International Journal of Analysis and Applications
ISSN 2291-8639
Volume 12, Number 1 (2016), 22-29
http://www.etamaths.com

ON THE GROWTH AND APPROXIMATION OF TRANSCENDENTAL ENTIRE FUNCTIONS ON ALGEBRAIC VARIETIES

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Abstract. Let X be a complete intersection algebraic variety of codimension $m > 1$ in $\mathbb{C}^{m+n}$. In this paper we characterized the classical growth parameters order and type for transcendental entire functions $f \in \oplus(X)$, the space of holomorphic functions on the complete intersection algebraic variety X, in terms of the best polynomial approximation error in L^p-norm, $0 < p \leq \infty$, on a $L-regular$ non-pluripolar compact subset K of $\mathbb{C}^{m+n}$.

1. Introduction

The growth of transcendental entire functions in one complex variable case is well represented in the work of B.Ja Levin [11] and Boas [2]. In several complex variables the standard reference is the work of P.Lelong and L.Gruman [10] and Ronkin's book [14]. Einstein-Matthews and Kasana [3] studied the growth parameters $(p, q) - order$ and $(p, q) - type$ introduced by Juneja et al.([6],[7]) of transcendental entire functions $f : \mathbb{C}^n \to \mathbb{C}$. Einstein-Matthews and Clement Lutterodt [4] extended the results studied in [3] to transcendental entire functions $f : X \to \mathbb{C}$, defined on a complete intersection algebraic variety X in $\mathbb{C}^{m+n}$ of codimension $m > 1$, and obtained the growth parameters in terms of the sequence of extremal polynomials occurring in the development of f. It has been noticed that the growth parameters of $f : X \to \mathbb{C}$ in terms of approximation errors is not studied so far. The aim of this paper is to bridge this gap and to study the results obtained in [4] in terms of the best approximation errors in L^p-norm, $0 < p \leq \infty$.

A.R. Reddy ([13],[14]) characterized the growth parameters in terms of approximation errors for a function continuous on [-1,1]. T. Winiarski ([21],[22]) studied the growth of entire functions in terms of Lagrange polynomial approximation errors with respect to sup norm on a compact subset K (positive capacity) of $\mathbb{C}$ and $\mathbb{C}^n, n > 1$. Kasana and Kumar [8] generalized the results of Winiarski [22] by using the concept of index-pair (p, q). Adam Janik [5] characterized the generalized order of entire functions by means of polynomial approximation and interpolation on compact subsets of $\mathbb{C}^n$, using the Siciak extremal function ([18],[19]). In [5] Adam Janik extended the results of S.M. Shah [17] in the case $n = 1, K = [-1, 1]$ and Winiarski [22]. Srivastava and Kumar [20] extended and improved the results of Adam Janik [5]. But our work is different from all these authors.

The text has been divided into four parts. Section 1 consists of an introductory exposition of the topic and Section 2 contains some definitions and notations. In Section 3, we have given Zeriahi's Bernstein-Markov type inequality with two lemmas in which first one is due to Zeriahi extending the classical Cauchy inequality and second is concerned with a sequence of extremal polynomials. Finally, in Section 4, we prove two theorems for a transcendental entire function $f \in \oplus(\mathbb{C}^{m+n})$, the space of holomorphic functions on the complete intersection algebraic variety X and studied the growth parameters order and type in terms of L^p-approximation error on a $L-regular$ non-pluripolar compact subset of $\mathbb{C}^{m+n}$.

2010 *Mathematics Subject Classification.* 41A20, 30E10.

Key words and phrases. algebraic variety; approximation error; extremal polynomials and plurisubharmonic functions.

2. Definitions and Notations

Following the definition of Einstein and Kasana [3], we have
Let $v : \mathbb{C}^{m+n} \to \mathbb{R}_+ := r \in \mathbb{R} : r > 0$ be a real-valued function such that the following properties hold:
(i) $v(z+w) \leq v(z) + v(w) : z, w \in \mathbb{C}^{m+n}$,
(ii)$v(bz) = |b|v(z) : z \in \mathbb{C}^{m+n}, b \in \mathbb{C}$,
(iii) $v(z) = 0 \Longleftrightarrow z = 0$.

