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ALPHA CONVEX FUNCTIONS ASSOCIATED WITH CONIC DOMAINS

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ABSTRACT. In this paper we define a new class $k-UM_{\alpha}[A,B]$ of Janowski type $k-$uniformly alpha convex functions. We use the method of differential subordinations theory to obtain some new results like sufficient condition, inclusion relations, coefficient estimate and covering properties. The results presented here include a number of well-known results as their special cases.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. Furthermore S represents class of all functions in A which are univalent in E.

For two functions $f(z)$ and $g(z)$ analytic in A, we say that $f(z)$ is subordinate to $g(z)$ in E (and write $f \prec g$ or $f(z) \prec g(z)$), if there exists an analytic function $w(z)$ such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$ for $z \in E$. If $g(z)$ is univalent in E then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$. The idea of subordination goes back to Lindelöf [9]. Subordination was more formally introduced and studied by Littelwood [10] and later by Rogosinski [20] and [19]. The concept of subordination was considered by Miller [12] and further investigated by Noor et al. [16] and many others see [9],[21].

Definition 1. *A function $p(z)$ is said to be in the class $P[A,B]$, if it is analytic in E with $p(0) = 1$ and*

$$p(z) \prec \frac{1+Az}{1+Bz}, \qquad -1 \leq B < A \leq 1.$$

This class was presented by Janowski [3] and explored by a few creators. Kanas and Wiśniowska [4],[5] presented and examined the class $k-ST$ of $k-$starlike functions and the relating class $k-UCV$ of $k-$uniformly convex functions. There were characterized subject to the conic region Ω_k, $k \geq 0$, as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

This domain represents the right half plane, a parabola, a hyperbola and an ellipse for $k = 0, k = 1, 0 < k < 1$ and $k > 1$ respectively. The extremal functions for these conic regions are

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \frac{2}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right\}, & 0 < k < 1, \\ 1 + \frac{2}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{dx}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\right) + \frac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.2}$$

where

$$u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tx}}, \qquad (z \in E),$$

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and $t \in (0,1)$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$. Here $R(t)$ is Legendre's complete elliptic integral of first kind and $R'(t)$ is the complementary integral of $R(t)$. If $p_k(z) = 1 + \delta_k z + \cdots$, then it is shown in [5] that from (1.2), one can have

$$\delta_k = \begin{cases} \frac{8(\arccos k)^2}{\pi^2(1-k^2)}, & 0 \leq k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)R^2(t)}, & k > 1. \end{cases} \tag{1.3}$$

Using the concepts of Janowski functions and the conic regions, Noor et al. [16] gave the following

Definition 2. [16]*A function $p(z)$ is said to be in the class $k - P[A,B]$, if and only if*

$$p(z) \prec \frac{(A+1)\,p_k(z) - (A-1)}{(B+1)\,p_k(z) - (B-1)}, \quad k \geq 0,$$

where $p_k(z)$ is defined in (1.2) *and* $-1 \leq B < A \leq 1$.

Geometrically, the function $p(z) \in k - P[A,B]$, takes all values from the domain $\Omega_k[A,B]$, $-1 \leq B < A \leq 1, k \geq 0$ which is defined as

$$\Omega_k[A,B] = \left\{ w : \Re\left(\frac{(B-1)\,w(z) - (A-1)}{(B+1)\,w(z) - (A+1)}\right) > k\left|\frac{(B-1)\,w(z) - (A-1)}{(B+1)\,w(z) - (A+1)} - 1\right|\right\}.$$

The domain $\Omega_k[A,B]$ retains the conic domain Ω_k inside the circular region defined by $\Omega[A,B]$. The impact of $\Omega[A,B]$, on the conic domain Ω_k, changes the original shape of the conic regions. The ends of hyperbola and parabola get closer to one another but never meet anywhere and the ellipse gets the oval shape. When $A \to 1, B \to -1$ the radius of the circular disk defined by $\Omega[A,B]$ tends to infinity, consequently the arm of the hyperbola and parabola expands to the oval terns into ellipse. We see that $\Omega_k[1,-1] = \Omega_k$, the conic domain defined by Kanas and Wiśniowska [4].

