International Journal of Analysis and Applications
ISSN 2291-8639
Volume 11, Number 2 (2016), 168-182
http://www.etamaths.com

ON MULTI-VALUED WEAKLY PICARD OPERATORS IN HAUSDORFF METRIC-LIKE SPACES

ABDELBASSET FELHI[1,2,*]

Abstract. In this paper, we study multi-valued weakly Picard operators on Hausdorff metric-like spaces. Our results generalize some recent results and extend several theorems in the literature. Some examples are presented making effective our results.

1. Introduction and preliminaries

Let (X,d) be a metric space and $CB(X)$ denotes the collection of all nonempty, closed and bounded subsets of X. Also, $CL(X)$ denotes the collection of nonempty closed subsets of X. For $A,B \in CB(X)$, define

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(x,A) := \inf\{d(x,a) : a \in A\}$ is the distance of a point x to the set A. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by d. Throughout the paper, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the set of positive integers, the set of all real numbers and the set of all non-negative real numbers, respectively.

Definition 1.1. ([1]) Let (X,d) be a metric space and $T : X \to CL(X)$ be a multi-valued operator. We say that T is a multi-valued weakly Picard operator (MWP operator) if for all $x \in X$ and $y \in Tx$, there exists a sequence $\{x_n\}$ such that:

(i) $x_0 = x, x_1 = y$;
(ii) $x_{n+1} \in Tx_n$ for all $n = 0,1,2,\ldots$;
(iii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of T.

The theory of MWP operators is studied by several authors (see for instance [1, 2]). In 2008 Suzuki [3] generalizes the Banach contraction principle by introducing a new type of mapping. Very recently, Jleli et al. [4] established Kikkawa-Suzuki type fixed point theorems for a new type of generalized contractive conditions on partial Hausdorff metric spaces. The purpose of this paper is to discuss multi-valued weakly Picard operators on partial Hausdorff metric spaces and on Hausdorff metric-like spaces. We will establish the above fixed point theorems for a new type of generalized contractive conditions which generalizes that of Jleli et al.

We recall that the study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [18] who proved the following theorem.

Theorem 1.2. *([18]) Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued mapping satisfying $H(Tx,Ty) \le kd(x,y)$ for all $x,y \in X$ and for some k in $[0,1)$. Then there exists $x \in X$ such that $x \in Tx$.*

We recall that the notion of partial metric spaces was introduced by Matthews [8] in 1994 as a part to study the denotational semantics of dataflow networks which play an important role in constructing models in the theory of computation. Moreover, the notion of metric-like spaces has been discovered by Amini-Harandi [12] which is an interesting generalization of the notion of partial metric spaces. For more fixed point results on metric-like spaces, see [7], [10], [11], [13], [15], [16], [17], [19], [20], [21], [22].

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Hausdorff metric-like: multi-valued operator; partial metric space; metric-like space.

Note that, every partial metric space is a metric-like space but the converse is not true in general. In what follows, we recall some definitions and results we will need in the sequel.

Definition 1.3. ([8]) A partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$

(PM1) $p(x, x) = p(x, y) = p(y, y)$, then $x = y$;
(PM2) $p(x, x) \le p(x, y)$;
(PM3) $p(x, y) = p(y, x)$;
(PM4) $p(x, z) + p(y, y) \le p(x, y) + p(y, z)$.

The pair (X, p) is then called a partial metric space (PMS).

According to [8], each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open $p-$balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Following [8], several topological concepts can be defined as follows. A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$ and is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite. Moreover, a partial metric space (X, p) is called to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$. It is known [8] that if p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

for all $x, y \in X$, is a metric on X.

Note that if a sequence converges in a partial metric space (X, p) with respect to τ_{p^s}, then it converges with respect to τ_p.

Also, a sequence $\{x_n\}$ in a partial metric space (X, p) is Cauchy if and only if it is a Cauchy sequence in the metric space (X, p^s). Consequently, a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Moreover, if $\{x_n\}$ is a sequence in a partial metric space (X, p) and $x \in X$, one has that

$$\lim_{n\to\infty} p^s(x_n, x) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

We have the following lemmas.

Lemma 1.4. *Let (X, p) be a partial metric space. Then,*
(1) *if $p(x, y) = 0$ then, $x = y$,*
(2) *if $x \ne y$ then, $p(x, y) > 0$.*

Following [9], let (X, p) be a partial metric space and $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X, p), induced by the partial metric p.

For $A, B \in CB^p(X)$ and $x \in X$, we define

$$p(x, A) = \inf\{p(x, a) : a \in A\},\ H_p(A, B) = \max\{\sup_{a\in A} p(a, B), \sup_{b\in B} p(b, A)\}.$$

Lemma 1.5. *([5]) Let (X, p) be a partial metric space and A any nonempty set in (X, p), then $a \in \overline{A}$ if and only if $p(a, A) = p(a, a)$, where $\overline{A}$ denotes the closure of A with respect to the partial metric p.*

Proposition 1.6. *([9]) Let (X, p) be a partial metric space. For all $A, B, C \in CB^p(X)$, we have*

(h1) $H_p(A, A) \le H_p(A, B)$;
(h2) $H_p(B, A) = H_p(A, B)$;
(h3) $H_p(A, B) \le H_p(A, C) + H_p(C, B) - \inf_{c\in C} p(c, c)$;
(h4) $H_p(A, B) = 0 \Rightarrow A = B$.

Definition 1.7. Let X be a nonempty set. A function $\sigma : X \times X \to \mathbb{R}^+$ is said to be a metric-like (dislocated metric) on X if for any $x, y, z \in X$, the following conditions hold:

(P_1) $\sigma(x, y) = 0 \Longrightarrow x = y$;
(P_2) $\sigma(x, y) = \sigma(y, x)$;
(P_3) $\sigma(x, z) \le \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is then called a metric-like (dislocated metric) space.

In the following example, we give a metric-like which is neither a metric nor a partial metric.

Example 1.8. Let $X=\{0,1,2\}$ and $\sigma : X\times X \to \mathbb{R}^+$ defined by

$$\sigma(0,0)=\sigma(1,1)=0,\ \sigma(2,2)=3,\ \sigma(0,1)=1,\ \sigma(0,2)=\sigma(1,2)=2,$$

and $\sigma(x,y)=\sigma(y,x)$ for all $x\in X$. Then, (X,σ) is a metric-like space. Note that σ is nor a metric as, $\sigma(2,2)>0$ and not a partial metric on X as, $\sigma(2,2)>\sigma(0,2)$.

Each metric-like σ on X generates a T_0 topology τ_σ on X which has as a base the family open σ-balls $\{B_\sigma(x,\varepsilon) : x\in X, \varepsilon>0\}$, where $B_\sigma(x,\varepsilon)=\{y\in X : |\sigma(x,y)-\sigma(x,x)|<\varepsilon\}$, for all $x\in X$ and $\varepsilon>0$.

Observe that a sequence $\{x_n\}$ in a metric-like space (X,σ) converges to a point $x\in X$, with respect to τ_σ, if and only if $\sigma(x,x)=\lim\limits_{n\to\infty}\sigma(x,x_n)$.

Definition 1.9. Let (X,σ) be a metric-like space.

(a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim\limits_{n,m\to\infty}\sigma(x_n,x_m)$ exists and is finite.

