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INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR HARMONIC (h, s)-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce a new concept of harmonic (h, s)-convex functions in the second sense which generalizes the harmonic convex functions. Some Hermite-Hadamard-Fejer type integral inequalities are derived. Some special cases also discussed. Results derived in this paper represent significant refinement and improvement of the known results.

1. Introduction

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. For recent applications, generalizations and other aspects of convex functions and their variant forms, see [1, 14–16] and the references therein. Varosanec [17] introduced a class of convex functions with respective to an arbitrary non-negative function h, which is known as h-convex function.. This class of functions unifies various classes of convex functions and is being used to discuss several concepts in a unified manners. An other important class of convex functions is known as harmonic convex functions, was investigated by Anderson et al. [1] and Iscan [8]. For the recent developments, see [8,9, 11,12,13,15 and the references therein. Nooe et al. . [13] introduced and investigated a new class of convex functions. It has been shown a number new and known classes of convex functions can be obtained as special cases.

Motivated and inspired by ongoing research in this filed, we introduce and study a new class of convex functions, which is called harmonic (h, s)-convex functions. One can easily show that harmonic (h, s)-convex functions include Godunova-Levin harmonic convex functions and harmonic s-convex functions as special cases. This is the main motivation of this paper. We also obtain several new Hermite-Hadamard-Fejer type inequalities for harmonic (h, s)-convex functions. Our results include several previously known and new results as special cases. The ideas and technique of this paper may be a starting point for further research in this dynamic field.

2. Preliminaries

First of all, we recall the following basic concepts.

Definition 2.1. [15]. *A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic convex set, if*

$$\frac{xy}{tx + (1-t)y} \in I, \qquad \forall x, y \in I, t \in [0, 1].$$

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Definition 2.2. [8]. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic convex function, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1].$$

We now introduce a new class of harmonic convex function in second sense, which is called the (h, s)-harmonic convex function.

Definition 2.3. *Let $h : J = [0, 1] \to \mathbb{R}$ a nonnegative function. A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic (h, s)-convex function in second sense, where $s \in [-1, 1]$, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq h((1-t)^s)f(x) + h(t^s)f(y), \qquad \forall x, y \in I, t \in (0, 1).$$

For $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{x + y}\right) \leq h\left(\frac{1}{2^s}\right)[f(x) + f(y)],$$

which is called Jensen type harmonic (h, s)-convex function.

We now discuss some special cases of harmonic (h, s) convex function.

I. If we take $h(t^s) = t^s$ and $s = -1$ in Definition 2.3, then it reduces to Godunova-Levin harmonic convex functions.

Definition 2.4. [9]. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be Godunova-Levin harmonic convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{1-t}f(x) + \frac{1}{t}f(y), \qquad \forall x, y \in I, t \in (0, 1).$$

II. If we take $h(t^s) = t^s$ in Definition 2.3, then it reduces to extended harmonic s-convex functions.

Definition 2.5. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be extended harmonic s-convex function in second sense, where $s \in [-1, 1]$, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)^s f(x) + t^s f(y), \qquad \forall x, y \in I, t \in [0, 1].$$

III. If $s = 1$ in Definition 2.3, then it reduces to the harmonic h-convex functions.

Definition 2.6. [11]. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic h-convex function, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq h(1-t)f(x) + h(t)f(y), \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 2.7. [14]. *Two functions f, g are said to be similarly ordered (f is g-monotone), if and only if,*

$$\langle f(x) - f(y), g(x) - g(y) \rangle \geq 0, \qquad \forall x, y \in \mathbb{R}^n.$$

We now show that the product of two harmonic (h, s)-convex functions is again harmonic (h, s)-convex function.

Lemma 2.1. *If $h(t^s) + h((1-t)^s) \leq 1$, then the product of two similarly ordered harmonic (h, s)-convex functions is harmonic $((h, s)$-convex function.*

Proof. Let f, g be two (h,s)-harmonic convex functions. Then

$$\begin{aligned} & f\left(\frac{xy}{tx+(1-t)y}\right)g\left(\frac{xy}{tx+(1-t)y}\right) \\ \leq\ & [h((1-t)^s)f(x)+h(t^s)f(y)][h((1-t)^s)g(x)+h(t^s)g(y))] \\ =\ & [h((1-t)^s)]^2 f(x)g(x)+h(t^s)h((1-t)^s)[f(x)g(y)+f(y)g(x)] \\ & +[h(t^s)]^2 f(y)g(y) \\ \leq\ & [h((1-t))^s]^2 f(x)g(x)+h(t^s)h((1-t)^s)[f(x)g(x)+f(y)g(y)] \\ & +[h(t^s)]^2 f(y)g(y) \\ =\ & [h((1-t)^s)f(x)g(x)+h(t^s)f(y)g(y)][h(t^s)+h((1-t)^s)] \\ \leq\ & h((1-t)^s)f(x)g(x)+h(t^s)f(y)g(y). \end{aligned} \tag{2.1}$$

This shows that product of two similarly ordered harmonic (h,s)-convex functions is again a harmonic (h,s)-convex function. □

We need the following well-known fact, which establishes a relationship between convex functions and harmonic convex functions. This fact plays a crucial part in deriving our results.