Now using Janowski functions and the conic regions, we give the following

Definition 3. *A function $f(z) \in A$ is said to be in the class $k - UM_\alpha[A,B]$, $k \geq 0, 0 \leq \alpha \leq 1, -1 \leq B < A \leq 1$, if and only if*

$$J(\alpha, f; z) \in k - P[A,B], \tag{1.4}$$

where

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}.$$

Special Cases:

(i) $k - UM_0[A,B] = k - ST[A,B]$, $k - UM_1[A,B] = k - UCV[A,B]$, the classes introduced by Noor et al. in [16].

(ii) $k - UM_0[1,-1] = k - ST$ and $k - UM_1[1,-1] = k - UCV$, we get the classes investigated by Kanas and Wisniowska [4], [5].

(iii) $k - UM_\alpha[1,-1] = k - UM_\alpha$, we have the class introduced and studied by Kanas [7].

(iv) $0 - UM_0[A,B] = S[A,B]$ and $0 - UM_1[A,B] = C[A,B]$, the well-known classes of Janowski starlike and Janowski convex functions, respectively, introduced by Janowski [3].

Definition 4. *Let $SS^*(\beta)$ denote the class of strongly starlike functions of order β,*

$$SS^*(\beta) = \left\{ f \in A : \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2} \quad z \in E \right\}, \quad \beta \in (0,1),$$

which was introduced in [24] *and* [1].

In this paper, several interesting subordination results are derived which yield sufficient condition, inclusion relations, coefficient estimate, covering result and order of strongly starlikeness in the class of uniformly alpha convex function.

To avoid repetitions, it is admitted once that $0 \leq \alpha \leq 1, k \geq 0$, and $-1 \leq B < A \leq 1$.

2. Preliminary results

To prove our main results we need the following Lemmas.

Lemma 1. [19]*Let $f(z)$ be subordinate to $g(z)$, with*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

If $g(z)$ is univalent in E and $g(E)$ is convex, then $|a_n| \leq |b_1|$.

Lemma 2. [11]*Let F be analytic and convex in E. If $f,g \in A$ and $f, g \prec F$, then for $t \in [0,1]$*

$$(1-t)f + tg \prec F.$$

Lemma 3. [14] *Let $k \geq 0$ and let δ, σ be any complex numbers with $\delta \neq 0$ and $\Re\left(\left(\frac{2k+1-A}{2k+1-B}\right)\delta + \sigma\right) > 0$. If $p(z)$ is analytic in E and $p(0) = 1$ and satisfies*

$$p(z) + \frac{zp^{'}(z)}{\delta p(z) + \sigma} \prec p_k(A, B; z), \tag{2.1a}$$

where

$$p_k(A, B; z) = \frac{(A+1)\, p_k\,(z) - (A-1)}{(B+1)\, p_k\,(z) - (B-1)},$$

and $q(z)$ is an analytic solution of

$$q(z) + \frac{zq(z)}{\delta q(z) + \sigma} = p_k(A, B; z)$$

then function $q(z)$ is univalent $p(z) \prec q(z) \prec p_k(A, B; z)$ and $q(z)$ is the best dominant of (2.1a) and is given as

$$q(z) = \left[\delta \int_0^1 \left(t^{\delta+\sigma-1} \exp \int_t^{tz} \frac{p_k(A, B, z) - 1}{u} du\right)^{\delta} dt\right]^{-1} - \frac{\sigma}{\delta}.$$