(b) (X,σ) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_σ to a point $x\in X$ such that $\lim\limits_{n\to\infty}\sigma(x,x_n)=\sigma(x,x)=\lim\limits_{n,m\to\infty}\sigma(x_n,x_m)$.

We have the following trivial inequality:

$$\sigma(x,x)\le 2\sigma(x,y)\quad \text{for all } x,y\in X. \tag{1.1}$$

Very recently, Aydi et al. [6] introduced the concept of Hausdorff metric-like. Let $CB^\sigma(X)$ be the family of all nonempty, closed and bounded subsets of the metric-like space (X,σ), induced by the metric-like σ. Note that the boundedness is given as follows: A is a bounded subset in (X,σ) if there exist $x_0\in X$ and $M\ge 0$ such that for all $a\in A$, we have $a\in B_\sigma(x_0,M)$, that is,

$$|\sigma(x_0,a)-\sigma(a,a)|<M.$$

The Closeness is taken in (X,τ_σ) (where τ_σ is the topology induced by σ).

For $A,B\in CB^\sigma(X)$ and $x\in X$, define

$$\begin{aligned}\sigma(x,A) &= \inf\{\sigma(x,a),\ a\in A\},\ \ \delta_\sigma(A,B)=\sup\{\sigma(a,B) : a\in A\} \quad \text{and}\\ \delta_\sigma(B,A) &= \sup\{\sigma(b,A) : b\in B\}.\end{aligned}$$

We have the the following useful lemmas.

Lemma 1.10. [6]
Let (X,σ) be a metric-like space and A any nonempty set in (X,σ), then

$$\text{if } \sigma(a,A)=0, \quad \text{then } a\in\bar{A},$$

where $\overline{A}$ denotes the closure of A with respect to the metric-like σ. Also, if $\{x_n\}$ is a sequence in (X,σ) that is τ_σ-convergent to $x\in X$, then

$$\lim_{n\to\infty}|\sigma(x_n,A)-\sigma(x,A)|=\sigma(x,x).$$

Lemma 1.11. *Let $A,B\in CB^\sigma(X)$ and $a\in A$. Suppose that $\sigma(a,B)>0$. Then, for each $h>1$, there exists $b=b(a)\in B$ such that $\sigma(a,b)<h\sigma(a,B)$.*

Proof. We argue by contradiction, that is, there exists $h>1$, such that for all $b\in B$, there is $\sigma(a,b)\ge h\sigma(a,B)$. Then, $\sigma(a,B)=\inf\{\sigma(a,B) : b\in B\}\ge h\sigma(a,B)$. Hence, $h\le 1$, which is a contradiction. □

Let (X,σ) be a metric-like space. For $A,B\in CB^\sigma(X)$, define

$$H_\sigma(A,B)=\max\{\delta_\sigma(A,B),\delta_\sigma(B,A)\}.$$

We have also some properties of $H_\sigma : CB^\sigma(X)\times CB^\sigma(X)\to[0,\infty)$.

Proposition 1.12. [6] *Let (X,σ) be a metric-like space. For any $A,B,C\in CB^{\sigma}(X)$, we have the following:*

$$\begin{aligned}(i) &: \quad H_\sigma(A,A)=\delta_\sigma(A,A)=\sup\{\sigma(a,A):a\in A\};\\ (ii) &: \quad H_\sigma(A,B)=H_\sigma(B,A);\\ (iii) &: \quad H_\sigma(A,B)=0 \text{ implies that } A=B;\\ (iv) &: \quad H_\sigma(A,B)\le H_\sigma(A,C)+H_\sigma(C,B).\end{aligned}$$

The mapping $H_\sigma : CB^\sigma(X)\times CB^\sigma(X)\to[0,+\infty)$ is called a Hausdorff metric-like induced by σ. Note that each partial hausdorff metric is a Hausdorff metric-like but the converse is not true in general as it is clear from the following example.

Example 1.13. Going back to Example 1.8, taking $A=\{2\}$, $B=\{0\}$ we have $H_\sigma(A,A)=\sigma(2,2)=3>2=\sigma(0,2)=H_\sigma(A,B)$.

We denote by Ψ the class of all functions $\psi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying
(ψ_1) ψ is nondecreasing;
(ψ_2) $\sum_n \psi^n(t)<\infty$ for each $t\in\mathbb{R}^+$, where ψ^n is the $n-$th iterate of ψ.

Also, we denote by Φ the class of all functions $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying
(φ_1) φ is nondecreasing;
(φ_2) $t\le\varphi(t)$ for each $t\in\mathbb{R}^+$.

Lemma 1.14. *(i) If $\psi\in\Psi$, then $\psi(t)<t$ for any $t>0$ and $\psi(0)=0$.*
(ii) If $\varphi\in\Phi$, then $t\le\varphi^n(t)$ for all $n\in\mathbb{N}\cup\{0\}$ and for any $t\in\mathbb{R}^+$.

We have the following useful lemma.

Lemma 1.15. *Let (X,σ) be a metric-like space, $B\in CB^\sigma(X)$ and $c>0$. If $a\in X$ and $\sigma(a,B)<c$ then there exists $b=b(a)\in B$ such that $\sigma(a,b)<c$.*

Proof. We argue by contradiction, that is, $\sigma(a,b)\ge c$ for all $b\in B$, then $\sigma(a,B)=\inf\{\sigma(a,b):b\in B\}\ge c$, which is a contradiction. Hence there exists $b=b(a)\in B$ such that $\sigma(a,b)<c$. □

2. Main Results

In this section, we give some fixed point results on metric-like spaces first and next we give some fixed point results on partial metric spaces.

Now, we need the following definition.

Definition 2.1. Let (X,σ) be a metric-like space. A multi-valued mapping $T:X\to CB^\sigma(X)$ is said to be $(\varphi,\psi)-$contractive multi-valued operator if there exist $\varphi\in\Phi$ and $\psi\in\Psi$ such that

$$\sigma(y,Tx)\le\varphi(\sigma(y,x))\Rightarrow H_\sigma(Tx,Ty)\le\psi(M_\sigma(x,y)) \tag{2.1}$$

for all $x,y\in X$, where

$$M_\sigma(x,y)=\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{4}[\sigma(x,Ty)+\sigma(Tx,y)]\}.$$

Now, we state and prove our first main result.

Theorem 2.2. *Let (X,σ) be a complete metric-like space and $T:X\to CB^\sigma(X)$ be $(\varphi,\psi)-$contractive multi-valued operator.*

If $2t\le\varphi(t)$ for each $t\in\mathbb{R}^+$, then T is an MWP operator.

Proof. Let $x_0\in X$ and $x_1\in Tx_0$. Let c a given real number such that $\sigma(x_0,x_1)<c$.

Clearly, if $x_1=x_0$ or $x_1\in Tx_1$, we conclude that x_1 is a fixed point of T and so the proof is finished. Now, we assume that $x_1\ne x_0$ and $x_1\notin Tx_1$. So then, $\sigma(x_0,x_1)>0$ and $\sigma(x_1,Tx_1)>0$.