Here v is a norm on $\mathbb{C}^{m+n}$ and it exhausts the complex space $\mathbb{C}^{m+n}$ by a family of sublevel sets $\{\Omega_c\}_{c\geq 1}$ which are defined by

$$\Omega_c = \{z \in \mathbb{C}^{m+n} : v(z) \leq c, c \in \mathbb{R}\}.$$

Let $\varphi : \mathbb{C}^{m+n} \to \mathbb{R}_+$. Define $M_{\varphi,v}(r) = \sup_{v(z)\leq r} \varphi(z)$, the maximum modulus of φ with respect to the norm v for each $r \in \mathbb{R}_+$. We say that the transcendental entire function $f : \mathbb{C}^{m+n} \to \mathbb{C}$ is of order ρ, if $\log |f|$ is of order ρ, where

$$\rho = \limsup_{r\to\infty} \frac{\log M_{f,v}(r)}{\log r} \tag{2.1}$$

If $\rho < +\infty$, f is said to have maximal, normal or minimal type if

$$\sigma = \limsup_{r\to\infty} \frac{M_{f,v}(r)}{r^\rho}, \tag{2.2}$$

is infinite, finite or zero.

Let K be a compact subset of $\mathbb{C}^{m+n}$, which is nonpluripolar on each irreducible component of a complete intersection variety X. The Siciak extremal function V_K associated to K has been studied extensibly by Siciak [18] and Sadullaev ([15],[16]) and is defined as:

$$V_K = \sup\{u(z) : u \in \curlywedge(X); u(\zeta) \leq 0, \zeta \in K, z \in X\}$$

where the subcone $\curlywedge(X)$ is given by

$$\curlywedge(X) = \{u(z) : u \in PSH(X); u(z) \leq \log(\parallel z \parallel +1) + C_u, z \in X\}$$

here C_u is a constant depending only on the cone of plurisubharmonic function $(PSH)u$ and $\parallel . \parallel$ is the Euclidean norm on $\mathbb{C}^{m+n}$.
The upper semi-continuous regularization of V_K is defined on X by

$$V_K^*(z) = \limsup_{\zeta\to z} V_K(\zeta), \zeta \in K, z \in X,$$

$V_K^*(z)$ is $PSH(X)$ and satisfies

$$V_K^*(z) \leq \log(\parallel z \parallel +1) + O(1), as \parallel z \parallel \to +\infty.$$

If V_K is continuous on $\mathbb{C}^{m+n}$, then $V_K = V_K^* \in \curlywedge$. It is given in [18] that if, for all $z \in K, V_K$ is continuous, then V_K is continuous on X. In this case we say that K is $L-regular$ in X. We define the sublevel sets of the extremal function V_K by setting

$$\Omega_\alpha = \{z \in X : V_K(z) \leq \alpha\}, \alpha > 1, \alpha \in \mathbb{R},$$

and sublevel sets of the upper semi-continuous regularization V_K^* of V_K by

$$\Omega_r = \{z \in \mathbb{C}^{m+n} : expV_K^*(z) \leq r\}, r > 1.$$

It has been observe that the sequence of sublevel sets $\{\Omega_r\}_{r>1}$ exhausts the complex space $\mathbb{C}^{m+n}$. For $f : \mathbb{C}^{m+n} \to \mathbb{C}$ a transcendental entire function, set

$$M_{K,f}(r) = \sup_{z\in\overline{\Omega}_r} |f(z)|, r > 1.$$

It can be easily shown that $\log^+ M_{K,f}(r)$ and $\log^+ M_{K,v}(r)$ give the same order given by

$$\rho \equiv \rho(f) = \limsup_{r\to\infty} \frac{\log\log^+ M_{K,f}(r)}{\log r}. \tag{2.3}$$

If $0 < \rho < +\infty$, the type of $f : \mathbb{C}^{m+n} \to \mathbb{C}$ is defined by

$$\sigma \equiv \sigma(f) = \limsup_{r\to\infty} \frac{\log^+ M_{K,f}(r)}{r^\rho}. \tag{2.4}$$

Zeriahi [23] constructed an orthogonal polynomial basis $\{A_k\}_{k\geq 1}$ for the space $\oplus(X)$. The basis is orthogonal in the Hilbert space $L^2(X,\mu)$, essentially by means of the Hilbert-Schmidt process, here μ be the extremal capacity measure on K given by $\mu = (dd^c V_K$. Further details on this positive Borel measure μ supported on K can be obtained from the paper of E. Bedford and B.A. Taylor [1].

