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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR THE SYSTEM OF NONLINEAR FRACTIONAL ORDER BOUNDARY VALUE PROBLEM

SABBAVARAPU NAGESWARA RAO*

ABSTRACT. This paper is concerned with boundary value problems for system of nonlinear fractional differential equations involving the Caputo fractional derivatives

$$\begin{cases} {}^cD^{q_1}u(t) + f_1(t, u(t), v(t)) = 0, \ t \in [0, 1], \\ {}^cD^{q_2}v(t) + f_2(t, u(t), v(t)) = 0, \ t \in [0, 1], \\ u(0) - \alpha u'(0) = u'(\eta) = \beta u(1) + \gamma u''(1) = 0, \\ v(0) - \alpha v'(0) = v'(\eta) = \beta v(1) + \gamma v''(1) = 0, \end{cases}$$

where ${}^cD^{q_1}$ and ${}^cD^{q_2}$ are the standard Caputo fractional derivatives of orders q_1 and q_2 respectively, with $2 < q_1, q_2 \le 3$. The functions $f_i : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous for $i = 1, 2$, $\alpha > 0, \beta > 0, \gamma > 0, \eta \in (0, 1)$. Under the suitable conditions, the existence and multiplicity of positive solutions are established by using abstract fixed point theorems.

1. INTRODUCTION

In recent years, the study of fractional order differential equations has emerged as an important area of mathematics. It has wide range of applications in various fields of science and engineering such as physics, mechanics, control systems, flow in porous media, electromagnetics and viscoelasticity. There has been much attention paid in developing the theory of existence of positive solutions for fractional order differential equations satisfying initial (or) boundary conditions to mention a few references [15, 16, 18, 24]. To mention a few references much interest has been created in establishing positive solutions and multiple positive solutions for two-point, multi-point fractional order boundary value problems (BVPs). To mention the related papers along these lines $-$ see, Bai and Sun [2], Bai, Sun and Zhang [3], Bai and Lü [4], Chai [5], Goodrich [10], Liang and Zhang [17], Nageswararao [21], Prasad and Krushna [23], and Tian and Liu [26].

Motivated by above papers, in this paper we are concerned with the existence of multiple positive solutions to the couple system of nonlinear fractional order differential equations

$$\begin{aligned} {}^cD^{q_1}u(t) + f_1(t, u(t), v(t)) = 0, \ t \in [0, 1], \\ {}^cD^{q_2}v(t) + f_2(t, u(t), v(t)) = 0, \ t \in [0, 1], \end{aligned} \tag{1}$$

with the three-point boundary conditions

$$\begin{aligned} u(0) - \alpha u'(0) = u'(\eta) = \beta u(1) + \gamma u''(1) = 0, \\ v(0) - \alpha v'(0) = v'(\eta) = \beta v(1) + \gamma v''(1) = 0, \end{aligned} \tag{2}$$

where ${}^cD^{q_1}$ and ${}^cD^{q_2}$ are the Caputo fractional derivatives of orders q_1 and q_2 respectively, with $2 < q_1, q_2 \le 3$. The functions $f_i : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous for $i = 1, 2$, $\alpha > 0, \beta > 0, \gamma > 0$, and $\eta \in (0, 1)$.

By a positive solution of the fractional order boundary value problem (1)-(2), we understand a pair of functions $(u, v) \in C([0, 1]) \times C([0, 1])$ satisfying (1)-(2) with $u(t) \ge 0, \ v(t) \ge 0$ for all $t \in [0, 1]$, and $\sup_{t\in[0,1]} u(t) > 0, \ \sup_{t\in[0,1]} v(t) > 0$.

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The rest of this paper is organized as follows, In Section 2, we present some definitions and background results. For sake of convenience, we also state the fixed point theorems. In Section 3, we construct the Green's function for the homogeneous BVP corresponding to (1)-(2), and estimate the bounds for the Green's function. In Section 4, we establish the existence and multiplicity positive solutions of the BVP (1)-(2). In Section 5, some examples are given to illustrate our existence results.