Since $x_1\in Tx_0$ and $2t\le\varphi(t)$, we get

$$\sigma(x_1,Tx_0)\le\sigma(x_1,x_1)\le2\sigma(x_1,x_0)\le\varphi(\sigma(x_1,x_0)).$$

Hence by (2.1) and triangular inequality, we have

$$\begin{aligned} 0<\sigma(x_1,Tx_1) &\leq H_\sigma(Tx_0,Tx_1)\leq \psi(M_\sigma(x_0,x_1)) \\ &\leq \psi(\max\{\sigma(x_0,x_1),\sigma(x_0,Tx_0),\sigma(x_1,Tx_1),\frac{1}{4}[\sigma(x_0,Tx_1)+\sigma(x_1,Tx_0)]\}) \\ &\leq \psi(\max\{\sigma(x_0,x_1),\sigma(x_0,x_1),\sigma(x_1,Tx_1),\frac{1}{4}[\sigma(x_0,Tx_1)+\sigma(x_1,x_1)]\}) \\ &\leq \psi(\max\{\sigma(x_0,x_1),\sigma(x_0,x_1),\sigma(x_1,Tx_1),\frac{1}{4}[\sigma(x_1,Tx_1)+3\sigma(x_0,x_1)]\}) \\ &= \psi(\max\{\sigma(x_0,x_1),\sigma(x_1,Tx_1)\}). \end{aligned}$$

If $\max\{\sigma(x_0,x_1),\sigma(x_1,Tx_1)\}=\sigma(x_1,Tx_1)$, then we obtain

$$0<\sigma(x_1,Tx_1)\leq \psi(\sigma(x_1,Tx_1))<\sigma(x_1,Tx_1)$$

wish is a contradiction. Then

$$0<\sigma(x_1,Tx_1)\leq \psi(\sigma(x_0,x_1))<\psi(c).$$

Thus, by Lemma 1.15, there exist $x_2\in Tx_1$ such that

$$\sigma(x_1,x_2)<\psi(c). \tag{2.2}$$

If $x_1=x_2$ or $x_2\in Tx_2$, we conclude that x_2 is a fixed point of T and so the proof is finished. Now, we assume that $x_2\neq x_1$ and $x_2\notin Tx_2$. Then we have

$$\sigma(x_2,Tx_1)\leq \sigma(x_2,x_2)\leq 2\sigma(x_2,x_1)\leq \varphi(\sigma(x_2,x_1)).$$

Hence by (2.1), triangular inequality and (2.2), we have

$$\begin{aligned} 0<\sigma(x_2,Tx_2) &\leq H_\sigma(Tx_1,Tx_2)\leq \psi(M_\sigma(x_1,x_2))\leq \psi(\max\{\sigma(x_1,x_2),\sigma(x_2,Tx_2)\}) \\ &= \psi(\sigma(x_1,x_2))<\psi^2(c). \end{aligned}$$

Then, by Lemma 1.15, there exist $x_3\in Tx_2$ such that

$$\sigma(x_2,x_3)<\psi^2(c). \tag{2.3}$$

Continuing in this fashion, we construct a sequence $\{x_n\}$ in X such that for all $n\in\mathbb{N}$

(i) $x_n\notin Tx_n,\ x_n\neq x_{n+1},\ x_{n+1}\in Tx_n$;

(ii)

$$\sigma(x_n,x_{n+1})\leq \psi^n(c). \tag{2.4}$$

Now, for $m>n$, we have

$$\sigma(x_n,x_m) \leq \sum_{i=n}^{m-1}\sigma(x_i,x_{i+1})\leq \sum_{i=n}^{m-1}\psi^i(c)\leq \sum_{i=n}^{\infty}\psi^i(c)\to 0 \text{ as } n\to\infty.$$

Thus,

$$\lim_{n,m\to\infty}\sigma(x_n,x_m)=0. \tag{2.5}$$

Hence, $\{x_n\}$ is $\sigma-$Cauchy. Moreover since (X,σ) is complete, it follows there exists $\nu\in X$ such that

$$\lim_{n\to\infty}\sigma(x_n,\nu)=\sigma(\nu,\nu)=\lim_{n,m\to\infty}\sigma(x_n,x_m)=0. \tag{2.6}$$

We will show that ν is a fixed point of T. First, we should prove that there exits a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\sigma(\nu,Tx_{n(k)})\leq \varphi(\sigma(\nu,x_{n(k)})),\quad \text{for all } k=0,1,2,\dots \tag{2.7}$$

Arguing by contradiction, that is, there exists $N\in\mathbb{N}$ such that $\sigma(\nu,Tx_n)>\varphi(\sigma(\nu,x_n))$ for all $n\geq N$. Since $x_{n+1}\in Tx_n$, it follows that $\sigma(\nu,x_{n+1})>\varphi(\sigma(\nu,x_n))$ for all $n\geq N$. Having φ nondecreasing, so by induction we get

$$\sigma(\nu,x_{n+m})>\varphi^m(\sigma(\nu,x_n)),\quad \text{for all } n\geq N \text{ and } m=1,2,3,\dots \tag{2.8}$$

Now, for all $n \geq N$ and $m \in \mathbb{N}$, we have

$$\sigma(x_n, x_{n+m}) \leq \sum_{i=n}^{n+m-1} \sigma(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^i(c) \leq \sum_{i=n}^{\infty} \psi^i(c).$$

Then for all $n \geq N$ and $m \in \mathbb{N}$, we obtain

$$\sigma(\nu, x_n) \leq \sigma(\nu, x_{n+m}) + \sigma(x_{n+m}, x_n) \leq \sigma(\nu, x_{n+m}) + \sum_{i=n}^{\infty} \psi^i(c).$$

Passing to the limit as $m \to \infty$, we get

$$\sigma(\nu, x_n) \leq \sum_{i=n}^{\infty} \psi^i(c).$$

This implies that for all $n \geq N$ and $m \in \mathbb{N}$,

$$\sigma(\nu, x_{n+m}) \leq \sum_{i=n+m}^{\infty} \psi^i(c). \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$\sigma(\nu, x_n) \leq \varphi^m(\sigma(\nu, x_n)) < \sigma(\nu, x_{n+m}) \leq \sum_{i=n+m}^{\infty} \psi^i(c).$$

Then for all $n \geq N$ and $m \in \mathbb{N}$, we obtain

$$\sigma(\nu, x_n) < \sum_{i=n+m}^{\infty} \psi^i(c). \tag{2.10}$$

Letting $m \to \infty$ in (2.10), we get $\sigma(\nu, x_n) = 0$ for all $n \geq N$ and so, $\sigma(\nu, x_{n+m}) = 0$ for all $n \geq N$ and $m \in \mathbb{N}$. Using (2.8), we have $0 \leq \varphi^m(0) < 0$, which is a contradiction. Therefore, (2.7) holds.

Now, we will show that $\sigma(\nu, T\nu) = 0$. Suppose in the contrary, that is $\sigma(\nu, T\nu) > 0$.

