorems appear in Dhage [5, 6].

Theorem 2.9 (Dhage [7])**.** *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that every compact chain C in E is Janhavi. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ and $\mathcal{C} : E \to E$ be three nondecreasing operators such that*

- (a) *$\mathcal{A}$ and $\mathcal{C}$ are partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ respectively.*
- (b) *$\mathcal{B}$ is partially continuous and uniformly partially compact,*
- (c) *$0 < M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C$ is a chain in $E\}$, and*
- (d) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0\mathcal{B}x_0 + \mathcal{C}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$.*

Then the operator equation $\mathcal{A}x\mathcal{B}x + \mathcal{C}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\mathcal{B}x_n + \mathcal{C}x_n$, $n = 0, 1, \ldots$ converges monotonically to x^*.

Remark 2.10. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E is held if every partially compact subset S of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact is used to prove the desired characterization of the mild solution of the problem (1.1) on J.

3. Main Existence Result

The equivalent integral form of the problem (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.1}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.2}$$

for all $t \in J$. Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and a lattice so that every pair of elements of E has a lower and an upper bound in it. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzellá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}),\leq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then every partially compact subset S of $C(J,\mathbb{R})$ is Janhavi.*

Proof. The proof of the lemma is given in Dhage and Dhage [11]. Since the proof is well-known, we omit the details of proof. □

We need the following definition in what follows.

Definition 3.2. *A function $u \in C^1(J,\mathbb{R})$ is said to be a lower solution of the problem (1.1) if the function $t \mapsto \dfrac{u(t)-I^\beta h(t,u(t))}{f(t,u(t))}$ is continuously differentiable and satisfies*

$$\left.\begin{aligned} {}^cD^\alpha\left(\frac{u(t)-I^\beta h(t,u(t))}{f(t,u(t))}\right) &\leq g\left(t,u(t),\int_0^t k(s,u(s))\,ds\right),\ t\in J,\\ u(0)&\leq x_0. \end{aligned}\right\} \tag{$*$}$$

Similarly, an upper solution $v \in C^1(J,\mathbb{R})$ to the problem (1.1) is defined on J, by the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

(A_0) The map $x \mapsto \dfrac{x}{f(t,x)}$ is injective for each $t \in J$.

(A_1) There exists a constant $M_f > 0$ such that $0 < f(t,x) \leq M_f$ for all $t \in J$ and $x \in \mathbb{R}$.

(A_2) There exists a $\mathcal{D}$-function φ such that

$$0 \leq f(t,x) - f(t,y) \leq \varphi(x-y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \geq y$.

(B_1) There exists a constant $M_g > 0$ such that $0 < g(t,x,y) \leq M_g$ for all $t \in J$ and $x, y \in \mathbb{R}$.

(B_2) The function $g(t,x,y)$ is monotone nondecreasing in x and y for each $t \in J$.

(B_3) The function $k(t,x)$ is monotone nondecreasing in x for each $t \in J$.

(C_1) There exists a constant $M_h > 0$ such that $0 \leq h(t,x) \leq M_h$ for all $t \in J$ and $x \in \mathbb{R}$.

(C_2) There exists a $\mathcal{D}$-function ω such that

$$0 \leq h(t,x) - h(t,y) \leq \omega(x-y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \geq y$.

(D_1) The problem (1.1) has a lower solution $u \in C^1(J,\mathbb{R})$.

(D_2) The problem (1.1) has an upper solution $v \in C^1(J,\mathbb{R})$.

Remark 3.3. *Notice that Hypothesis (A_0) holds in particular if the function $x \mapsto \dfrac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ for each $t \in J$.*

The following lemma is useful in what follows and may be found in Kilbas *et.al.* [16] and Podlubny [17].