Let $P_d(\mathbb{C}^{m+n})$ denote the $\mathbb{C}$-vector space of polynomials $\pi_d : \mathbb{C}^{m+n} \to \mathbb{C}$ of degree $\leq d$ for $d \geq 1$. Let $L^2_P(K,\mu)$ denote the closed subspace of the Hilbert space $L^2(K,\mu)$ generated by the restriction to K of polynomials $\pi_d \in P_d(\mathbb{C}^{m+n})$ of degree $(\pi_d) \leq d$, for $d \geq 1$. Then every function $f \in L^2_P(K,\mu)$ has a power series expansion of the form

$$f = \Sigma_{k\geq 1} f_k A_k, \tag{2.5}$$

with

$$f_k = \frac{1}{\Delta_k^2(K)} \int_K f.\overline{A}_k d\mu, \Delta_k(K) = (\int_K |A_k|^2 d\mu)^{\frac{1}{2}}, k \geq 1,$$

here . is the dot product of vectors.

Let $L^p(K,\mu), p \geq 1$ denote the class of all functions such that

$$\| f \|_{L^p(K,\mu)} = (\int_K |f|^p d\mu)^{\frac{1}{p}} < \infty,$$

then we define the best polynomial approximation error in L^p-norm, $p \geq 1$, by

$$E_d^p(K,f) = \inf\{\| f - \pi_d \|_{L^p(K,\mu)}, \pi_d \in P_d(\mathbb{C}^{m+n})\}. \tag{2.6}$$

If the extremal function V_K associated with K is continuous for every $z \in K$, then V_K is continuous on X and $L - regular$, so instead of defining sublevel sets for the upper semi-continuous regularization, we define the same for V_K by setting

$$\Omega_r = \{z \in X : V_K(z) < \log r, r \in \mathbb{R}, r > 1\}.$$

Then we have

$$V_K(z) \geq \frac{1}{s_k} \log(\frac{|A_k|}{a_k(K)}),$$

where

$$a_k(K) = \max_{z\in K} |A_k(z)|, |A_k|_{\overline{\Omega}_r} \leq a_k(K) r^{s_k}, s_k = degree(A_k).$$

Following the Siciak [18] we observe that if K is $L - regular$ then

$$\limsup_{d\to\infty} (E_d^p(K,f))^{\frac{1}{d}} = \frac{1}{R} < 1$$

if and only if f has an analytic continuous to

$$\{z \in \mathbb{C}^{m+n}; V_K(z) < \log(\frac{1}{R})\}.$$

3. Auxiliary Results

In this section we shall state some preliminary results which will be used in the sequel.

First we state Zeriahi's Bernstein-Markov type inequality [23]:
BM:For all $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\sup_{z\in K} |f(z)| \leq C_\epsilon (1+\epsilon)^{deg(f)} (\int_K |f|^2 d\mu)^{\frac{1}{2}} \tag{3.1}$$

for every holomorphic function f with polynomial growth on the complete intersection algebraic variety X and K is a nonpluripolar compact subset of X.
Now we state the following lemmas of Zeriahi extending the classical Cauchy inequality.

Lemma 3.1. Let $f=\Sigma_{k\geq 0}f_kA_k$ be a holomorphic function on X. Then for every $\theta>1$, there exist an integer N_θ and a constant $C_\theta>0$ such that

$$|f_k|r^{s_k}\Delta_k(K)\leq C_\theta\frac{(r+1)^{N_\theta}}{(r-1)^{2n-1}}|f_k|_{\overline{\Omega}_{r\theta}}, \tag{3.2}$$

for every $r>1,k\geq 1$, where C_θ and N_θ are independent of r,k and f.