By (2.1) and (2.7), we have for all $k \in \mathbb{N}$

$$\begin{aligned}
\sigma(\nu, T\nu) &\leq \sigma(\nu, x_{n(k)+1}) + \sigma(x_{n(k)+1}, T\nu) \leq \sigma(\nu, x_{n(k)+1}) + H_\sigma(Tx_{n(k)}, T\nu) \\
&\leq \sigma(\nu, x_{n(k)+1}) + \psi(M_\sigma(x_{n(k)}, \nu)) \\
&\leq \sigma(\nu, x_{n(k)+1}) \\
&+ \psi(\max\{\sigma(x_{n(k)}, \nu), \sigma(x_{n(k)}, Tx_{n(k)}), \sigma(\nu, T\nu), \frac{1}{4}[\sigma(x_{n(k)}, T\nu) + \sigma(\nu, Tx_{n(k)})]\}) \\
&\leq \sigma(\nu, x_{n(k)+1}) \\
&+ \psi(\max\{\sigma(x_{n(k)}, \nu), \sigma(x_{n(k)}, x_{n(k)+1}), \sigma(\nu, T\nu), \frac{1}{4}[\sigma(x_{n(k)}, T\nu) + \sigma(\nu, x_{n(k)+1})]\}).
\end{aligned}$$

We know that

$$\lim_{k\to\infty} \sigma(x_{n(k)}, \nu) = \lim_{k\to\infty} \sigma(x_{n(k)}, x_{n(k)+1}) = \lim_{k\to\infty} \sigma(x_{n(k)+1}, \nu) = 0, \ \lim_{k\to\infty} \sigma(x_{n(k)}, T\nu) = \sigma(\nu, T\nu).$$

Then there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$\max\{\sigma(x_{n(k)}, \nu), \sigma(x_{n(k)}, x_{n(k)+1}), \sigma(\nu, T\nu), \frac{1}{4}[\sigma(x_{n(k)}, T\nu) + \sigma(\nu, x_{n(k)+1})]\}) = \sigma(\nu, T\nu).$$

It follows that for all $k \geq N$

$$0 < \sigma(\nu, T\nu) \quad \leq \quad \sigma(\nu, x_{n(k)+1}) + \psi(\sigma(\nu, T\nu)).$$

Passing to the limit as $k \to \infty$, we get

$$0 < \sigma(\nu, T\nu) \leq \psi(\sigma(\nu, T\nu)) < \sigma(\nu, T\nu)$$

which is a contradiction. Hence $\sigma(\nu, T\nu) = 0$ and so, by Lemma 1.10 we have $\nu \in \overline{T\nu} = T\nu$, that is ν is a fixed point of T. □

We give an example to illustrate the utility of Theorem 2.2.

Example 2.3. Let $X = \{0,1,2\}$ and $\sigma : X \times X \to \mathbb{R}^+$ defined by:

$$\sigma(0,0) = 0,\ \sigma(1,1) = 3,\ \sigma(2,2) = 1$$
$$\sigma(0,1) = \sigma(1,0) = 7, \qquad \sigma(0,2) = \sigma(2,0) = 3, \qquad \sigma(1,2) = \sigma(2,1) = 4.$$

Then (X,σ) is a complete metric-like space. Note that $\sigma$is not a partial metric on X because $\sigma(0,1) \nleq \sigma(2,0) + \sigma(2,1) - \sigma(2,2)$.
Define the map $T : X \to CB^\sigma(X)$ by

$$T0 = T2 = \{0\} \quad \text{and } T1 = \{0,2\}.$$

Note that Tx is bounded and closed for all $x \in X$ in metric-like space (X,σ). Take $\varphi(t) = st$ with $s \geq 7$ and $\psi(t) = rt$ with $r \in [\frac{3}{4}, 1)$.

It is easy tho show that

$$\begin{aligned}\max\{\sigma(y,Tx),\, x,y \in X\} = \sigma(1,0) = 7 &\leq 7\min\{\sigma(y,x),\, x,y \in X, (x,y) \neq (0,0)\}\\ &\leq \varphi(\min\{\sigma(y,x),\, x,y \in X, (x,y) \neq (0,0)\}).\end{aligned}$$

This implies that, for all $x,y \in X$ with $(x,y) \neq (0,0)$

$$\sigma(y,Tx) \leq \varphi(\sigma(y,x)).$$

Now, we shall show that for all $x,y \in X$ with $(x,y) \neq (0,0)$

$$H_\sigma(Tx,Ty) \leq \psi(M_\sigma(x,y)). \tag{2.11}$$

For this, we consider the following cases:
case1 : $x,y \in \{0,2\}$. We have

$$H_\sigma(Tx,Ty) = \sigma(0,0) = 0 \leq \psi(M_\sigma(x,y)).$$

case2 : $x \in \{0,2\}$, $y = 1$. We have

$$\begin{aligned}H_\sigma(Tx,Ty) = H_\sigma(\{0\},\{0,2\}) &= max\{\sigma(0,\{0,2\}), max\{\sigma(0,0),\sigma(0,2)\}\}\\ &= max\{0,3\} = 3 \leq \frac{3}{4}\sigma(x,y) \leq \psi(M_\sigma(x,y)).\end{aligned}$$

case3 : $x = y = 1$. We have

$$\begin{aligned}H_\sigma(Tx,Ty) = H_\sigma(\{0,2\},\{0,2\}) &= max\{\sigma(0,\{0,2\}),\sigma(2,\{0,2\})\}\\ &= min\{\sigma(0,2),\sigma(2,2)\} = 1 \leq \frac{3}{4}\sigma(1,1) \leq \psi(M_\sigma(x,y)).\end{aligned}$$

Note that (2.11) is also true for $(x,y) = (0,0)$. Then, all the required hypotheses of Theorem 2.2 are satisfied. Here, $x = 0$ is the unique fixed point of T

We state the following corollaries as consequences of Theorem 2.2.

Corollary 2.4. *Let (X,σ) be a complete metric-like space and $T : X \to CB^\sigma(X)$ be a multi-valued mapping. Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that, for all $x,y \in X$*

$$H_\sigma(Tx,Ty) \leq \psi(M_\sigma(x,y)) - \varphi(\sigma(y,x)) + \sigma(y,Tx), \tag{2.12}$$

where $M_\sigma(x,y) = \max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{4}[\sigma(x,Ty) + \sigma(Tx,y)]\}$.

If $2t \leq \varphi(t)$ for each $t \in \mathbb{R}^+$, then T is an MWP operator.

Proof. Let $x,y \in X$ such that $\sigma(y,Tx) \leq \varphi(\sigma(y,x))$. Then, if (2.12) holds, we have

$$H_\sigma(Tx,Ty) \leq \psi(M_\sigma(x,y)) - \varphi(\sigma(y,x)) + \sigma(y,Tx) \leq \psi(M_\sigma(x,y)).$$

Thus, the proof is concluded by Theorem 2.2. □

Corollary 2.5. *Let (X,σ) be a complete metric-like space and $T : X \to CB^\sigma(X)$ be a multi-valued mapping. Assume that there exist $r \in [0,1)$ and $s \geq 2$ such that, for all $x,y \in X$*

$$\sigma(y,Tx) \leq s\sigma(y,x) \Rightarrow H_\sigma(Tx,Ty) \leq rM_\sigma(x,y),$$

where $M_\sigma(x,y) = \max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{4}[\sigma(x,Ty) + \sigma(Tx,y)]\}$.

Then T is an MWP operator.

Proof. It suffice to take $\varphi(t) = st$ and $\psi(t) = rt$ in Theorem 2.2. □

Corollary 2.6. *Let (X,σ) be a complete metric-like space and $T : X \to CB^{\sigma}(X)$ be a multi-valued mapping. Assume that there exist $r \in [0,1)$ and $s \geq 2$ such that, for all $x, y \in X$*

$$\sigma(y,Tx) \leq s\sigma(y,x) \Rightarrow H_{\sigma}(Tx,Ty) \leq r\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty)\}.$$

Then T is an MWP operator.

