International Journal of Analysis and Applications
ISSN 2291-8639
Volume 1, Number 2 (2013), 100-105
http://www.etamaths.com

A SUBORDINATION THEOREM INVOLVING A MULTIPLIER TRANSFORMATION

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Abstract. We, here, study a certain differential subordination involving a multiplier transformation which unifies some known differential operators. As a special case to our main result, we find some new results providing the best dominant for $z^p/f(z),\ z/f(z)$ and $z^{p-1}/f'(z),\ 1/f'(z)$.

1. Introduction

Let $\mathcal{A}$ be the class of all functions f analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{A}_p$ denote the class of functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, p \in \mathbb{N} = \{1, 2, 3, \cdots\}$, which are analytic and multivalent in the open unit disk $\mathbb{E}$. Note $\mathcal{A}_1 = \mathcal{A}$. For $f \in \mathcal{A}_p$, define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n, \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k + \lambda}{p + \lambda}\right)^n a_k z^k,\ (\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

The operator $I_p(n, \lambda)$ has been recently studied by Aghalary et al. [3]. $I_1(n, 0)$ is the well-known Sălăgean [1] derivative operator D^n, defined for $f \in \mathcal{A}$ as under:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to: $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

2010 *Mathematics Subject Classification.* 30C80, 30C45.
Key words and phrases. Differential subordination, Multiplier transformation, Analytic function; Univalent function, p-valent function.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$ such that $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for each dominant q of (1), is said to be the best dominant of (1).

Obradovič [2], introduced and studied the class $\mathcal{N}(\alpha),\ 0 < \alpha < 1$ of functions $f \in \mathcal{A}$ satisfying the following inequality

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha}\right\} > 0,\ z \in \mathbb{E}.$$

He called it, the class of non-Bazilevič functions.

In 2005, Wang et al. [6] introduced the generalized class $\mathcal{N}(\lambda, \alpha, A, B)$ of non-Bazilevič functions which is analytically defined as:

$$\mathcal{N}(\lambda, \alpha, A, B) = \left\{f \in \mathcal{A} : (1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz},\right\}$$

where $0 < \alpha < 1,\ \lambda \in \mathbb{C},\ -1 \le B \le 1,\ A \ne B,\ A \in \mathbb{R}$.

Wang et al. [6] studied the class $\mathcal{N}(\lambda, \alpha, A, B)$ and made some estimates on $\left(\frac{z}{f(z)}\right)^{\alpha}$.

Using the concept of differential subordination, Shanmugam et al. [5] studied the differential operator $(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}$ and obtained the best dominant for $\left(\frac{z}{f(z)}\right)^{\alpha}$.

The main objective of this paper is to unify the above mentioned differential operators. For this, we establish a differential subordination involving the multiplier transformation $I_p(n, \lambda)$, defined above. As special cases of main theorem, we obtain best dominant for $z^p/f(z),\ z/f(z)$ and $z^{p-1}/f'(z),\ 1/f'(z)$ and some known results also appear as special cases to our main result.

To prove our main result, we shall make use of the following lemma of Miller and Macanu [4].

Lemma 1.1. *Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \ne 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

(i) h is convex, or

(ii) Q is starlike.

In addition, assume that

(iii) $\Re\,\frac{zh'(z)}{Q(z)} > 0,\ z \in \mathbb{E}$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. Main Results

In what follows, all the powers taken are the principal ones.

Theorem 2.1. *Let α and β be non-zero complex numbers such that $\Re(\beta/\alpha) > 0$ and let $f \in \mathcal{A}_p$, $\left(\frac{z^p}{I_p(n,\lambda)f(z)}\right)^\beta \neq 0,\ z \in \mathbb{E}$, satisfy the differential subordination*

(2)
$$\left(\frac{z^p}{I_p(n,\lambda)f(z)}\right)^\beta \left[1+\alpha-\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right] \prec \frac{1+Az}{1+Bz}+\frac{\alpha}{\beta(p+\lambda)}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{z^p}{I_p(n,\lambda)f(z)}\right)^\beta \prec \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1,\ z \in \mathbb{E},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof: On writing $u(z) = \left(\frac{z^p}{I_p(n,\lambda)f(z)}\right)^\beta$, a little calculation yields that

$$\left(\frac{z^p}{I_p(n,\lambda)f(z)}\right)^\beta \left[1+\alpha-\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right] = u(z)+\frac{\alpha}{\beta(p+\lambda)}zu'(z), \tag{3}$$