Lemma 3.2. If K is an $L-regular$, then the sequence of extremal polynomials $\{A_k\}_{k\geq 1}$ satisfies

$$\lim_{k\to\infty}(\frac{|A_k(z)|}{\nu_k})^{\frac{1}{s_k}}=exp(V_K(z)),\nu_k=\| A_k \|_{L^2(K,\mu)}, \tag{3.3}$$

for every $z\in\mathbb{C}^{m+n}$ and

$$\lim_{k\to\infty}(\frac{|A_k(z)|}{\nu_k})^{\frac{1}{s_k}}=1. \tag{3.4}$$

4. Main Results

In this section we shall prove our main theorems. Moreover, we shall characterize the classical growth parameters order and types of transcendental entire function in terms of L^p-approximation error defined by (2.6).

Theorem 4.1. If $f:X\to\mathbb{C}$ is a transcendental entire function on X with a series expansion (2.5) with respect to the orthogonal polynomial basis $\{A_k\}_{k\geq 1}$, then $f\in L^p(K,\mu),1<p\leq\infty$ is of finite order if and only if

$$\rho=\limsup_{k\to\infty}\frac{s_k\log s_k}{-\log(E^p_{s_k}(K,f))}<+\infty, \tag{4.1}$$

and $\rho=\rho_1$, where $E^p_{s_k}(K,f)$ is defined by (2.6).

Proof. First we have to prove that $\rho\leq\rho_1$. If $\rho_1=\infty$, then nothing to be prove. Assume that $\rho_1<\infty$ and let $\epsilon>0$. For a sufficiently large k, from (4.1) we have

$$0\leq\frac{s_k\log s_k}{-\log(E^p_{s_k}(K,f))}\leq\rho_1+\epsilon$$

or

$$E^p_{s_k}(K,f)\leq(s_k)^{\frac{-s_k}{\rho_1+\epsilon}}. \tag{4.2}$$

Since adding a polynomial will not change the order of a function. Thus, for $r\geq 1$ and $a_0(K)=0$, we can assume that following inequality holds for every $k\geq 0$,

$$M_{K,f}(r)\leq\Sigma_{k\geq 1}|f_k|a_k(K)r^{s_k}. \tag{4.3}$$

Now we will proceed the proof in two steps ($p\geq 2$) and ($1<p<2$). Let $f=\Sigma_{k\geq 0}f_k.A_k$ be an element of $L^p(K,\mu)$.

Step 1. If $f\in L^p(K,\mu)$ with $p\geq 2$, then $f=\Sigma_{k=0}^{\infty}f_k.A_k$ with convergence in $L^2(K,\mu)$,

$$f_k=\frac{1}{\nu_k^2}\int_K f.\overline{A}_kd\mu,k\geq 1,\nu_k\equiv\Delta_k(K),$$

or

$$=\frac{1}{\nu_k^2}\int_K(f-P_{s_k-1}).\overline{A}_kd\mu.$$

It gives

$$|f_k|\leq\frac{1}{\nu_k^2}\int_K|(f-P_{s_k-1})|.|\overline{A}_k|d\mu,$$

now using $Bernstein - Walsh$ inequality and $H\ddot{o}lder's$ inequality we have for any $\epsilon > 0$

$$|f_k|\nu_k \leq C_\epsilon(1+\epsilon)^{s_k} E^p_{s_k-1}(K,f), k \geq 0. \tag{4.4}$$

Step 2. If $1 \leq p < 2$, let p' such that $\frac{1}{p} + \frac{1}{p'} = 1$ then $p' \geq 2$. By $H\ddot{o}lder's$ inequality we get

$$|f_k|\nu_k^2 \leq \| f - P_{s_k-1} \|_{L^p(K,\mu)} \| A_k \|_{L^{p'}(K,\mu)} .$$

But $\| A_k \|_{L^{p'}(K,\mu)} \leq C \| A_k \|_K = Ca_k(K)$, now by $Bernstein - Markov$ inequality we have

$$|f_k|\nu_k^2 \leq CC_\epsilon(1+\epsilon)^{s_k} \| f - P_{s_k-1} \|_{L^p(K,\mu)},$$

it gives

$$|f_k|\nu_k^2 \leq C'_\epsilon(1+\epsilon)^{s_k} E^p_{s_k}(K,f). \tag{4.5}$$

From (4.4) and (4.5), we get for $p \geq 1$

$$|f_k|\nu_k \leq A_\epsilon(1+\epsilon)^{s_k} E^p_{s_k}(K,f), \tag{4.6}$$

where A_ϵ is a constant depends only on ϵ.