Corollary 2.7. *Let (X,σ) be a complete metric-like space and $T : X \to CB^{\sigma}(X)$ be a multi-valued mapping. Assume that there exist $r \in [0,1)$ and $s \geq 2$ such that, for all $x, y \in X$*

$$\sigma(y,Tx) \leq s\sigma(y,x) \Rightarrow H_{\sigma}(Tx,Ty) \leq \frac{r}{3}\{\sigma(x,y)+\sigma(x,Tx)+\sigma(y,Ty)\}.$$

Then T is an MWP operator.

Corollary 2.8. [6] *Let (X,σ) be a complete metric-like space. If $T : X \to CB^{\sigma}(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_{\sigma}(Tx,Ty) \leq k\,M(x,y), \tag{2.13}$$

where $k \in [0,1)$ and

$$M(x,y) = \max\left\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{4}\left(\sigma(x,Ty)+\sigma(y,Tx)\right)\right\}.$$

Then T has a fixed point.

Proof. Let $\varphi(t) = 2t$ and $\psi(t) = kt$. Then, if (2.13) holds, we have

$$H_{\sigma}(Tx,Ty) \leq \psi(M(x,y)),$$

for all $x, y \in X$ satisfying $\sigma(y,Tx) \leq 2\sigma(y,x)$. Thus, the proof is concluded by Theorem 2.2. □

If T is a single-valued mapping, we deduce the following results.

Corollary 2.9. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be a mapping. Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that, for all $x, y \in X$*

$$\sigma(y,Tx) \leq \varphi(\sigma(y,x)) \Rightarrow \sigma(Tx,Ty) \leq \psi(M_{\sigma}(x,y)),$$

where $M_{\sigma}(x,y) = \max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{4}[\sigma(x,Ty)+\sigma(Tx,y)]\}$.
If $2t \leq \varphi(t)$ for each $t \in \mathbb{R}^+$ and if $\psi(2t) < t$ for each $t > 0$, then T has a unique fixed point.

Proof. The existence follows immediately from Theorem 2.2. Thus, we need to prove uniqueness of fixed point. We assume that there exist $x, y \in X$ such that $x = Tx$ and $y = Ty$ with $x \neq y$.
Since $\sigma(y,Tx) = \sigma(y,x) \leq \varphi(\sigma(y,x))$, then by (2.1) and since $\psi(2t) < t$, we get

$$\begin{aligned}0 < \sigma(x,y) = \sigma(Tx,Ty) &\leq \psi(\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{4}[\sigma(x,Ty)+\sigma(Tx,y)]\})\\ &= \psi(\max\{\sigma(x,y),\sigma(x,x),\sigma(y,y),\frac{1}{2}\sigma(x,y)\})\\ &\leq \psi(2\sigma(x,y)) < \sigma(x,y).\end{aligned}$$

which is a contradiction. Hence $x = y$, so the uniqueness of the fixed point of T.

□

Corollary 2.10. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be a mapping. Assume that there exist $r \in [0,\frac{1}{2})$ and $s \geq 2$ such that, for all $x, y \in X$*

$$\sigma(y,Tx) \leq s(\sigma(y,x)) \Rightarrow \sigma(Tx,Ty) \leq r(M_{\sigma}(x,y)),$$

where $M_{\sigma}(x,y) = \max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{4}[\sigma(x,Ty)+\sigma(Tx,y)]\}$.
Then T has a unique fixed point.

Now, we need the following definition.

Definition 2.11. Let (X,σ) be a metric-like space. A multi-valued mapping $T: X \to CB^{\sigma}(X)$ is said to be $(r,s)-$contractive multi-valued operator if there exist $r, s \in [0,1)$, such that

$$\frac{1}{1+r}\sigma(x,Tx) \leq \sigma(y,x) \leq \frac{1}{1-s}\sigma(x,Tx) \Rightarrow H_{\sigma}(Tx,Ty) \leq rM_{\sigma}(x,y) \tag{2.14}$$

for all $x, y \in X$, where

$$M_{\sigma}(x,y) = \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{1}{4}[\sigma(x,Ty) + \sigma(Tx,y)]\}.$$

We give the following result.

Theorem 2.12. *Let (X,σ) be a complete metric-like space and $T: X \to CB^{\sigma}(X)$ be $(r,s)-$contractive multi-valued operator with $r < s$. Then T is an MWP operator.*

Proof. Let r_1 be a real number such that $0 \leq r \leq r_1 < s$. Let $x_0 \in X$. Clearly, if $x_0 \in Tx_0$, then x_0 is a fixed point of T and so, the proof is finished. Now, we assume that $x_0 \notin Tx_0$. Then $\sigma(x_0, Tx_0) > 0$. By Lemma 1.11, there exists $x_1 \in Tx_0$ such that

$$\sigma(x_0, x_1) \leq \frac{1-r_1}{1-s}\sigma(x_0, Tx_0).$$

If $x_1 \in Tx_1$, then x_1 is a fixed point of T and also, the proof is finished. Now, we assume that $x_1 \notin Tx_1$. Then $\sigma(x_1, Tx_1) > 0$. Since

$$\frac{1}{1+r}\sigma(x_0, Tx_0) \leq \sigma(x_0, x_1)) \leq \frac{1-r_1}{1-s}\sigma(x_0, Tx_0),$$

then, by (2.14), we have

$$\begin{aligned}
&\sigma(x_1, Tx_1) \leq H_{\sigma}(Tx_0, Tx_1) \leq \\
&r\max\{\sigma(x_0,x_1), \sigma(x_0,Tx_0), \sigma(x_1,Tx_1), \frac{1}{4}[\sigma(x_0,Tx_1) + \sigma(x_1,Tx_0)]\} \\
&\leq r\max\{\sigma(x_0,x_1), \sigma(x_0,x_1), \sigma(x_1,Tx_1), \frac{1}{4}[\sigma(x_1,Tx_1) + 3\sigma(x_0,x_1)]\} \\
&\leq r\max\{\sigma(x_0,x_1), \sigma(x_1,Tx_1)\}.
\end{aligned}$$

If $\max\{\sigma(x_0,x_1), \sigma(x_1,Tx_1)\} = \sigma(x_1,Tx_1)$, then we obtain $\sigma(x_1,Tx_1) \leq r\sigma(x_1,Tx_1) < \sigma(x_1,Tx_1)$, which is a contradiction. Thus, we get

$$\sigma(x_1, Tx_1) \leq r\sigma(x_0, x_1).$$

By Lemma 1.11, there exists $x_2 \in Tx_1$ such that

$$\sigma(x_1,x_2) \leq \frac{r_1}{r}\sigma(x_1,Tx_1) \quad \text{and } \sigma(x_1,x_2) \leq \frac{1-r_1}{1-s}\sigma(x_1,Tx_1).$$

This implies that

$$\sigma(x_1,x_2) \leq r_1\sigma(x_0,x_1) \quad \text{and } \sigma(x_1,x_2) \leq \frac{1-r_1}{1-s}\sigma(x_1,Tx_1).$$

It follows that

$$\frac{1}{1+r}\sigma(x_1,Tx_1) \leq \sigma(x_1,Tx_2) \leq \frac{1}{1-s}\sigma(x_1,Tx_1).$$