We assume the following conditions hold throughout the paper:

(A1) The functions $f_i : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and $f_i(t,0,0) \equiv 0,$ for $1 \le i \le 2$;

(A2) $\alpha > 0,\ \beta > 0,\ 0 < \gamma < \beta(1-2\eta(1+\alpha))$ and $1-2\eta(1+\alpha) > 0$;

(A3) $\lim_{u+v\to 0^+} \sup_{t\in[0,1]} \frac{f_1(t,u,v)}{u+v} = 0,\ \lim_{u+v\to 0^+} \sup_{t\in[0,1]} \frac{f_2(t,u,v)}{u+v} = 0$;

(A4) $\lim_{u+v\to \infty} \inf_{t\in[0,1]} \frac{f_1(t,u,v)}{u+v} = \infty,\ \lim_{u+v\to \infty} \inf_{t\in[0,1]} \frac{f_2(t,u,v)}{u+v} = \infty$;

(A5) $\lim_{u+v\to 0^+} \inf_{t\in[0,1]} \frac{f_1(t,u,v)}{u+v} = \infty,\ \lim_{u+v\to 0^+} \inf_{t\in[0,1]} \frac{f_2(t,u,v)}{u+v} = \infty$;

(A6) $\lim_{u+v\to \infty} \sup_{t\in[0,1]} \frac{f_1(t,u,v)}{u+v} = 0,\ \lim_{u+v\to \infty} \sup_{t\in[0,1]} \frac{f_2(t,u,v)}{u+v} = 0$;

(A7) For each $t \in [0,1]$, $f_i(t,u,v)$ are nondecreasing with respect to u, v and there exists a constant $N > 0$ such that $f_i(t,u,v) < \frac{N}{2N'}$, for $1 \le i \le 2$, where $N' = \frac{(1-2\eta(1+\alpha))(\beta+2\gamma)}{2d}$.

2. **Preliminaries**

In this section, we recall some definitions and properties of the fractional calculus. We also state a fixed point theorem of Krasnosel'skii [14] is yield the existence of positive and multiple positive solutions.

Definition: For a continuous function $f : [0,\infty) \to R$, the Caputo derivative of fractional order q is defined by

$$ {}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds,\ n-1 < q < n,\ n = [q]+1 $$

provided that $f^{(n)}(t)$ exists, where $[q]$ denotes the integer part of the real number q

Definition: The Riemann-Liouville fractional integral of order q for a continuous function $f(t)$ is defined as

$$ I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds,\ q > 0 $$

provided that such integral exists.

Definition: The Riemann-Liouville fractional derivative of order q for a continuous function $f(t)$ is defined by

$$ D^q f(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-q-1} f(s)ds,\ n = [q]+1 $$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Furthermore, we note that the Riemann-Liouville fractional derivative of a constant is usually nonzero which can cause serious problems in real world applications. Actually, the relationship between the two-types of fractional derivative is as follows

$$ \begin{aligned} {}^cD^q f(t) &= \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds \\ &= D^q f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(s)}{\Gamma(k-q+1)} t^{k-q} \\ &= D^q \Big[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}}{k!} t^k \Big],\ t > 0, n-1 < q < n. \end{aligned} $$

So, we prefer to use Caputos definition which gives better results than those of Riemann- Liouville.

Lemma 2.1[25] Let $q > 0$, then the fractional differential equation ${}^cD^q u(t) = 0$ has solution $u(t) =$

$c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},\ c_i \in R,\ i = 0, 1, 2, \cdots n-1$ where n is the smallest integer greater than or equal to q.

Lemma 2.2[25] Let $q > 0$, then $I^{q\,c}D^q u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},\ \textit{for some}\ c_i \in R,\ i = 0, 1, 2, \cdots n-1$ where n is the smallest integer greater than or equal to q.

Theorem 2.1([6, 9, 14]) Let $(E, \|\cdot\|)$ be a Banach space, and let $P \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. If $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous operator such that either

(i) $\| Tu \| \le \| u \|,\ u \in P \cap \partial\Omega_1$ and $\| Tu \| \ge \| u \|,\ u \in P \cap \partial\Omega_2$,($or$)

(ii) $\| Tu \| \ge \| u \|,\ u \in P \cap \partial\Omega_1$ and $\| Tu \| \le \| u \|,\ u \in P \cap \partial\Omega_2$,

holds. Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 2.2([6, 9, 14]) Let $(E, \|\cdot\|)$ be a Banach space, and let $P \subset E$ be a cone in E. Assume that $\Omega_1,\ \Omega_2$ and Ω_3 are open bounded subsets of E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2,\ \overline{\Omega}_2 \subset \Omega_3$. If $T : P \cap (\overline{\Omega}_3 \backslash \Omega_1) \to P$ is completely continuous operator such that:

(i) $\| Tu \| \ge \| u \|,\ \forall u \in P \cap \partial\Omega_1$;

(ii) $\| Tu \| \le \| u \|,\ Tu \ne u,\ \forall u \in P \cap \partial\Omega_2$;

(iii) $\| Tu \| \ge \| u \|,\ \forall u \in P \cap \partial\Omega_3$,

then T has at least two fixed points $x^*,\ x^{**}$ in $P \cap (\overline{\Omega}_3 \backslash \Omega_1)$, and furthermore $x^* \in P \cap (\overline{\Omega}_2 \backslash \Omega_1),\ x^{**} \in P \cap (\overline{\Omega}_3 \backslash \Omega_2)$.