Remark 2.1. *Let $I=[a,b]\subseteq\mathbb{R}\setminus\{0\}$ and consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$ defined by $g(x)=f\left(\frac{1}{x}\right)$, then f is harmonic (h,s)-convex on $[a,b]$, if and only if, g is (h,s)-convex in the usual sense on $\left[\frac{1}{b},\frac{1}{a}\right]$.*

3. Main results

In this section, we obtain Hermite-Hadamard inequalities for harmonic (h,s)-convex function.

Theorem 3.1. *Let $f: I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic (h,s)-convex function, where $s\in(-1,1]$. If $f\in L[a,b]$, then*

$$\frac{1}{2h\left(\frac{1}{2^s}\right)}f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\leq[f(a)+f(b)]\int_0^1 h(t^s)\mathrm{d}t.$$

Proof. Let f be harmonic (h,s)-convex function with $t=\frac{1}{2}$. Then

$$f\left(\frac{2xy}{x+y}\right)\leq h\left(\frac{1}{2^s}\right)[f(x)+f(y)].$$

Taking $x=\frac{ab}{ta+(1-t)b}$ and $y=\frac{ab}{(1-t)a+tb}$, we have

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq h\left(\frac{1}{2^s}\right)\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)\right] \\ &= h\left(\frac{1}{2^s}\right)\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t\right] \\ &\leq h\left(\frac{1}{2^s}\right)\int_0^1\Big[h((1-t)^s)f(a)+h(t^s)f(b)+h((1-t)^s)f(b) \\ &\quad +h(t^s)f(a)\Big]\mathrm{d}t \\ &= 2h\left(\frac{1}{2^s}\right)[f(a)+f(b)]\int_0^1 h(t^s)\mathrm{d}t. \end{aligned}$$

Using the fact that

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t=\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x,$$

we have

$$\frac{1}{2h\left(\frac{1}{2^s}\right)}f\left(\frac{2ab}{a+b}\right) \quad \leq \quad \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq [f(a)+f(b)]\int_0^1 h(t^s)\mathrm{d}t.$$

□

Theorem 3.2. *Let $f : I = [a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic (h,s)-convex function, where $s \in (-1,1]$. If $f \in L[a,b]$, then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \quad \leq \quad \frac{1}{2}[f(a)+f(b)]\int_0^1 [h((1-t)^s)+h(t^s)]\mathrm{d}t.$$

Proof. Let f be harmonic (h,s)-convex function. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq h((1-t)^s)f(a)+h(t^s)f(b)$$
$$f\left(\frac{ab}{(1-t)a+tb}\right) \leq h(t^s)f(a)+h((1-t)^s)f(b)$$
$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq h((1-t)^s)f(a)+h(t^s)f(b)$$

and

$$f\left(\frac{ab}{(1-t)a+tb}\right) \leq h(t^s)f(a)+h((1-t)^s)f(b).$$

Adding the above inequalities, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)$$
$$+f\left(\frac{ab}{(1-t)a+tb}\right) \leq 2[f(a)+f(b)][h((1-t)^s)+h(t^s)]$$

Integrating the above inequality over $t \in [0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t \\
&+\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t \\
\leq\quad & 2[f(a)+f(b)]\int_0^1 [h((1-t)^s)+h(t^s)]\mathrm{d}t,
\end{aligned}$$

which implies

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \quad \leq \quad \frac{1}{2}[f(a)+f(b)]\int_0^1 [h((1-t)^s)+h(t^s)]\mathrm{d}t,$$

which is the required result. □

Theorem 3.3. *Let $f, g : I = [a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be two harmonic (h,s)-convex functions, where $s \in (-1,1]$. If $f \in L[a,b]$, then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \quad \leq \quad M(a,b)\int_0^1 [h(t^s)]^2\mathrm{d}t+N(a,b)\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t,$$

where

$$M(a,b) \quad = \quad f(a)g(a)+f(b)g(b), \tag{3.1}$$

$$N(a,b) \quad = \quad f(a)g(b)+f(b)g(a). \tag{3.2}$$

Proof. Let f, g be two harmonic (h, s)-convex functions. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq h((1-t)^s)f(a) + h(t^s)f(b)$$
$$g\left(\frac{ab}{ta+(1-t)b}\right) \leq h((1-t)^s)g(a) + h(t^s)g(b).$$