Now using $Zeriahi'sBernstein - Markov$ type inequality in (4.3) and (4.6), we obtain

$$M_{K,f}(r) \leq \Sigma_{k\geq 0}|f_k|C_\epsilon(1+\epsilon)^{s_k}\nu_k r^{s_k} \leq \Sigma_{k\geq 0}A_\epsilon C_\epsilon(1+\epsilon)^{2s_k} E^p_{s_k}(K,f)r^{s_k},$$

using inequality (4.2) in above, we get

$$M_{K,f}(r) \leq C'_\epsilon \Sigma_{k\geq 0}(1+\epsilon)^{2s_k}(s_k)^{\frac{-s_k}{\rho_1+\epsilon}} r^{s_k} = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \Sigma_{1\leq k\leq (2r(1+\epsilon)^2)^{(\rho_1+\epsilon)}}(1+\epsilon)^{2s_k}(s_k)^{\frac{-s_k}{(\rho_1+\epsilon)}} r^{s_k}$$

and

$$\Sigma_1 = \Sigma_{k\geq (2r(1+\epsilon)^2)^{(\rho_1+\epsilon)}}(1+\epsilon)^{2s_k}(s_k)^{\frac{-s_k}{(\rho_1+\epsilon)}} r^{s_k}.$$

In Σ_2, we have $(r(1+\epsilon)^2 k)^{\frac{-1}{(\rho_1+\epsilon)}} \leq \frac{1}{2}$, so that $\Sigma_2 \leq 1$, and

$$\Sigma_1 \leq (F(r,\epsilon))^{(\rho_1+\epsilon)} \Sigma_{k\geq 1}(s_k)^{\frac{-s_k}{(\rho_1+\epsilon)}}$$

where $F(r,\epsilon) = (r(1+\epsilon)^2)^{(2r(1+\epsilon)^2)}$
or

$$\Sigma_1 \leq K_1 \exp((2r(1+\epsilon)^2)^{(\rho_1+\epsilon)} \log(r(1+\epsilon)^2)) \leq K_2 \exp(r^{(\rho_1+\epsilon)}),$$

for some constants $K_1 > 0, K_2 > 0$. Hence it follows from definition of order given by (2.3) that $\rho \leq \rho_1 + \epsilon$, since $\epsilon > 0$ is arbitrary, it gives

$$\rho \leq \rho_1. \tag{4.7}$$

In order to prove the reverse inequality i.e., $\rho_1 \leq \rho$ we consider the polynomial of degree s_k as

$$P_{s_k}(z) = \Sigma^k_{j=0} f_j A_j,$$

then

$$E^p_{s_k-1}(K,f) \leq \Sigma^\infty_{s_j=s_k}|f_j| \| A_j \|_{L^p(K,\mu)} \leq C_0 \Sigma^\infty_{s_j=s_k+1}|f_j| \| A_j \|_K, k \geq 0, p \geq 1. \tag{4.8}$$

In the consequences of Lemmas 3.1 and 3.2, we obtain the following inequality

$$|f_k|a_k(K) \leq \frac{M_{K,f}(r)}{r^{s_k}}, r > 0. \tag{4.9}$$

Using (4.9)in (4.8), we get

$$E^p_{s_k-1}(K,f) \leq C_0 \Sigma^\infty_{s_j=s_k+1} M_{K,f}(r) r^{-s_j} = C_0 M_{K,f}(r) \frac{(r^*/r)^{(s_k+1)}}{1-(r^*/r)} \tag{4.10}$$

for all sufficiently large s_k and all $r > r^*, r^* > 1$. Here C_0 is some fixed number.
For all sufficiently large s_k and $r > 2r^*$, (4.10) gives

$$E^p_{s_k-1}(K,f) \leq \gamma M_{K,f}(r)(r^*/r)^{s_k}, \tag{4.11}$$

where γ is a constant independent of s_k and r.