Define the functions θ and ϕ as follows:

$$\theta(w) = w \text{ and } \phi(w) = \frac{\alpha}{\beta(p+\lambda)}.$$

Clearly, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C}$ and $\phi(w) \neq 0,\ w \in \mathbb{D}$. Select $q(z) = \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1,\ z \in \mathbb{E}$ and define the functions Q and h as follows:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\alpha}{\beta(p+\lambda)}zq'(z) = \frac{\alpha}{\beta(p+\lambda)}\frac{(A-B)z}{(1+Bz)^2},$$

and

$$h(z) = \theta(q(z))+Q(z) = q(z)+\frac{\alpha}{\beta(p+\lambda)}zq'(z) = \frac{1+Az}{1+Bz}+\frac{\alpha}{\beta(p+\lambda)}\frac{(A-B)z}{(1+Bz)^2}. \tag{4}$$

A little calculation yields

$$\Re\left(\frac{zQ'(z)}{Q(z)}\right) = \Re\left(1+\frac{zq''(z)}{q'(z)}\right) = \Re\left(\frac{1-Bz}{1+Bz}\right) > 0,\ z \in \mathbb{E},$$

i.e. Q is starlike in $\mathbb{E}$ and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(1+\frac{zq''(z)}{q'(z)}+(p+\lambda)\frac{\beta}{\alpha}\right) = \Re\left(\frac{1-Bz}{1+Bz}\right)+(p+\lambda)\Re\left(\frac{\beta}{\alpha}\right) > 0,\ z \in \mathbb{E}.$$

Thus conditions (ii) and (iii) of Lemma 1.1, are satisfied. In view of (2), (3) and (4), we have

$$\theta[u(z)]+zu'(z)\phi[u(z)] \prec \theta[q(z)]+zq'(z)\phi[q(z)].$$

Therefore, the proof follows from Lemma 1.1.

For $p = 1$ and $\lambda = 0$ in above theorem, we get the following result involving Sălăgean operator.

Theorem 2.2. *If* $\alpha,\ \beta$ *are non-zero complex numbers such that* $\Re(\beta/\alpha) > 0$. *If* $f \in \mathcal{A},\ \left(\frac{z}{D^n f(z)}\right)^{\beta} \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$\left(\frac{z}{D^n f(z)}\right)^{\beta}\left[1+\alpha-\alpha\frac{D^{n+1}f(z)}{D^n f(z)}\right] \prec \frac{1+Az}{1+Bz}+\frac{\alpha}{\beta}\frac{(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\left(\frac{z}{D^n f(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

3. Dominant for $z^p/f(z),\ z/f(z)$

This section is concerned with the results giving the best dominant for $z^p/f(z)$ and $z/f(z)$. Select $\lambda = n = 0$ in Theorem 2.1, we obtain the following result.

Corollary 3.1. *Let* $\alpha,\ \beta$ *be non-zero complex numbers such that* $\Re(\beta/\alpha) > 0$ *and let* $f \in \mathcal{A}_p,\ \left(\frac{z^p}{f(z)}\right)^{\beta} \neq 0,\ z \in \mathbb{E}$, *satisfy*

$$(1+\alpha)\left(\frac{z^p}{f(z)}\right)^{\beta} - \alpha\frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz}+\frac{\alpha}{p\beta}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\left(\frac{z^p}{f(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

Taking $\beta = 1$ in above theorem, we obtain:

Corollary 3.2. *Suppose that* α *is a non-zero complex number such that* $\Re(1/\alpha) > 0$ *and suppose that* $f \in \mathcal{A}_p,\ \frac{z^p}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$(1+\alpha)\frac{z^p}{f(z)} - \alpha\frac{z^{p+1}f'(z)}{p(f(z))^2} \prec \frac{1+Az}{1+Bz}+\frac{\alpha}{p}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\frac{z^p}{f(z)} \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

On writing $\alpha = -1$ in Corollary 3.1, we get:

Corollary 3.3. *Let* β *be a complex number with* $\Re(\beta) < 0$ *and let* $f \in \mathcal{A}_p,\ \left(\frac{z^p}{f(z)}\right)^{\beta} \neq 0,\ z \in \mathbb{E}$, *satisfy*

$$\frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz}-\frac{1}{p\beta}\frac{(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\left(\frac{z^p}{f(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