Lemma 4. [18] *Let a function $p\,(z)$ be analytic in E and has the form*

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n \quad , \quad c_m \neq 0,$$

with $p\,(z) \neq 0$ for $|z| < 1$. If there exists a point $z_\circ$, $|z_\circ| < 1$, such that

$$|\arg p(z)| < \frac{\pi}{2}\theta \quad \textit{for} \quad |z| < |z_\circ|$$

and

$$|\arg p(z_\circ)| = \frac{\pi}{2}\theta,$$

for some $\theta > 0$, then we have

$$\frac{z_\circ p'(z_\circ)}{p(z_\circ)} = il\theta,$$

where

$$l \geq \frac{m}{2}(x + \frac{1}{x}) \geq m \quad \textit{when} \quad \arg\{p(z_\circ)\} = \frac{\pi}{2}\theta$$

and

$$l \leq -\frac{m}{2}(x + \frac{1}{x}) \leq -m \quad \textit{when} \ \arg\{p(z_\circ)\} = -\frac{\pi}{2}\theta,$$

where

$$(p\,(z_\circ))^{\frac{1}{\theta}} = \pm ix \quad \textit{and} \quad x > 0.$$

3. Main results

Theorem 1. *A function $f(z) \in k - UM_{\alpha}[A,B]$, if it satisfies the condition*

$$\sum_{n=1}^{\infty} F_n(k,\alpha,A,B) < |B-A|,$$

where

(3.1)

$$F_n(k,\alpha,A,B)$$

$$= \sum_{n=2}^{\infty}[|\{2(k+1)(1-n)(1-\alpha(1-n)) + ((A+1)(n+1)$$

$$-(B+1)\left(2n+\alpha\left(1-n^2\right)\right))\}|\,|a_n| + \sum_{j=2}^{n-1}|\{2(k+1)((1-j)-\alpha(n+1-2j))$$

$$+((A+1)-(B+1)((1-2\alpha)j+\alpha(1+n)))\}(n+1-j)|\,|a_j a_{n+1-j}|].$$

Proof. Assume that (3.1) holds, then it suffices to show that

$$k\left|\frac{(B-1)J(\alpha,f;z)-(A-1)}{(B+1)J(\alpha,f;z)-(A+1)}-1\right| - \mathfrak{Re}\left[\frac{(B-1)J(\alpha,f;z)-(A-1)}{(B+1)J(\alpha,f;z)-(A+1)}-1\right] < 1. \tag{3.2}$$

We have

$$\begin{aligned} & k\left|\frac{(B-1)J(\alpha,f;z)-(A-1)}{(B+1)J(\alpha,f;z)-(A+1)}-1\right| - \mathfrak{Re}\left[\frac{(B-1)J(\alpha,f;z)-(A-1)}{(B+1)J(\alpha,f;z)-(A+1)}-1\right] \\ \le\ & (k+1)\left|\frac{(B-1)\left((1-\alpha)zf'(z)f'(z)+\alpha f(z)(zf'(z))'\right)-(A-1)f(z)f'(z)}{(B+1)\left((1-\alpha)zf'(z)f'(z)+\alpha f(z)(zf'(z))'\right)-(A+1)f(z)f'(z)}-1\right| \\ =\ & 2(k+1)\left|\frac{(1-\alpha)f(z)f'(z)-(1-\alpha)zf'(z)f'(z)-\alpha zf(z)f''(z)}{(B+1)\left((1-\alpha)zf'(z)f'(z)+\alpha f(z)(zf'(z))'\right)-(A+1)f(z)f'(z)}\right|. \end{aligned}$$

(3.3)