If $\rho_1 = 0$, then nothing to be prove. Let us assume that $0 < \rho_1 < \infty$. If $\rho_1 < \infty$, define $\rho^* = \rho_1 - \epsilon$, for small $\epsilon > 0$, so that $\rho_1 > 0$. Let $\rho^* > 0$ be arbitrary if $\rho_1 = +\infty$. Then for infinitely many indices $k \geq 1$, from (4.1) we have

$$s_k \log s_k \geq \rho^* \log(E^p_{s_k}(K,f))^{-1}$$

or

$$\log E^p_{s_k}(K,f) \geq \frac{-s_k \log s_k}{\rho^*}. \tag{4.12}$$

Using (4.12) in (4.11) we get

$$\log M_{K,f}(r) \geq \frac{-s_k \log s_k}{\rho^*} + s_k \log(\frac{r}{r^*}) - \log\gamma. \tag{4.13}$$

The minimum value of right hand side of (4.13) is obtained at $\frac{r_{s_k}}{r^*} = (es_k)^{\frac{1}{\rho^*}}$ and substituting the value of $(\frac{r_{s_k}}{r^*})$ in (4.13) we obtain the following inequality

$$\log M_{K,f}(r) \geq \frac{s_k}{\rho^*} - \log\gamma$$

or

$$\frac{\log\log M_{K,f}(r_{s_k})}{\log r_{s_k} - \log r^*} \geq \rho^*(\frac{\log s_k - \log \rho^*}{\log s_k + 1}).$$

Proceeding the limits and taking the definition (2.3) into account, we get

$$\rho = \limsup_{r\to\infty} \frac{\log\log M_{K,f}(r)}{\log r} \geq \limsup_{r\to\infty} \frac{\log\log M_{K,f}(r_{s_k})}{\log r_{s_k}} \geq \rho^*. \tag{4.14}$$

Since ρ^* is arbitrary real number, smaller than ρ, it gives that $\rho \geq \rho_1$. Now in view of (4.7) the result is immediate. Hence the proof is completed.

Theorem 4.2 If $f : X \to \mathbb{C}$ is a transcendental entire function on X with a series expansion (2.5) with respect to the orthogonal polynomial basis $\{A_k\}_{k\geq 1}$, then $f \in L^p(K,\mu)$ with a finite order $\rho(0 < \rho < \infty)$ has finite type $\sigma(0 < \sigma < \infty)$ if and only if

$$e\rho\sigma = \limsup_{k\to\infty} s_k(E^p_{s_k}(K,f))^{\frac{\rho}{s_k}} < +\infty,$$

and $\sigma_1 = \sigma$, where $E^p_{s_k}(K,f)$ is given by (2.6).

Proof. Let $\delta_1 = e\rho\sigma_1$. For given $\epsilon > 0$ and $\delta_1 > 0$, we have for sufficiently large k

$$s_k(E^p_{s_k}(K,f))^{\frac{\rho}{s_k}} \leq \delta_1 + \epsilon \tag{4.15}$$

or

$$\frac{s_k \log s_k}{-\log(E^p_{s_k}(K,f))} \leq \frac{\rho}{1 - \log(\frac{\delta_1+\epsilon}{s_k})}.$$

Now it follows from Theorem 4.1 that the order of f is at most ρ.
Now let us consider that $0 < \delta_1 < \infty$ and we have to show that $\sigma \leq \frac{\delta_1}{e\rho} = \sigma_1$. From (4.15) we get

$$E^p_{s_k}(K,f) \leq (\frac{\delta_1+\epsilon}{s_k})^{\frac{s_k}{\rho}}. \tag{4.16}$$

Consider

$$|f(z)| \leq \Sigma_{k\geq 1}|f_k||A_k|_{\overline{\Omega}_r} \leq \Sigma_{k\geq 1}|f_k|a_k(K)r^{s_k}. \tag{4.17}$$

Further we will proceed the proof by considering two cases:

Case 1. Let $p \geq 2$, then we have $f = \Sigma_{k\geq 0} f_k Ak$ because $f \in L^2(K,\mu), L^p(K,\mu) \subset L^2(K,\mu)$ and $\{A_k\}_k$ is a basis of $L^2(K,\mu)$. Consider the series $\Sigma_{k\geq 0} f_k Ak$ in $\mathbb{C}^{m+n}$ and it can be easily seen that this series converges uniformly on every compact subsets of $\mathbb{C}^{m+n}$ to an entire function. Using the $Bernstein - Markov$ inequality (BM) in (4.17) we get

$$|f(z)| \leq C_\epsilon \Sigma_{k\geq 1} |f_k| (1+\epsilon)^{s_k} \nu_k r^{s_k},$$

it gives from (4.6) that

$$|f(z)| \leq C_\epsilon \Sigma_{k\geq 1} |f_k| (1+\epsilon)^{2s_k} E^p_{s_k}(K,f) r^{s_k}.$$