Lemma 3.4. *For a given continuous function $h : J \to \mathbb{R}$, a function $u \in C^1(J,\mathbb{R})$ is a solution of the QFDE*

$$\left.\begin{aligned} {}^cD^q x(t) &= h(t),\ t\in J,\ 0<q<1,\\ x(0) &= \alpha_0, \end{aligned}\right\} \tag{3.3}$$

if and only if it is a solution of the nonlinear integral equation,

$$x(t) = \alpha_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \, h(s) \, ds, \;\; t \in J. \tag{3.4}$$

As an application of Lemma 3.4, we obtain

Lemma 3.5. *Assume that the hypothesis* (A_0) *holds. If a function* $x \in C^1(J, \mathbb{R})$ *is a solution of the QFDE*

$$\left.\begin{aligned} {}^cD^q \left[\frac{x(t) - I^\beta h(t, x(t))}{f(t, x(t))} \right] &= g\left(s, x(s), \int_0^s k(\tau, x(\tau)) \, d\tau \right), \;\; t \in J, \;\; 0 < q < 1, \\ x(t_0) &= \alpha_0, \end{aligned}\right\} \tag{3.5}$$

then it satisfies the nonlinear integral equation,

$$\begin{aligned} x(t) = \big[f(t, x(t))\big] &\left[\frac{\alpha_0}{f(t_0, \alpha_0)} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \, g\left(s, x(s), \int_0^s k(\tau, x(\tau)) \, d\tau \right) \, ds \right] \\ &+ I^\beta h(t, x(t)), \;\; t \in J. \end{aligned} \tag{3.6}$$

Proof. Assume first that $x \in C^1(J, \mathbb{R})$ is a solution to the QFDE (1.1) defined on J. By Lemma 1.5, we have

$$\frac{x(t) - I^\beta h(t, x(t))}{f(t, x(t))} = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x(s), \int_0^s k(\tau, x(\tau)) \, d\tau \right) \, ds + c_0, \tag{3.7}$$

where $c_0 \in \mathbb{R}$. Since $x(0) = \alpha_0, f(0, \alpha_0) \neq 0$, it follows $c_0 = \dfrac{\alpha_0}{f(0, \alpha_0)}$. Thus (3.6) holds. □

Definition 3.6. *A function* $x \in C^1(J, \mathbb{R})$ *which satisfies the QFIE (3.7) is called a* **mild solution** *of the QFDE (1.1) defined on* J.

Theorem 3.7. *Assume that the hypotheses* (A_0)*-*(A_2)*,* (B_1)*-*(B_2)*,* (C_1)*-*(C_7) *and* (D_1) *hold. Furthermore, if*

$$\left[\left| \frac{\alpha_0}{f(t_0, \alpha_0)} \right| + \frac{M_g T^\alpha}{\Gamma(\alpha+1)} \right] \varphi(r) + \frac{T^\beta}{\Gamma(\beta+1)} \omega(r) < r, \tag{3.8}$$

then the problem (1.1) has a mild solution x^* *defined on* J *and the sequence* $\{x_n\}_{n=1}^\infty$ *of successive approximations defined by*

$$\begin{aligned} x_{n+1}(t) = &\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, x_n(s)) ds \\ &+ f(t, x_n(t)) \left[\frac{\alpha_0}{f(t_0, \alpha_0)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x(s), \int_0^s k(\tau, x(\tau)) \, d\tau \right) \, ds \right] \end{aligned} \tag{3.9}$$

for all $t \in J$*, where* $x_1 = u$*, converges monotonically to* x^**.*

Proof. By Lemma 3.5, the mild solution x of the problem (1.1) satisfies the nonlinear integral equation

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, x(s)) ds \\ &+ f(t, x(t)) \left[\frac{\alpha_0}{f(t_0, \alpha_0)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x(s), \int_0^s k(\tau, x(\tau)) \, d\tau \right) \, ds \right] \end{aligned} \tag{3.10}$$

for all $t \in J$.

Set $E = C(J, \mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ on E by

$$\mathcal{A}x(t) = f(t, x(t)), \;\; t \in J, \tag{3.11}$$

$$\mathcal{B}x(t) = \frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\left(s, x(s), \int_0^s k(\tau, x(\tau))\,d\tau\right) ds,\ t \in J, \tag{3.12}$$

and

$$\mathcal{C}x(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} h(s, x(s))\,ds,\ t \in J. \tag{3.13}$$

From the continuity of the integrals, it follows that $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ define the maps $\mathcal{A}, \mathcal{B}:\ E \to \mathcal{K}$ and $\mathcal{C}:\ E \to E$. Then, the problem (1.1) is equivalent to the operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t) + \mathcal{C}x(t) = x(t),\ \ t \in J. \tag{3.14}$$

We shall show that the operators $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.9. This is achieved in the series of following steps.