Now we have

$$\begin{aligned} zf'(z)f'(z) &= z\left(\sum_{n=0}^{\infty} na_n z^{n-1}\right)\left(\sum_{n=0}^{\infty} na_n z^{n-1}\right) \qquad (3.4) \\ &= \frac{1}{z}\left(\sum_{n=0}^{\infty} na_n z^{n}\right)\left(\sum_{n=0}^{\infty} na_n z^{n}\right) \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} j(n-j)a_j a_{n-j}\right)z^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} j(n-j)a_j a_{n-j}\right)z^{n-1} \\ &= z+\sum_{n=3}^{\infty}\left(\sum_{j=0}^{n} j(n-j)a_j a_{n-j}\right)z^{n-1} \\ &= z+\sum_{n=2}^{\infty}\left(\sum_{j=0}^{n+1} j(n+1-j)a_j a_{n+1-j}\right)z^{n} \\ &= z+\sum_{n=2}^{\infty}\left(2na_n+\sum_{j=2}^{n-1} j(n+1-j)a_j a_{n+1-j}\right)z^{n}. \qquad (3.5) \end{aligned}$$

Proceeding on the same way we have

$$f(z)\,f'(z) = z + \sum_{n=2}^{\infty}\left((n+1)\,a_n + \sum_{j=2}^{n-1}(n+1-j)\,a_j a_{n+1-j}\right)z^n \tag{3.6}$$

and

$$zf(z)\,f''(z) = \sum_{n=2}^{\infty}\left(n\,(n-1)\,a_n + \sum_{j=2}^{n-1}(n+1-j)\,(n-j)\,a_j a_{n+1-j}\right)z^n. \tag{3.7}$$

Using the equalities (3.5),(3.6) and (3.7) in (3.3), the equation (3.3) in simplified form can be written as

$$\begin{aligned}
& k\left|\frac{(B-1)J(\alpha,f;z)-(A-1)}{(B+1)J(\alpha,f;z)-(A+1)}-1\right| - \mathfrak{Re}\left[\frac{(B-1)J(\alpha,f;z)-(A-1)}{(B+1)J(\alpha,f;z)-(A+1)}-1\right] \\
\le\ & 2(k+1)\left|\frac{(1-\alpha)\,f(z)\,f'(z)-(1-\alpha)\,zf'(z)f'(z)-\alpha zf(z)\,f''(z)}{(B+1)\left((1-\alpha)\,zf'(z)f'(z)+\alpha f(z)\,(zf'(z))'\right)-(A+1)f(z)\,f'(z)}\right| \\
\le\ & \frac{2(k+1)\left[\begin{array}{c}\sum_{n=2}^{\infty}[|(1-n)\,(1-\alpha\,(1-n))|\,|a_n| \\ +\sum_{j=2}^{n-1}|[(1-j)-\alpha\,(n+1-2j)]\,(n+1-j)|\,|a_j a_{n+1-j}|]\end{array}\right]}{\left[\begin{array}{c}|B-A|-\sum_{n=2}^{\infty}[\left|\left((A+1)\,(n+1)-(B+1)\left(2n+\alpha\left(1-n^2\right)\right)\right)\right|\,|a_n| \\ -\sum_{j=2}^{n-1}|((A+1)-(B+1)\,((1-2\alpha)\,j+\alpha\,(1+n)))\,(n+1-j)|\,|a_j a_{n+1-j}|]\end{array}\right]}.
\end{aligned}$$

The last expression is bounded by 1, if

$$\begin{aligned}
& \sum_{n=2}^{\infty}[|\{2(k+1)\,(1-n)\,(1-\alpha\,(1-n))+((A+1)\,(n+1) \\
& -(B+1)\left(2n+\alpha\left(1-n^2\right)\right))\}|\,|a_n|+\sum_{j=2}^{n-1}|\{2(k+1)\,((1-j)-\alpha\,(n+1-2j)) \\
& +((A+1)-(B+1)\,((1-2\alpha)\,j+\alpha\,(1+n)))\}\,(n+1-j)\,|\,|a_j a_{n+1-j}|] \\
& <|B-A|.
\end{aligned}$$

This completes the proof. □

Putting $\alpha = 0$, in Theorem 1, we have the result below which is comparable to the one obtained by Noor and Malik [15].

Corollary 1. *A function $f \in k-ST[A,B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{2(k+1)\,(n-1)+|n\,(B+1)+(A+1)|\}\,|a_n| < |B-A|.$$

Putting $\alpha = 0$, $A = 1$ and $B = -1$ in Theorem 1, we can obtain the following result which improves the result of Kanas and Wiśniowska [4].