Then, by (2.14), we get $\sigma(x_2,Tx_2) \leq r\sigma(x_1,x_2)$. Continuing this process, we construct a sequence $\{x_n\}$ in X such that

(i) $x_{n+1} \in Tx_n$;
(ii) $\sigma(x_n, Tx_n) \leq r\sigma(x_{n-1}, x_n)$;
(iii) $\sigma(x_n, x_{n+1}) \leq r_1\sigma(x_{n-1}, x_n)$;
(iv) $\sigma(x_n, x_{n+1}) \leq \frac{1-r_1}{1-s}\sigma(x_n, Tx_n)$

for all $n = 1, 2, \ldots$

Since $\sigma(x_n, x_{n+1}) \le r_1 \sigma(x_{n-1}, x_n)$, by induction we obtain

$$\sigma(x_n, x_{n+1}) \le r_1^n \sigma(x_0, x_1) \quad \text{for all } n = 1, 2, \ldots$$

Now, for $m > n$, we have

$$\sigma(x_n, x_m) \le \sum_{i=n}^{m-1} \sigma(x_i, x_{i+1}) \le \sigma(x_0, x_1) \sum_{i=n}^{m-1} r_1^i \le \sigma(x_0, x_1) \sum_{i=n}^{\infty} r_1^i \to 0 \text{ as } n \to \infty.$$

Thus,

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.15}$$

Hence, $\{x_n\}$ is $\sigma-$Cauchy. Moreover since (X, σ) is complete, it follows there exists $z \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.16}$$

For all $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \sigma(x_n, x_{n+m}) &\le \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \ldots + \sigma(x_{n+m-1}, x_{n+m}) \\ &\le [1 + r_1 + r_1^2 + \ldots + r_1^{m-1}]\sigma(x_n, x_{n+1}) = \frac{1 - r_1^m}{1 - r_1}\sigma(x_n, x_{n+1}). \end{aligned}$$

It follows that for all $m, n \in \mathbb{N}$

$$\sigma(x_n, z) \le \sigma(x_n, x_{n+m}) + \sigma(x_{n+m}, z) \le \sigma(x_{n+m}, z) + \frac{1 - r_1^m}{1 - r_1}\sigma(x_n, x_{n+1}).$$

Passing to limit as $m \to \infty$, we get for all $n \in \mathbb{N}$

$$\sigma(x_n, z) \le \frac{1}{1 - r_1}\sigma(x_n, x_{n+1}) \le \frac{1}{1 - r_1} \cdot \frac{1 - r_1}{1 - s}\sigma(x_n, Tx_n) = \frac{1}{1 - s}\sigma(x_n, Tx_n).$$

Thus, we have for all $n \in \mathbb{N}$

$$\sigma(x_n, z) \le \frac{1}{1 - s}\sigma(x_n, Tx_n). \tag{2.17}$$

Now, we assume that there exists $N \in \mathbb{N}$ such that

$$\frac{1}{1 + r}\sigma(x_n, Tx_n) > \sigma(x_n, z)$$

for all $n \ge N$. Then we have

$$\begin{aligned} \sigma(x_n, x_{n+1}) &\le \sigma(x_n, z) + \sigma(z, x_{n+1}) < \frac{1}{1 + r}[\sigma(x_n, Tx_n) + \sigma(x_{n+1}, Tx_{n+1})] \\ &< \frac{1}{1 + r}[\sigma(x_n, x_{n+1}) + r\sigma(x_n, x_{n+1})] = \sigma(x_n, x_{n+1}). \end{aligned}$$

which is a contradiction. Thus, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\frac{1}{1 + r}\sigma(x_{n(k)}, Tx_{n(k)}) \le \sigma(x_{n(k)}, z) \tag{2.18}$$

for all $k \in \mathbb{N}$. Now, we should show that z is a fixed point of T.

Using (2.17), (2.18) and (2.14), we have for all $k \in \mathbb{N}$

$$\begin{aligned} &\sigma(x_{n(k)+1}, Tz) \le H_\sigma(Tx_{n(k)}, Tz) \le \\ &r \max\{\sigma(x_{n(k)}, z), \sigma(x_{n(k)}, Tx_{n(k)}), \sigma(z, Tz), \frac{1}{4}[\sigma(x_{n(k)}, Tz) + \sigma(z, Tx_{n(k)})]\} \le \\ &r \max\{\sigma(x_{n(k)}, z), \sigma(x_{n(k)}, x_{n(k)+1}), \sigma(z, Tz), \frac{1}{4}[\sigma(x_{n(k)}, Tz) + \sigma(z, x_{n(k)+1})]\}. \end{aligned}$$

Passing to limit as $k \to \infty$, we get

$$\sigma(z, Tz) \le r\sigma(z, Tz).$$

Since $r < 1$, it follows that $\sigma(z, Tz) = 0$. Thus, by Lemma 1.10 we obtain $z \in Tz$, that is, z is a fixed point of T. □

Corollary 2.13. *Let (X,σ) be a complete metric-like space and $T: X \to X$ be a mapping. Assume that there exist $r \in [0,1)$ such that, for all $x, y \in X$*

$$\frac{1}{1+r}\sigma(x,Tx) \le \sigma(x,y) \le \frac{1}{1-r}\sigma(x,Tx) \Rightarrow \sigma(Tx,Ty) \le rM_\sigma(x,y),$$

where $M_\sigma(x,y) = \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{1}{4}[\sigma(x,Ty)+\sigma(Tx,y)]\}$.

Then T has a fixed point.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n = 0,1,2,\dots$ We have for all $n = 0,1,2,\dots$

$$\frac{1}{1+r}\sigma(x_n,Tx_n) \le \sigma(x_n,x_{n+1}) \le \frac{1}{1-r}\sigma(x_n,Tx_n)$$

It follows that for all $n = 0,1,2,\dots$

$$\sigma(x_{n+1},x_{n+2}) = \sigma(Tx_n,Tx_{n+1}) \le r\sigma(x_n,x_{n+1}).$$

Thus the sequence $\{x_n\}$ is Cauchy in (X,σ). By completeness of (X,σ) there exists $z \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n,z) = \sigma(z,z) = \lim_{n,m\to\infty} \sigma(x_n,x_m) = 0. \tag{2.19}$$

We have for all $n, m \in \mathbb{N}$

$$\sigma(x_n,x_{n+m}) \le \frac{1-r^m}{1-r}\sigma(x_n,x_{n+1}).$$

It follows that

$$\sigma(x_n,z) \le \sigma(x_n,x_{n+m}) + \sigma(x_{n+m},z) \le \frac{1-r^m}{1-r}\sigma(x_n,x_{n+1}) + \sigma(x_{n+m},z).$$

Passing to limit as $m \to \infty$, we get

$$\sigma(x_n,z) \le \frac{1}{1-r}\sigma(x_n,x_{n+1})$$

Proceeding as in the proof of Theorem 2.12, we can find a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\frac{1}{1+r}\sigma(x_{n(k)},Tx_{n(k)}) \le \sigma(x_{n(k)},z) \tag{2.20}$$

for all $k \in \mathbb{N}$. Then as in the proof of Theorem 2.12 we get z is a fixed point of T. □

We give the following illustrative example inspired from [4].