Step I: *$\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are nondecreasing operators on E.*

Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (A_2), we obtain

$$\mathcal{A}x(t) = f(t, x(t)) \geq f(t, y(t)) = \mathcal{A}y(t),$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E. Similarly, we have by (A_5),

$$\begin{aligned}\mathcal{B}x(t) &= \frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\left(s, x(s), \int_0^s k(\tau, x(\tau))\,d\tau\right) ds \\ &\geq \frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\left(s, x(s), \int_0^s k(\tau, x(\tau))\,d\tau\right) ds \\ &= \mathcal{B}y(t),\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}$ is nondecreasing operator on E into itself. The proof that $\mathcal{C}$ is nondecreasing operator on E into itself is similar.

Step II: *$\mathcal{A}$ and $\mathcal{C}$ are partially bounded and partially $\mathcal{D}$-contraction on E.*

Let $x \in E$ be arbitrary. Then by (A_1),

$$|\mathcal{A}x(t)| \leq |f(t, x(t))| \leq M_f,$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\| \leq M_f$ and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then,

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |f(t, x(t)) - f(t, y(t))| \leq \varphi(|x(t) - y(t)|) \leq \varphi(\|x - y\|).$$

Then, $\|\mathcal{A}x - \mathcal{A}y\| \leq \varphi(\|x - y\|)$ for all $x, y \in E$ with $x \geq y$ and hence $\mathcal{A}$ is a partially $\mathcal{D}$-Lipschitz on E with $\mathcal{D}$-functions $\varphi(r)$, which further implies that $\mathcal{A}$ is also a partially continuous on E.

Again, we have

$$\begin{aligned}|\,\mathcal{C}x(t)| &\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |h(s, x(s))| ds \\ &\leq M_h \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \\ &\leq \frac{M_h\, t^\beta}{\Gamma(\beta+1)} \\ &\leq \frac{M_h\, T^\beta}{\Gamma(\beta+1)},\end{aligned}$$

which means that $\mathcal{C}$ is bounded and consequently partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then,

$$\begin{aligned} |\mathcal{C}x(t) - \mathcal{C}y(t)| &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |h(s, x(s)) - h(s, y(s))| ds \\ &\leq \frac{T^\beta}{\Gamma(\beta+1)} \omega(\|x-y\|). \end{aligned}$$

Hence $\mathcal{C}$ is a partially $\mathcal{D}$-Lipschitz on E with $\mathcal{D}$-functions $\dfrac{T^\beta}{\Gamma(\beta+1)}\omega(r)$, which further implies that $\mathcal{C}$ is a partially continuous on E.

Step III: *$\mathcal{B}$ is a partially continuous operator on E.*

Let $\{x_n\}$ be a sequence of points of a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[\frac{\alpha_0}{f(t_0, \alpha_0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\left(s, x_n(s), \int_0^s k(\tau, x_n(\tau))\, d\tau\right) ds \right] \\ &= \frac{\alpha_0}{f(t_0, \alpha_0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\lim_{n\to\infty} g\left(s, x_n(s), \int_0^s k(\tau, x_n(\tau))\, d\tau\right) \right] ds \\ &= \frac{\alpha_0}{f(t_0, \alpha_0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\left(s, x(s), \int_0^s k(\tau, x(\tau))\, d\tau\right) ds \\ &= \mathcal{B}x(t), \end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{B}x_n\}$ converges to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned} &|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \\ &\qquad \leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}| \left| g\left(s, x(s), \int_0^s k(\tau, x(\tau))\, d\tau\right) \right| ds \right| \\ &\qquad\quad + \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_1-s)^{\alpha-1} \left| g\left(s, x(s), \int_0^s k(\tau, x(\tau))\, d\tau\right) \right| ds \right| \\ &\qquad \leq \frac{M_g}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha). \end{aligned}$$

Consequently, we obtain

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniformly and hence $\mathcal{B}$ is a partially continuous on E.