Selecting $\alpha = \beta = 1/2$ in Corollary 3.1, we get:

Corollary 3.4. *If* $f \in \mathcal{A}_p,\ \sqrt{\dfrac{z^p}{f(z)}} \neq 0,\ z \in \mathbb{E},$ *satisfies*

$$\sqrt{\frac{z^p}{f(z)}}\left(3 - \frac{zf'(z)}{pf(z)}\right) \prec \frac{2(1+Az)}{1+Bz} + \frac{2}{p}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\sqrt{\frac{z^p}{f(z)}} \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

Taking $p = 1$ in Corollary 3.2, we have the following result.

Corollary 3.5. *If* α *is a non-zero complex number such that* $\Re(1/\alpha) > 0$ *and if* $f \in \mathcal{A},\ \dfrac{z}{f(z)} \neq 0,\ z \in \mathbb{E},$ *satisfies*

$$(1+\alpha)\frac{z}{f(z)} - \alpha\frac{z^2 f'(z)}{(f(z))^2} \prec \frac{1+Az}{1+Bz} + \alpha\frac{(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\frac{z}{f(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

Setting $p = 1$ in Corollary 3.3, we have the following result.

Corollary 3.6. *If* β *is a complex number with* $\Re(\beta) < 0$ *and if* $f \in \mathcal{A},\ \left(\dfrac{z}{f(z)}\right)^\beta \neq 0,\ z \in \mathbb{E},$ *satisfies*

$$\frac{z^{\beta+1}f'(z)}{(f(z))^{\beta+1}} \prec \frac{1+Az}{1+Bz} - \frac{1}{\beta}\frac{(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\left(\frac{z}{f(z)}\right)^\beta \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

Setting $p = 1$ in Corollary 3.1, we obtain, below, the result of Shanmugam et al. [5].

Corollary 3.7. *If* $\alpha,\ \beta$ *are non-zero complex numbers such that* $\Re(\beta/\alpha) > 0$. *If* $f \in \mathcal{A},\ \left(\dfrac{z}{f(z)}\right)^\beta \neq 0,\ z \in \mathbb{E},$ *satisfies*

$$(1+\alpha)\left(\frac{z}{f(z)}\right)^\beta - \alpha f'(z)\left(\frac{z}{f(z)}\right)^{1+\beta} \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\beta}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\left(\frac{z}{f(z)}\right)^\beta \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

4. Dominant for $z^{p-1}/f'(z),\ 1/f'(z)$

We obtain here, the best dominant for $z^{p-1}/f'(z)$ and $1/f'(z)$ as special cases to our main result. Select $\lambda = 0$ and $n = 1$ in Theorem 2.1, we obtain:

Corollary 4.1. *Let α, β be non-zero complex numbers such that $\Re(\beta/\alpha) > 0$ and let $f \in \mathcal{A}_p$, $\left(\frac{pz^{p-1}}{f'(z)}\right)^{\beta} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$(1+\alpha)\left(\frac{pz^{p-1}}{f'(z)}\right)^{\beta} - \frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{pz^{p-1}}{f'(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{p\beta}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\left(\frac{pz^{p-1}}{f'(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

Taking $\beta = 1$ in above theorem, we obtain:

Corollary 4.2. *Suppose that α is a non-zero complex number such that $\Re(1/\alpha) > 0$ and suppose that $f \in \mathcal{A}_p$, $\frac{pz^{p-1}}{f'(z)} \neq 0$, $z \in \mathbb{E}$, satisfies*

$$(1+\alpha)\frac{pz^{p-1}}{f'(z)} - \alpha\frac{z^{p-1}}{f'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{p}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\frac{z^{p-1}}{f'(z)} \prec \frac{1+Az}{p(1+Bz)},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

Taking $p = 1$ in Corollary 4.2, we have the following result.

Corollary 4.3. *If α is a non-zero complex number such that $\Re(1/\alpha) > 0$ and if $f \in \mathcal{A}$, $\frac{1}{f'(z)} \neq 0$, $z \in \mathbb{E}$, satisfies*

$$\frac{1}{f'(z)}\left(1-\alpha\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+Az}{1+Bz} + \alpha\frac{(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\frac{1}{f'(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

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