Corollary 2. *A function $f \in k-ST$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n+k\,(n-1)\}\,|a_n| < 1.$$

Putting $\alpha = 0, A = 1-2\beta$, $B = -1$ with $0 \le \beta < 1$ in Theorem 1, we have the result below which is comparable to the one obtained by Shams et al. [22].

Corollary 3. *A function $f(z) \in SD(k,\beta)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n\,(k+1)-(k+\beta)\}\,|a_n| < 1-\beta.$$

Putting $\alpha = 0, A = 1 - 2\beta$, $B = -1$ with $0 \leq \beta < 1$ and $k = 0$ in Theorem 1, we get the following result proved by Silverman [23].

Corollary 4. *A function $f(z) \in S^*(\beta)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n - \beta\} |a_n| < 1 - \beta.$$

Putting $\alpha = 1$, in Theorem 1, we can obtain Corollary 5, below which is comparable to the result obtained by Noor and Malik [15].

Corollary 5. *A function $f \in k - UCV[A, B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n \{2(k+1)(n-1) + |n(B+1) + (A+1)|\} |a_n| < |B - A|.$$

The following is an inclusion result stating the fact that $k - UM_\alpha[A, B] \subset k - ST[A, B]$.

Theorem 2. *Let $f(z) \in k - UM_\alpha[A, B]$. Then $f(z) \in k - ST[A, B]$.*

Proof. Let $f(z) \in k - UM_\alpha[A, B]$ and let

$$\frac{zf'(z)}{f(z)} = p(z), \tag{3.8}$$

where $p(z)$ is analytic in E with $p(0) = 1$.

Differentiating logarithmically we have

$$\frac{(zf'(z))'}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}. \tag{3.9}$$

Using (3.8) and (3.9), we have

$$J(\alpha, f; z) = p(z) + \frac{\alpha z p'(z)}{p(z)}.$$

Since $f(z) \in k - UM_\alpha[A, B]$, so we obtain

$$J(\alpha, f; z) = p(z) + \frac{zp'(z)}{\frac{1}{\alpha} p(z)} \in k - UM_\alpha[A, B].$$

Since $\Re\left(\left(\frac{2k+1-A}{2k+1-B}\right)\frac{1}{\alpha}\right) > 0$, $z \in E$, therefore applying Lemma 3, with $\delta = \frac{1}{\alpha}$ and $\sigma = 0$, we have

$$\frac{zf'(z)}{f(z)} = p(z) \prec p_k(A, B, z),$$

which implies that $f(z) \in k - ST[A, B]$. □

By giving special values to the parameters in Theorem 2, we get the following well-known result proved by Mocanu in [13].

Corollary 6. *Let $f(z) \in 0 - UM_\alpha[1, -1]$. Then $f(z) \in 0 - ST[1, -1]$. That is*

$$M_\alpha \subset S^*, \quad \alpha \geq 0.$$

Theorem 3. *If* $0 \leq \alpha_1 < \alpha_2$, *then*

$$k - UM_{\alpha_2}[A, B] \subset k - UM_{\alpha_1}[A, B].$$

Proof. Let $f(z) \in k - UM_{\alpha_2}[A, B]$. Then consider

$$\begin{aligned} J(\alpha_1, f; z) &= \left[(1-\alpha_1)\frac{zf'(z)}{f(z)} + \alpha_1\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \\ &= \left(1 - \frac{\alpha_1}{\alpha_2}\right)\frac{zf'(z)}{f(z)} + \frac{\alpha_1}{\alpha_2}\left[(1-\alpha_2)\frac{zf'(z)}{pf(z)} + \alpha_2\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \\ &= \left(1 - \frac{\alpha_1}{\alpha_2}\right)J(0, f; z) + \frac{\alpha_1}{\alpha_2}\left(J(\alpha_2, f; z)\right). \end{aligned}$$