3. Green's Function and Bounds

In this section, we construct the Green's function and bounds for the homogeneous boundary value problem corresponding (1)-(2) that will be used to prove our main theorems.

Lemma 3.1 Let $d = \beta + 2\gamma - 2\eta\beta(1+\alpha) > 0$. If $h \in C[0,1]$, then the fractional order boundary value problem

$$^cD^{q_1} u(t) + h(t) = 0,\ 0 < t < 1, \tag{3}$$

$$u(0) - \alpha u'(0) = u'(\eta) = \beta u(1) + \gamma u''(1) = 0 \tag{4}$$

has a unique solution

$$u(t) = \int_0^1 G_1(t,s) h(s) ds,$$

where $G_1(t,s)$ is the Green's function for the problem (3)-(4) and is given by

$$G_1(t,s) = \begin{cases} \underset{t\in[0,\eta]}{G_1(t,s)} = \begin{cases} G_{11}(t,s), & 0 \le t \le s \le \eta < 1, \\ G_{12}(t,s), & 0 \le s \le \min\{t,\eta\} < 1, \end{cases} \\ \\ \underset{t\in[\eta,1]}{G_1(t,s)} = \begin{cases} G_{13}(t,s), & 0 \le \max\{t,\eta\} \le \mathrm{s} \le 1, \\ G_{14}(t,s), & 0 < \eta \le s \le t \le 1, \end{cases} \end{cases} \tag{5}$$

$$
\begin{aligned}
G_{11}(t,s) =& \frac{1}{d}\Big[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\\
&-(1+\alpha)\Big(\beta\big((t-\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}\Big]\\
G_{12}(t,s) =& \frac{1}{d}\Big[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\\
&-(1+\alpha)\Big(\beta\big((t-\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}\Big]-\frac{(t-s)^{q_1-1}}{\Gamma(q_1)}\\
G_{13}(t,s) =& \frac{1}{d}\Big[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\Big]\\
G_{14}(t,s) =& \frac{1}{d}\Big[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\Big]-\frac{(t-s)^{q_1-1}}{\Gamma(q_1)}
\end{aligned}
$$

Proof. Assume that $u \in C^{[q_1]+1}[0,1]$ is a solution of fractional order boundary value problem by (3)-(4) and is uniquely expressed as $I^{q_1}\,{}^cD^{q_1}u(t) = -I^{q_1}h(t)$, so that

$$u(t) = \frac{-1}{\Gamma(q_1)}\int_0^t (t-s)^{q_1-1}h(s)ds + c_1 + c_2 t + c_3 t^2.$$

Using the boundary conditions (4), we obtain that

$$
\begin{aligned}
c_1 =& \frac{-2\alpha\eta}{d}\int_0^1\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)h(s)ds\\
&+\frac{1}{d}\Big(2\eta\beta(1+\alpha)-\alpha(\beta+2\gamma-2\eta\beta(1+\alpha))\Big)\int_0^\eta\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}h(s)ds\\
c_2 =& \frac{-2\eta}{d}\int_0^1\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)h(s)ds\\
&+\frac{1}{d}\Big(2\eta\beta(1+\alpha)-(\beta+2\gamma-2\eta\beta(1+\alpha))\Big)\int_0^\eta\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}h(s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
c_3 =& \frac{1}{d}\int_0^1\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)h(s)ds\\
&-\frac{1}{d}\big(\beta(1+\alpha)\big)\int_0^\eta\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}h(s)ds.
\end{aligned}
$$

Hence, the unique solution of (3) and (4) is

$$
\begin{aligned}
u(t) =& \frac{1}{d}\Big[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\int_0^1\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)h(s)ds\Big]\\
&-\frac{1}{d}\Big[(1+\alpha)\Big(\beta\big((t-\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\int_0^\eta\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}h(s)ds\Big]\\
&-\frac{1}{\Gamma(q_1)}\int_0^t(t-s)^{q_1-1}h(s)ds\\
=& \int_0^1 G_1(t,s)h(s)ds
\end{aligned}
$$

where $G_1(t,s)$ is given in (5).

Lemma 3.2 Assume that the condition $(A2)$ is satisfied. Then the Green's function $G_1(t,s)$ given in (5) is nonnegative, for all $(t,s)\in[0,1]\times[0,1]$.