Step IV: *$\mathcal{B}$ is a partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned} |\mathcal{B}x(t)| &\leq \left| \frac{\alpha_0}{f(t_0, \alpha_0)} \right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| g\left(s, x(s), \int_0^s k(\tau, x(\tau))\, d\tau\right) \right| ds \\ &\leq \left| \frac{\alpha_0}{f(t_0, \alpha_0)} + \right| + \frac{M_g T^\alpha}{\Gamma(\alpha+1)} \\ &= r, \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{B}x\| \leq r$ for all $x \in C$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let

$t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then,

$$\begin{aligned}
&|\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^{t_2} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \left| g\left(s, x(s), \int_0^s k(\tau, x(\tau))\, d\tau\right) \right| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} \left| g\left(s, x(s), \int_0^s k(\tau, x(\tau))\, d\tau\right) \right| ds \right| \\
&\quad \le \frac{M_g}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha).
\end{aligned}$$

Thus, we have that

$$|\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1$$

uniformly for all $x \in C$. This shows that $\mathcal{B}(C)$ is an equicontinuous set in E. Hence $\mathcal{B}(C)$ is compact subset of E and consequently $\mathcal{B}$ is a partially compact operator on E into itself.

Step V: *$\mathcal{D}$-functions φ and ω satisfy the inequality* $0 < M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r, \ r > 0$.

We have

$$M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) = \left[\left| \frac{\alpha_0}{f(t_0, \alpha_0)} \right| + \frac{M_g T^\alpha}{\Gamma(\alpha+1)} \right] \varphi(r) + \frac{T^\beta}{\Gamma(\beta+1)} \omega(r) < r,$$

by assumption.

Step VI: *u satisfies the operator inequality $u \le \mathcal{A}u\mathcal{B}u + \mathcal{C}u$.*

Since the hypothesis (A_6) holds, u is a lower solution of (1.1) defined on J. Then,

$$ {}^cD^\alpha \left(\frac{u(t) - I^\beta h(t, u(t))}{f(t, u(t))} \right) \le g(t, u(t)), \tag{3.15}$$

satisfying,

$$u(0) \le \alpha_0, \tag{3.16}$$

for all $t \in J$.

Taking the Riemann-Livoulle Integration of fractional order α from 0 to t on both sides of the above inequality (3.15), we obtain

$$\begin{aligned}
u(t) &\le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, u(s)) ds \\
&+ \left[f(t, u(t)) \right] \left[\frac{\alpha_0}{f(0, \alpha_0)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x(s), \int_0^s k(\tau, x(\tau))\, d\tau\right) ds \right],
\end{aligned} \tag{3.17}$$

for all $t \in J$. This show that u is a solution of the operator inequality $u \le \mathcal{A}u\mathcal{B}u + \mathcal{C}u$.

Thus, the operators $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.9 in view of Remark 2.9 and we apply it to conclude that the operator equation $\mathcal{A}x\mathcal{B}x + \mathcal{C}x = x$ has a solution defined on J. Consequently the integral equation has a solution x^* defined on J which is also a mild solution of the QFDE (1.1) and the sequence $\{x_n\}$ of successive approximations defined by (3.9) converges monotonically to x^*. This completes the proof. □

Remark 3.8. *The conclusion of Theorem 3.7 alsi ramains true if we replace the hypothesis (D_1) with (D_2). The oroof under this new hypothesis is similar with obvious modifications.*