Now as $f(z) \in k - UM_{\alpha_2}[A, B]$ so

$$J(\alpha_2, f; z) \in k - p[A, B],$$

also from Theorem 2,

$$J(0, f; z) \in k - p[A, B].$$

Using theses along with Lemma 2, we have

$$J(\alpha_1, f; z) \in k - p[A, B],$$

which implies that

$$f(z) \in k - UM_{\alpha_1}[A, B].$$

□

Theorem 4. *A function* $f(z)$ *is in* $k - UM_\alpha[A, B], \alpha > 0$, *if and only if there exists a function* $g(z)$ *belonging to the class* $k - ST[A, B]$, *such that*

$$f(z) = \left[\frac{1}{\alpha}\int_0^z \{g(z)\}^{\frac{1}{\alpha}} t^{-1} dt\right]^\alpha. \tag{3.10}$$

Proof. Let us set

$$g(z) = f(z)\left\{\frac{zf'(z)}{f(z)}\right\}^\alpha,$$

so that (3.10) is satisfied. Logarithmically differentiation gives us

$$\frac{zg'(z)}{g(z)} = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}.$$

Hence $f \in k - UM_\alpha[A, B]$ if and only if $g \in k - ST[A, B]$. □

Theorem 5. *Let the function* $f(z) \in k - UM_\alpha[A, B]$. *Then*

$$|a_2| \leq \frac{(A-B)\delta_k}{2(1+\alpha)},$$

where δ_k *is given by* (1.3).

Proof. Let $f(z) \in k - UM_\alpha[A, B]$. Then

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = p(z) \qquad z \in E,$$

where

$$\begin{aligned} p(z) &\prec \frac{(A+1)p_k(z) - (A-1)}{(B+1)p_k(z) - (B-1)} \\ &= 1 + \frac{1}{2}(A-B)\delta_k z + \cdots, \end{aligned}$$

where $p_k(z) = 1 + \delta_k z + \cdots$.

Now using the definition of subordination we can see that there exists a function $\omega(z)$ analytic in E with $\omega(0)=0$ and $|\omega(z)|<1$ such that

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{f'(z)}=1+\frac{1}{2}(A-B)\delta_k\omega(z)+\cdots$$

$$\begin{aligned} & 1+(1+\alpha)a_2z+\left(2(1+2\alpha)a_3-(1+3\alpha)a_2^2\right)z^2\cdots \\ = \; & 1+\frac{1}{2}(A-B)\delta_k\left(c_1z+c_2z^2+\cdots\right)+\cdots . \end{aligned}$$

Comparing the coefficient of z both sides and using well known result due to Janowski and Lemma 1, we have

$$|(1+\alpha)a_2|\leq\frac{1}{2}(A-B)\delta_k.$$

This gives

$$|a_2|\leq\frac{(A-B)\delta_k}{2(1+\alpha)}$$

and the proof is complete. □

Taking $\alpha=0, A=1,\ B=-1$ in Theorem 5, we can obtain the following result proved in [5].

Corollary 7. *Let $f\in k-ST$. Then*

$$|a_2|\leq\delta_k,$$

where δ_k is given by (1.3).

Putting $k=0, \delta_k=2,\ \alpha=0, A=1, B=-1$ in Theorem 5, we can obtain Corollary 8 below which is the result obtained in [2].

Corollary 8. *Let $f\in S^*$. Then*

$$|a_2|\leq 2.$$

Putting $\alpha=1, A=1,\ B=-1$ in Theorem 5, we can obtain Corollary 9 below which is comparable to the result obtained in [4].

Corollary 9. *Let $f\in k-UCV$. Then*

$$|a_2|\leq\frac{\delta_k}{2},$$

where δ_k is given by (1.3).

Putting $k=0, \delta_k=2,\ \alpha=0, A=1,\ B=-1$ in Theorem 5, we can obtain Corollary 10 below which is the result obtained in [2].