Now in view of (4.16) we have

$$|f(z)| \leq C'_\epsilon \Sigma_{k\geq 1} ((1+\epsilon)^{2\rho} (\frac{\delta_1+\epsilon}{s_k}) r^\rho)^{\frac{s_k}{\rho}} = C'_\epsilon \Sigma_1 + \Sigma_2.$$

Let us assume the function

$$\phi(s) = ((r(1+\epsilon)^2)^\rho (\frac{\delta_1+\epsilon}{s}))^{\frac{s}{\rho}}, s > 0.$$

This function attains its maximum value at

$$s = (\frac{\delta_1+\epsilon}{e})(r(1+\epsilon)^2)^\rho$$

and the value is equal to $\exp((\frac{\delta_1+\epsilon}{e\rho})(r(1+\epsilon)^2)^\rho)$. Hence for any constant $K_1 > 0$,

$$\begin{aligned}
C'_\epsilon \Sigma_1 &= C'_\epsilon \Sigma_{1\leq k\leq 2(\delta_1+\epsilon)(1+\epsilon)^{2\rho} r^\rho} ((\frac{\delta_1+\epsilon}{s_k})(1+\epsilon)^{2\rho} r^\rho)^{\frac{s_k}{\rho}} \\
&\leq 2(\delta_1+\epsilon)(1+\epsilon)^{2\rho} r^\rho \exp((\frac{\delta_1+\epsilon}{e\rho})(r(1+\epsilon)^2)^\rho) \\
&\leq K_1 \exp((\frac{\delta_1+\epsilon}{e\rho})(r(1+\epsilon)^2)^\rho),
\end{aligned}$$

and

$$\begin{aligned}
C'_\epsilon \Sigma_2 &= C'_\epsilon \Sigma_{k > 2(\delta_1+\epsilon)(1+\epsilon)^{2\rho} r^\rho} ((\frac{\delta_1+\epsilon}{s_k})(1+\epsilon)^{2\rho} r^\rho)^{\frac{s_k}{\rho}} \\
&\leq C'_\epsilon \Sigma_{k\geq 1} \frac{1}{2^k} = K_2 < \infty.
\end{aligned}$$

Thus from above discussion we get $\sigma \leq \frac{\delta_1}{e\rho}$.

Case 2. For the case $1 \leq p < 2$ and $f \in L^p(K,\mu)$ by (BM) inequality and $H\ddot{o}lder's$ inequality we get again the inequality (4.6). Now proceeding on the lines of proof of Case 1, the result is immediate.

In order to prove the reverse inequality, we note that if $\delta_1 > \epsilon > 0$, then for infinitely many indices k

$$E^p_{s_k}(K,f) \geq (\frac{\delta_1-\epsilon}{s_k})^{\frac{s_k}{\rho}}. \tag{4.18}$$

Now using (4.18) in (4.11) we obtain

$$\log M_{K,f}(r_{s_k}) \geq \frac{s_k}{\rho} \log(\frac{\delta_1-\epsilon}{s_k}) + s_k \log(r_{s_k}/r^*) - \log\gamma.$$

The minimum value of right hand side is attains at $\frac{r_{s_k}}{r^*} = \frac{e s_k}{(\delta_1-\epsilon)}$. Thus we get

$$M_{K,f}(r_{s_k}) \geq e^{\frac{s_k}{\rho}} = \exp((\frac{\delta_1-\epsilon}{e\rho}) r^\rho_{s_k}) + O(1).$$

Proceeding to limits and using the definition (2.4) of type of $f \in L^p(K,\mu)$, we get

$$\sigma \geq \frac{\delta_1}{e\rho}.$$

This completes the proof of theorem.
Remark 4.3. Theorem 4.1 and 4.2 also holds for $(0 < p < 1)$ (see[9]).

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