Now

$$\begin{aligned} & f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{(1-t)a+tb}\right) \\ \leq\ & \big[h((1-t)^s)f(a) + h(t^s)f(b)\big]\big[h((1-t)^s)g(a) + h(t^s)g(b)\big] \\ =\ & [h((1-t)^s)]^2[f(a)g(a)] + h(t^s)h((1-t)^s)[f(a)g(b) + f(b)g(a)] \\ & + [h(t^s)]^2[f(b)g(b)] \end{aligned}$$

Integrating the above inequality over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t \\ \leq\ & [f(a)g(a)]\int_0^1 [h((1-t)^s)]^2\mathrm{d}t + [f(a)g(b) + f(b)g(a)]\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t \\ & + [f(b)g(b)]\int_0^1 [h(t^s)]^2\mathrm{d}t \\ =\ & [f(a)g(a) + f(b)g(b)]\int_0^1 [h(t^s)]^2\mathrm{d}t \\ & + [f(a)g(b) + f(b)g(a)]\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t, \end{aligned}$$

Thus

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \quad \leq \quad M(a,b)\int_0^1 [h(t^s)]^2\mathrm{d}t + N(a,b)\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t,$$

the required result. □

Theorem 3.4. *Let $f, g : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic (h, s)-convex functions, where $s \in (-1, 1]$. If $f, g \in L[a, b]$, then*

$$\begin{aligned} & \left(\frac{ab}{b-a}\right)^{s+1}\int_{\frac{1}{b}}^{\frac{1}{a}} h\left(\left(x-\frac{1}{b}\right)^s\right)\left[f(a)g\left(\frac{1}{x}\right) + g(a)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\ & \left(\frac{ab}{b-a}\right)^{s+1}\int_{\frac{1}{b}}^{\frac{1}{a}} h\left(\left(\frac{1}{a}-x\right)^s\right)\left[f(b)g\left(\frac{1}{x}\right) + g(b)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\ \leq\ & M(a,b)\int_0^1 [h(t^s)]^2\mathrm{d}t + N(a,b)\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t \\ & + \frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x, \end{aligned}$$

where $M(a, b)$ and $N(a, b)$ are given by (3.1) *and* (3.2) *respectively.*

Proof. Let f, g be harmonic (h, s)-convex functions. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq h((1-t)^s)f(a) + h(t^s)f(b)$$
$$g\left(\frac{ab}{ta+(1-t)b}\right) \leq h((1-t)^s)g(a) + h(t^s)g(b).$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned}
& f\left(\frac{ab}{ta+(1-t)b}\right)[h((1-t)^s)g(a)+h(t^s)g(b)] \\
& +g\left(\frac{ab}{ta+(1-t)b}\right)[h((1-t)^s)f(a)+h(t^s)f(b)] \\
\leq\ & [h((1-t)^s)f(a)+h(t^s)f(b)][h((1-t)^s)g(a)+h(t^s)g(b)] \\
& +f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
& g(a)h((1-t)^s)f\left(\frac{ab}{ta+(1-t)b}\right)+g(b)h(t^s)f\left(\frac{ab}{ta+(1-t)b}\right) \\
& +f(a)h((1-t)^s)g\left(\frac{ab}{ta+(1-t)b}\right)+f(b)h(t^s)g\left(\frac{ab}{ta+(1-t)b}\right) \\
\leq\ & [h((1-t)^s)]^2[f(a)g(a)]+h(t^s)h((1-t)^s)[f(b)g(a) \\
& +f(a)g(b)]+[h(t^s)]^2[f(b)g(b)] \\
& +f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)
\end{aligned}$$

Integrating the above inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
& g(a)\int_0^1 h((1-t)^s)f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+g(b)\int_0^1 h(t^s)f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t \\
& +f(a)\int_0^1 h((1-t)^s)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+f(b)\int_0^1 h(t^s)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t \\
\leq\ & [f(a)g(a)]\int_0^1[h((1-t)^s)]^2\mathrm{d}t+[f(a)g(b)+f(b)g(a)]\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t \\
& +[f(b)g(b)]\int_0^1[h(t^s)]^2\mathrm{d}t \\
& +\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t,
\end{aligned}$$