Corollary 10. *Let $f\in\mathcal{C}$. Then*

$$|a_2|\leq 1.$$

Theorem 6. *The range of every univalent functions $f\in k-UM_\alpha[A,B]$, contains the unit disk*

$$R_{\alpha,\delta_k}=\frac{2(1+\alpha)}{4(1+\alpha)+(A-B)\delta_k},$$

where δ_k is given by (1.3).

Proof. Let $\omega_\circ$ be any complex number such that $f(z)\neq\omega_\circ$. Then

$$\frac{\omega_\circ f(z)}{\omega_\circ-f(z)}=z+\left(a_2+\frac{1}{\omega_\circ}\right)z^2+\cdots,$$

is univalent in E so that

$$\left|a_2+\frac{1}{\omega_\circ}\right|\leq 2.$$

Therefore

$$\left|\frac{1}{\omega_\circ}\right|\leq\frac{4(1+\alpha)+(A-B)\delta_k}{2(1+\alpha)}.$$

Hence using Theorem 5, we have

$$|\omega_\circ| \leq \frac{2(1+\alpha)}{4(1+\alpha)+(A-B)\delta_k} = R_{\alpha,\delta_k}.$$

□

Putting $\alpha = 0, A = 1, B = -1$ in Theorem 6, we can obtain Corollary 11.

Corollary 11. *The range of every univalent functions $f \in k - ST$ contains the unit disk*

$$R_{\delta_k} = \frac{1}{2+\delta_k},$$

where δ_k is given by (1.3).

Putting $\alpha = 1, A = 1, B = -1$ in Theorem 6, we can obtain Corollary 12.

Corollary 12. *The range of every univalent functions $f \in k - UCV$ contains the unit disk*

$$R_{\delta_k} = \frac{2}{4+\delta_k},$$

where δ_k is given by (1.3).

Letting $k = 1, A = 1$ and $B = -1$, we have the following Theorem.

Theorem 7. *Let $f \in UM_\alpha$ and let it be of the form*

$$f(z) = z + \sum_{n=m+1}^{\infty} a_n z^n \quad a_{m+1} \neq 0.$$

Then $f(z)$ is strongly starlike of order $\theta_\circ$, where

$$\theta_\circ = \min_{\theta\in(0,1)} \left\{ 1 - 2x^\theta \cos\left(\frac{\theta\pi}{2}\right) + \left(\frac{\alpha m\left(x^2+1\right)\theta}{2x} + x^\theta \sin\left(\frac{\theta\pi}{2}\right)\right) \geq 0 \quad \textit{for all } x > 0 \right\}. \tag{3.11}$$

Proof. From the assumption we have

$$\Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right\} > \left|(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} - 1\right|. \tag{3.12}$$

Let $p(z) = \frac{zf'(z)}{f(z)}$, then by $p(z)$ has of the form

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n,$$

and (3.12) , becomes

$$\Re\left\{p(z) + \alpha\frac{z_\circ p'(z_\circ)}{p(z)}\right\} > \left|p(z) + \alpha\frac{z_\circ p'(z_\circ)}{p(z)} - 1\right|. \tag{3.13}$$

If there exists a point $z_\circ$, $|z_\circ| < 1$, such that

$$|\arg\{p(z)\}| < \frac{\pi}{2}\theta \quad \text{for} \quad |z| < |z_\circ|,$$

and

$$|\arg p(z_\circ)| = \frac{\pi}{2}\theta.$$

Then, applying Lemma 4, we have

$$\frac{z_\circ p'(z_\circ)}{p(z_\circ)} = il\theta,$$

where $(p(z_\circ))^{\frac{1}{\theta}} = \pm ix \quad (x > 0)$,

$$l \geq \frac{m}{2}(x + \frac{1}{x}) \qquad \text{when } \arg\{p(z_\circ)\} = \frac{\pi}{2}\theta,$$

and

$$l \leq -\frac{m}{2}(x + \frac{1}{x}) \qquad \text{when } \arg\{p(z_\circ)\} = -\frac{\pi}{2}\theta.$$