Example 2.14. Let $X = \{0,1,2\}$ and $\sigma : X \times X \to \mathbb{R}^+$ defined by:

$$\sigma(0,0) = \sigma(2,2) = \frac{1}{4}, \ \ \sigma(1,1) = 0, \quad \sigma(0,1) = \sigma(1,0) = \frac{1}{3},$$
$$\sigma(0,2) = \sigma(2,0) = \frac{2}{5}, \quad \sigma(1,2) = \sigma(2,1) = \frac{11}{15}.$$

Then (X,σ) is a complete metric-like space. Note that $\sigma$is not a partial metric on X as $\sigma(1,2) > \sigma(1,0) + \sigma(0,2) - \sigma(0,0)$.

Define the map $T : X \to CB^\sigma(X)$ by

$$T0 = T1 = \{1\} \quad \text{and } T2 = \{0,1\}.$$

Note that Tx is bounded and closed for all $x \in X$ in metric-like space (X,σ).

We have

$$\max\{\sigma(x,Tx),\, x \in X\} = \max\{\sigma(0,1), \sigma(1,1), \sigma(2,0)\} = \frac{2}{5},$$
$$\min\{\sigma(x,Tx),\, x \in X - \{1\}\} = \frac{1}{3}.$$

Therefore, we have

$$\frac{1}{4} \le \sigma(x,y) \le \frac{11}{15}$$

for all $x, y \in X$ with $(x, y) \neq (1, 1)$. It follows that

$$\frac{1}{1+r}\sigma(x, Tx) \leq \sigma(x, y) \leq \frac{1}{1-s}\sigma(x, Tx)$$

for all $x, y \in X$ with $x \neq 1$ and for some $\frac{3}{5} \leq r < s < 1$. Observe that the above inequalities are also true for $x = y = 1$ but not hold for $x = 1$ and $y \in \{0, 2\}$.

Now, we shall show that

$$H_\sigma(Tx, Ty) \leq rM_\sigma(x, y) \tag{2.21}$$

for all $x, y \in X$ for some $\frac{5}{6} \leq r < 1$. For this, we consider the following cases:

<u>case1</u> : $x, y \in \{0, 1\}$, with $(x, y) \neq (1, 0)$. We have

$$H_\sigma(Tx, Ty) = \sigma(1, 1) = 0 \leq rM_\sigma(x, y).$$

<u>case2</u> : $x = 0$, $y = 2$. We have

$$\begin{aligned} H_\sigma(Tx, Ty) = H_\sigma(\{1\}, \{0, 1\}) &= max\{\sigma(1, \{0, 1\}), max\{\sigma(1, 1), \sigma(1, 0)\}\} \\ &= \frac{1}{3} \leq \frac{5}{6}\sigma(x, y) \leq rM_\sigma(x, y) \end{aligned}$$

<u>case3</u> : $x = y = 2$. We have

$$\begin{aligned} H_\sigma(Tx, Ty) = H_\sigma(\{0, 1\}, \{0, 1\}) &= max\{\sigma(0, \{0, 1\}), \sigma(1, \{0, 1\})\} \\ &= min\{\sigma(0, 0), \sigma(0, 1)\} = \frac{1}{4} \end{aligned}$$

Moreover, we have $M_\sigma(2, 2) = \max\{\sigma(2, 2), \sigma(2, T2)\} = \max\{\frac{1}{4}, \frac{2}{5}\} = \frac{2}{5}$. Then for $x = y = 2$ we get

$$H_\sigma(T2, T2) = \frac{1}{4} \leq \frac{5}{6}.\frac{2}{5} \leq rM_\sigma(2, 2).$$

Then, all the required hypotheses of Theorem 2.12 are satisfied. Here, $x = 1$ is the unique fixed point of T.

Now, we need the following definition.

Definition 2.15. Let (X, p) be a partial metric space. A multi-valued mapping $T : X \to CB^p(X)$ is said to be $(\varphi, \psi)-$contractive multi-valued operator if there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$p(y, Tx) \leq \varphi(p(y, x)) \Rightarrow H_p(Tx, Ty) \leq \psi(M_p(x, y)) \tag{2.22}$$

for all $x, y \in X$, where

$$M_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.$$

We give the following result.

Theorem 2.16. *Let (X, p) be a complete partial metric space and $T : X \to CB^p(X)$ be $(\varphi, \psi)-$contractive multi-valued operator. Then T is an MWP operator.*

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. Let c a given real number such that $p(x_0, x_1) < c$.

Clearly, if $x_1 = x_0$ or $x_1 \in Tx_1$, we conclude that x_1 is a fixed point of T and so the proof is finished. Now, we assume that $x_1 \neq x_0$ and $x_1 \notin Tx_1$. So then, $p(x_0, x_1) > 0$ and $p(x_1, Tx_1) > 0$.

Since $x_1 \in Tx_0$, we get

$$p(x_1, Tx_0) \leq p(x_1, x_1) \leq p(x_1, x_0) \leq \varphi(p(x_1, x_0)).$$

Hence by (2.22) and triangular inequality, we have

$$\begin{aligned} 0 < p(x_1, Tx_1) &\leq H_p(Tx_0, Tx_1) \leq \psi(M_p(x_0, x_1)) \\ &\leq \psi(\max\{p(x_0, x_1), p(x_0, Tx_0), p(x_1, Tx_1), \frac{1}{2}[p(x_0, Tx_1) + p(x_1, Tx_0)]\}) \\ &\leq \psi(\max\{p(x_0, x_1), p(x_1, Tx_1), \frac{1}{2}[p(x_0, Tx_1) + p(x_1, x_1)]\}) \\ &\leq \psi(\max\{\sigma(x_0, x_1), \sigma(x_0, x_1), \sigma(x_1, Tx_1), \frac{1}{2}[p(x_1, Tx_1) + p(x_0, x_1)]\}) \\ &= \psi(\max\{p(x_0, x_1), p(x_1, Tx_1)\}) = \psi(p(x_0, x_1)) < \psi(c). \end{aligned}$$

Proceeding as in the proof of Theorem 2.2, we construct a sequence $\{x_n\}$ in X such that for all $n \in \mathbb{N}$

(i) $x_n \notin Tx_n,\ x_n \neq x_{n+1},\ x_{n+1} \in Tx_n$;

(ii)

$$p(x_n, x_{n+1}) \leq \psi^n(c). \tag{2.23}$$

Now, for $m > n$, we have

$$p(x_n, x_m) \leq \sum_{i=n}^{m-1} p(x_i, x_{i+1}) - \sum_{i=n+1}^{m-1} p(x_i, x_i) \leq \sum_{i=n}^{m-1} \psi^i(c) \leq \sum_{i=n}^{\infty} \psi^i(c) \to 0 \text{ as } n \to \infty.$$

Thus,

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.24}$$

Hence, $\{x_n\}$ is Cauchy in (X, p). Moreover since (X, p) is complete, it follows there exists $z \in X$ such that

$$\lim_{n\to\infty} p(x_n, z) = p(z, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.25}$$

Proceeding again as in the proof of Theorem 2.2, we prove that z is a fixed point of T. □

Analogously, we can derive the following results.