from which, it follows that

$$\begin{aligned}
& \left(\frac{ab}{b-a}\right)^{s+1}\int_{\frac{1}{b}}^{\frac{1}{a}} h\left(\left(x-\frac{1}{b}\right)^s\right)\left[f(a)g\left(\frac{1}{x}\right)+g(a)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\
& \left(\frac{ab}{b-a}\right)^{s+1}\int_{\frac{1}{b}}^{\frac{1}{a}} h\left(\left(\frac{1}{a}-x\right)^s\right)\left[f(b)g\left(\frac{1}{x}\right)+g(b)f\left(\frac{1}{x}\right)\right]\mathrm{d}x \\
\leq\ & [f(a)g(a)+f(b)g(b)]\int_0^1[h(t^s)]^2\mathrm{d}t \\
& +[f(b)g(a)+f(a)g(b)]\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t \\
& +\frac{ab}{b-a}\int_a^b\frac{f(x)g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

which is the required result. □

We need the following Lemma in order to obtain the Fejer type Hermite-Hadamard inequality for Harmonic (h,s)-convex functions.

Lemma 3.1. *Let $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic (h, s)-convex function, where $s \in (-1, 1]$. Then*

$$f\left(\frac{abx}{(a+b)x - ab}\right) \leq [h((1-t)^s) + h(t^s)][f(a) + f(b)] - f(x).$$

Proof. It is known that that $x \in [a, b]$, can be represented as $x = \frac{ab}{ta+(1-t)b}$, $\forall t \in [0, 1]$. Thus

$$\begin{aligned} f\left(\frac{abx}{(a+b)x - ab}\right) &= f\left(\frac{ab}{(1-t)a + tb}\right) \\ &\leq h(t^s)f(a) + h((1-t)^s)f(b) \\ &= h(t^s)[f(a) + f(b)] + h((1-t)^s)[f(a) + f(b)] \\ &\quad -[h(t^s)f(b) + h((1-t)^s)f(a)] \\ &\leq h(t^s)[f(a) + f(b)] + h((1-t)^s)[f(a) + f(b)] - f(x), \end{aligned}$$

the required result. □

Theorem 3.5. *Let $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic (h, s)-convex function, where $s \in (-1, 1]$. If $f \in L[a, b]$, then*

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2^s})} f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{g(x)}{x^2} \mathrm{d}x \\ \leq\ &\int_a^b \frac{f(x)g(x)}{x^2} \mathrm{d}x \\ \leq\ &\frac{[f(a) + f(b)]}{2} \int_a^b \left[h\left(\frac{b(x-a)}{x(b-a)}\right)^s + h\left(\frac{a(b-x)}{x(b-a)}\right)^s \right] \frac{g(x)}{x^2} \mathrm{d}x, \end{aligned}$$

where $g : [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is a nonnegative, integrable function and satisfies

$$g(x) = g\left(\frac{abx}{[a+b]x - ab}\right), \qquad \forall x \in [a, b].$$

Proof. Using the given fact and Lemma 3.1, we have

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2^s})} f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{g(x)}{x^2} \mathrm{d}x \\ =\ &\frac{1}{2h(\frac{1}{2^s})} \int_a^b f\left(\frac{2abx}{(a+b)x - ab + ab}\right) \frac{g(x)}{x^2} \mathrm{d}x \\ \leq\ &\frac{1}{2h(\frac{1}{2^s})} \int_a^b h\left(\frac{1}{2^s}\right) \left[f\left(\frac{abx}{(a+b)x - ab}\right) + f(x) \right] \frac{g(x)}{x^2} \mathrm{d}x \\ =\ &\frac{1}{2} \int_a^b f\left(\frac{abx}{(a+b)x - ab}\right) \frac{g(x)}{x^2} \mathrm{d}x + \frac{1}{2} \int_a^b \frac{f(x)g(x)}{x^2} \mathrm{d}x \\ =\ &\int_a^b \frac{f(x)g(x)}{x^2} \mathrm{d}x \end{aligned}$$

To prove the other part of the inequality, we consider

$$\begin{aligned}
&\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \\
= \;& \frac{1}{2}\int_a^b f\left(\frac{abx}{(a+b)x-ab}\right)\frac{g(x)}{x^2}\mathrm{d}x + \frac{1}{2}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \\
\leq \;& \frac{1}{2}\int_a^b \Big[[h((1-t)^s)+h(t^s)][f(a)+f(b)]-f(x)\Big]\frac{g(x)}{x^2}\mathrm{d}x \\
&+\frac{1}{2}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \\
\leq \;& \frac{[f(a)+f(b)]}{2}\int_a^b \left[h\left(\frac{b(x-a)}{x(b-a)}\right)^s + h\left(\frac{a(b-x)}{x(b-a)}\right)^s\right]\frac{g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

This completes the proof.

□

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