Therefore, for the case $\arg\{p(z_\circ)\} = \frac{\pi}{2}\theta$, we have

$$\Re\left(p(z_\circ) + \alpha\frac{z_\circ p'(z_\circ)}{p(z_\circ)}\right) = \Re\left\{(ix)^\theta + \alpha l\theta\right\} = x^\theta\cos\left(\frac{\theta\pi}{2}\right), \tag{3.14}$$

and

$$\begin{aligned}\left|p(z_\circ) + \alpha\frac{z_\circ p'(z_\circ)}{p(z_\circ)} - 1\right| &= \left|(ix)^\theta + i\alpha l\theta - 1\right| \\ &= \left|x^\theta\cos\left(\frac{\theta\pi}{2}\right) - 1 + i\left(\alpha l\theta + x^\theta\sin\left(\frac{\theta\pi}{2}\right)\right)\right| \\ &= \sqrt{\left(x^\theta\cos\left(\frac{\theta\pi}{2}\right) - 1\right)^2 + \left(\alpha l\theta + x^\theta\sin\left(\frac{\theta\pi}{2}\right)\right)^2}.\end{aligned} \tag{3.15}$$

From (3.11) and then from $l \geq \frac{m}{2}(x + \frac{1}{x})$ for $\theta \geq \theta_\circ$, we have

$$\begin{aligned} 0 &\leq 1 - 2x^\theta\cos\left(\frac{\theta\pi}{2}\right) + \left(\frac{\alpha\theta m}{2x}(x^2+1) + x^\theta\sin\left(\frac{\theta\pi}{2}\right)\right)^2 \\ &\leq 1 - 2x^\theta\cos\left(\frac{\theta\pi}{2}\right) + \left(\alpha\theta l + x^\theta\sin\left(\frac{\theta\pi}{2}\right)\right)^2. \end{aligned} \tag{3.16}$$

Therefore,

$$0 \leq 1 - 2x^{\theta_\circ}\cos\left(\frac{\theta_\circ\pi}{2}\right) + \left(\alpha\theta_\circ l + x^{\theta_\circ}\sin\left(\frac{\theta_\circ\pi}{2}\right)\right)^2, \tag{3.17}$$

by (3.14) and (3.15) is equivalent to the inequality

$$\Re\left\{p(z_\circ) + \alpha\frac{z_\circ p'(z_\circ)}{p(z_\circ)}\right\} \leq \left|p(z_\circ) + \alpha\frac{z_\circ p'(z_\circ)}{p(z_\circ)} - 1\right|,$$

which contradicts with (3.11). Therefore, $|\arg\{p(z_\circ)\}| < \frac{\pi}{2}\theta_\circ$ for $|z| < 1$.

For the case $\arg\{p(z_\circ)\} = -\frac{\pi}{2}\theta_\circ$, applying the same method as the above we will get a contradiction. In this way we have proved that f is strongly starlike of order $\theta_\circ$. This completes the proof. □

Letting $\alpha = 1$, in Theorem 7, we have the result 13 below which is comparable to the one obtained in [18].

Corollary 13. *Let $f \in UCV$ and let it be of the form*

$$f(z) = z + \sum_{n=m+1}^{\infty} a_n z^n \qquad a_{m+1} \neq 0.$$

Then $f(z)$ is strongly starlike of order $\theta_\circ$, where

$$\theta_\circ = \min_{\theta\in(0,1)}\left\{1 - 2x^\theta\cos\left(\frac{\theta\pi}{2}\right) + \left(m\frac{(x^2+1)\,\theta}{2x} + x^\theta\sin\left(\frac{\theta\pi}{2}\right)\right) \geq 0 \quad \textit{for all } x > 0\right\}.$$

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