Corollary 2.17. *Let (X, p) be a complete partial metric space and $T : X \to CB^p(X)$ be a multi-valued mapping. Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that, for all $x, y \in X$*

$$H_p(Tx, Ty) \leq \psi(M_p(x, y)) + p(y, Tx) - \varphi(p(y, x)),$$

where $M_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}$.

Then T has a unique fixed point.

Corollary 2.18. *([4], Theorem 2.2) Let (X, p) be a complete partial metric space and $T : X \to CB^p(X)$ be a multi-valued mapping. Assume that there exist $r \in [0, 1)$ and $s \geq 1$ such that, for all $x, y \in X$*

$$p(y, Tx) \leq sp(y, x) \Rightarrow H_p(Tx, Ty) \leq rM_p(x, y),$$

where $M_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}$.

Then T is an MWP operator.

Proof. It suffice to take $\varphi(t) = st$ and $\psi(t) = rt$ in Theorem 2.16. □

Corollary 2.19. *Let (X, p) be a complete partial metric space and $T : X \to CB^p(X)$ be a multi-valued mapping. Assume that there exist $r \in [0, 1)$ and $s \geq 1$ such that, for all $x, y \in X$*

$$p(y, Tx) \leq sp(y, x) \Rightarrow H_p(Tx, Ty) \leq r \max\{p(x, y), p(x, Tx), p(y, Ty)\}.$$

Then T is an MWP operator.

Corollary 2.20. *Let (X, p) be a complete partial metric space and $T : X \to CB^p(X)$ be a multi-valued mapping. Assume that there exist $r \in [0, 1)$ and $s \geq 1$ such that, for all $x, y \in X$*

$$p(y, Tx) \leq sp(y, x) \Rightarrow H_p(Tx, Ty) \leq \frac{r}{3}\{p(x, y) + p(x, Tx) + p(y, Ty)\}.$$

Then T is an MWP operator.

If T is a single-valued mapping, we deduce the following results.

Corollary 2.21. *Let (X, p) be a complete partial metric space and $T : X \to X$ be a mapping. Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that, for all $x, y \in X$*

$$p(y, Tx) \leq \varphi(p(y, x)) \Rightarrow p(Tx, Ty) \leq \psi(M_p(x, y)),$$

where $M_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}$.

Then T has a unique fixed point.

Proof. The existence follows immediately also from Theorem 2.16. Thus, we need to prove uniqueness of fixed point. We assume that there exist $x, y \in X$ such that $x = Tx$ and $y = Ty$ with $x \neq y$.

Since $p(y, Tx) = p(y, x) \leq \varphi(p(y, x))$, then by (2.22), we get

$$\begin{aligned} 0 < p(x,y) = p(Tx,Ty) &\leq \psi(\max\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(Tx,y)]\}) \\ &= \psi(\max\{p(x,y), p(x,x), p(y,y), p(x,y)\}) = \psi(p(x,y)) \\ &< p(x,y) \end{aligned}$$

which is a contradiction. Hence $x = y$, so the uniqueness of the fixed point of T.

□

Corollary 2.22. *Let (X, p) be a complete partial metric space and $T : X \to X$ be a mapping. Assume that there exist $r \in [0, 1)$ and $s \geq 1$ such that, for all $x, y \in X$*

$$p(y, Tx) \leq sp(y, x) \Rightarrow p(Tx, Ty) \leq rM_p(x, y),$$

where $M_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}$.

Then T has a unique fixed point.

References

[1] IA. Rus, A. Petrusel and A. Sintamarian, Data dependence of the fixed points set of multi-valued weakly Picard operators, Stud. Univ. Babe?s-Bolyai, Math. 46 (2001), 111-121.

[2] IA. Rus, A. Petrusel and A. Sintamarian, Data dependence of the fixed points set of some multi-valued weakly Picard operators, Nonlinear Anal. 52 (2003), 1947-1959.

[3] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Am. Math. Soc. 136 (2008), 1861-1869.

[4] M. Jleli, H. K. Nashine, B. Samet and C. Vetro, On multi-valued weakly Picard operators in partial Hausdorff metric spaces, Fixed Point Theory Appl. 20015 (2015), Art. ID 52.

[5] I. Altun and H. Simsek, Some fixed point theorems on dualistic partial metric spaces, J. Adv. Math. Stud. 1 (2008), 1C8.

[6] E. Karapinar, H. Aydi, A. Felhi and S. Sahmim, Hausdorff Metric-Like, generalized Nadler's Fixed Point Theorem on Metric-Like Spaces and application, Miskolc Math. Notes (in press).

[7] C.T. Aage and J.N. Salunke, The results on fixed points in dislocated and dislocated quasi-metric space, Appl. Math. Sci, 2(59) (2008), 2941-2948.

[8] S.G. Matthews, Partial metric topology, Proc. 8th Summer Conference on General Topology and Applications, Annals of the New York Academi of Sciences, 728 (1994), 183-197.

[9] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadlers fixed point theorem on partial metric spaces, Topol. Appl. 159 (2012), 3234-3242.

[10] R.D. Daheriya, R. Jain and M. Ughade, Some fixed point theorem for expansive type mapping in dislocated metric space, ISRN Math. Anal. 2012 (2012), Art. ID 376832.

[11] A. Isufati, Fixed point theorems in dislocated quasi-metric space, Appl. Math. Sci. 4(5) (2010), 217-233.

[12] Amini A. Harandi, Metric-like spaces, partial metric spaces and fixed points, Fixed Point Theory Appl. 2012 (2012), Art. ID 204.

[13] R. George, Cyclic contractions and fixed points in dislocated metric spaces, Int. J. Math. Anal. 7(9) (2013), 403-411.

[14] G.E. Hardy and T.D. Rogers, A generalization of a fixed point theorem of Reich, Canadian Mathematical Bulletin, 16 (1973), 201C206.

[15] E. Karapınar and P. Salimi, Dislocated metric space to metric spaces with some fixed point theorems, Fixed Point Theory Appl. 2013 (2013), Art. ID 222.

[16] P.S. Kumari, W. Kumar and I.R. Sarma, Common fixed point theorems on weakly compatible maps on dislocated metric spaces, Math. Sci. 6 (2012), 71.

[17] PS. Kumari, Some fixed point theorems in generalized dislocated metric spaces, Math. Theory Model. 1(4) (2011), 16-22.

[18] S.B. Nadler, Multi-valued contraction mappings, Pacific J. Math. 30 (1969), 475-488.

[19] I.R. Sarma and P.S. Kumari, On dislocated metric spaces, Int. J. Math. Arch. 3(1) (2012), 72-77.

[20] R. Shrivastava, ZK. Ansari and M. Sharma, Some results on fixed points in dislocated and dislocated quasi-metric spaces, J. Adv. Stud. Topol. 3(1) (2012), 25-31.

[21] M. Shrivastava, K. Qureshi and A.D. Singh, A fixed point theorem for continuous mapping in dislocated quasi-metric spaces, Int. J. Theor. Appl. Sci. 4(1) (2012), 39-40.

[22] K. Zoto, Some new results in dislocated and dislocated quasi-metric spaces. Appl. Math. Sci. 6(71) (2012), 3519-3526.

[1]Department of Mathematics, College of Sciences, King Faisal University, Hafouf, Saudi Arabia

[2]Department of Mathematics, Preparatory Engineering Institute, Bizerte, Carthage University, Tunisia

*Corresponding author: afelhi@kfu.edu.sa