Example 3.9. Given a closed and bounded interval $J=[0,1]$ in $\mathbb{R}$, consider the initial value problem of quadratic fractional nonlinear integro-differential equation,

$$\begin{cases} {}^cD^{1/2}\left[\dfrac{x(t)-I^{3/2}(\arctan x(t))}{f(t,x(t))}\right]=\dfrac{2+\tanh x(t)+\tanh\left(\displaystyle\int_0^t k(s,x(s))\,ds\right)}{16}, \\ \\ x(0)=0, \end{cases} \tag{3.18}$$

for all $t\in J:=[0,1]$, where ${}^cD^{1/2}$ denotes the Caputo fractional derivative of order $1/2$, $k: J\times\mathbb{R}\to\mathbb{R}$ and $f: J\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ are two continuous functions defined by

$$f(t,x)=\begin{cases} 1, & \text{if} \quad x\le 0, \\ 1+\dfrac{x}{1+x}, & \text{if} \quad x>0. \end{cases}$$

and

$$k(t,x)=\begin{cases} 0 & \text{if} \ \ x\le 0, \\ \log(1+x) & \text{if} \ \ x>0 \end{cases}$$

for $t\in J$ and $x\in\mathbb{R}$.

If we take $h(t,x)=\arctan x$, $g(t,x,y)=\dfrac{2+\tanh x+\tanh y}{16}$ and then it is easy to check that the conditions of Theorem 3.7 are satisfied with the lower solution u defined by $u(t)=-\dfrac{4t^{3/2}}{3\sqrt{\pi}}+\dfrac{t^{1/2}}{6\sqrt{\pi}}$, $t\in J$. Therefore, the problem (3.17) has a mild solution defined on $[0,1]$.

References

[1] M. Ammi, E. El Kinani, D. Torres, Existence and uniqueness of solutions to functional integro-differential fractional equations, Electron. J. Diff. Equ., 2012 (2012), Art. ID 103.

[2] M. El Borai, M. Abbas, On some integro-differential equations of fractional orders involving Carathéodory nonlinearities, Int. J. Mod. Math. 2 (2007), 41-52.

[3] B.C. Dhage, Fixed point theorems in ordered Banach algebras and applications, PanAmer. Math. J. 9(4) (1999), 93–102.

[4] B.C. Dhage, Basic results in the theory of hybrid differential equations with mixed perturbations of second type, Funct. Diff. Equ. 19 (2012), 1-20.

[5] B.C. Dhage, Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, Differ. Equ. Appl. 2 (2010), 465–486.

[6] B.C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, Differ. Equ. Appl. 5 (2013), 155-184.

[7] B.C. Dhage, Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations, Tamkang J. Math. 45(4) (2014), 397-426.

[8] B.C. Dhage, Nonlinear $\mathcal{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations, Malaya J. Mat. 3(1)(2015), 62-85.

[9] B.C. Dhage, Operator theoretic techniques in the theory of nonlinear hybrid differential equations, Nonlinear Anal. Forum 20 (2015), 15-31.

[10] B.C. Dhage, A new monotone iteration principle in the theory of nonlinear fractional differential equations, Intern. J. Anal. Appl. 8(2) (2015), 130-143.

[11] B.C. Dhage, S.B. Dhage, Approximating solutions of nonlinear first order ordinary differential equations, Glob. J. Math. Sci. 2 (2014), 25-35.

[12] B.C. Dhage, S.B. Dhage, S.K. Ntouyas, Existence and approximate solutions for fractional differential equations with nonlocal conditions, Journal of Fractional Calculus and Applications 7(1) (2016), 24-35.

[13] B.C. Dhage, S.B. Dhage, S.K. Ntouyas, Dhage iteration method for existence and approximate solutions of nonlinear quadratic fractional differential equations, Journal of Fractional Calculus and Applications, 7(2) (2016), 132-144.

[14] B.C. Dhage, S.K. Ntouyas, Existence results for boundary value problems for fractional hybrid differential inclucions, Topol. Methods Nonlinear Anal. 44 (2014), 229-238.

[15] S. Heikkilä and V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Marcel Dekker inc., New York 1994.

[16] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[17] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[18] S. Sun, Y. Zhao, Z. Han, Y. Li, The existence of solutions for boundary value problem of fractional hybrid differential equations, Commun. Nonlinear Sci. Numer. Simul., 17 (2012), 4961-4967.
[19] Y. Zhao, S. Sun, Z. Han, Q. Li, Theory of fractional hybrid differential equations, Comput. Math. Appl. 62 (2011), 1312-1324.

[1]Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India

[2]School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, nanded, Maharashtra, India

*Corresponding author: bcdhage@email.email