Proof. Consider the Green's function $G_1(t,s)$ given by (5)
Let $0 \le t \le s \le \eta < 1$. Then

$$\begin{aligned}
G_{11}(t,s) &= \frac{1}{d}\bigg[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big) \\
&\quad -(1+\alpha)\Big(\beta\big((t-\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}\bigg] \\
&\ge \frac{1}{d}\bigg[\Big((t-t\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta}{\Gamma(q_1)}+\frac{\gamma(1-s)^{-2}}{\Gamma(q_1-2)}\Big)(1-s)^{q_1-1} \\
&\quad -(1+\alpha)\Big(\beta\big((t-t\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\frac{(\eta-\eta s)^{q_1-2}}{\Gamma(q_1-1)}\bigg] \\
&= \frac{1}{d}\bigg[\Big(t^2(1-\eta)^2-(\eta+\alpha)^2+\alpha^2\Big)\Big(\frac{\beta}{\Gamma(q_1)}+\frac{\gamma(1+2s+O(s^2))}{\Gamma(q_1-2)}\Big) \\
&\quad -(1+\alpha)\Big(\beta\big(t^2(1-\eta)^2+(1+\eta)^2-1\big)+d\Big)\frac{\eta^{q_1-2}(1+s+s^2)}{\Gamma(q_1-1)}\bigg](1-s)^{q_1-1} \ge 0
\end{aligned}$$

Let $0 \le s \le \min\{t,\eta\} < 1$. Then

$$\begin{aligned}
G_{12}(t,s) &= \frac{1}{d}\bigg[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big) \\
&\quad -(1+\alpha)\Big(\beta\big((t-\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}\bigg]-\frac{(t-s)^{q_1-1}}{\Gamma(q_1)} \\
&\ge \frac{1}{d}\bigg[\Big((t-t\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big) \\
&\quad -(1+\alpha)\Big(\beta\big((t-t\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\frac{(\eta-\eta s)^{q_1-2}}{\Gamma(q_1-1)}\bigg]-\frac{(t-ts)^{q_1-1}}{\Gamma(q_1)} \\
&= \frac{1}{d}\bigg[\Big(t^2(1-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta}{\Gamma(q_1)}+\frac{\gamma(1+2s+O(s^2))}{\Gamma(q_1-2)}\Big) \\
&\quad -(1+\alpha)\Big(\beta\big(t^2(1-\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\frac{\eta^{q_1-1}}{\Gamma(q_1-1)}-\frac{dt^{q_1-1}}{\Gamma(q_1)}\bigg]\times \\
&\quad \times(1-s)^{q_1-1} \ge 0
\end{aligned}$$

Let $0 \le \max\{t,\eta\} \le s \le 1$. Then

$$\begin{aligned}
G_{13}(t,s) &= \frac{1}{d}\bigg[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\bigg] \\
&\ge \frac{1}{d}\bigg[\Big(t^2(1-\eta)^2-(\eta+\alpha)^2+\alpha^2\Big)\Big(\frac{\beta}{\Gamma(q_1)}+\frac{\gamma(1+2s+O(s^2))}{\Gamma(q_1-2)}\Big)\bigg]\times \\
&\quad \times(1-s)^{q_1-1} \ge 0
\end{aligned}$$

Let $0 < \eta \le s \le t \le 1$. Then

$$\begin{aligned}
G_{14}(t,s) &= \frac{1}{d}\bigg[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\bigg]-\frac{(t-s)^{q_1-1}}{\Gamma(q_1)} \\
&\ge \frac{1}{d}\bigg[\Big(t^2(1-\eta)^2-(\eta+\alpha)^2+\alpha^2\Big)\Big(\frac{\beta}{\Gamma(q_1)}+\frac{\gamma(1+2s+O(s^2))}{\Gamma(q_1-2)}\Big)-\frac{dt^{q_1-1}}{\Gamma(q_1)}\bigg]\times \\
&\quad \times(1-s)^{q_1-1} \ge 0
\end{aligned}$$

Lemma 3.3 Assume that the condition $(A2)$ is satisfied. Then the Green's function satisfies the following inequality,

$$m_1 G_1(1,s) \le G_1(t,s) \le G_1(1,s), \; \textit{for all } (t,s) \in [0,1]\times[0,1], \tag{6}$$

where $0 < m_1 = \min\Big\{\frac{\eta^2}{1-2\eta(1+\alpha)}, \frac{2\alpha\eta\gamma}{\eta^2+\eta\alpha\beta+2\gamma}, \frac{2\gamma\eta}{2\gamma(1+\eta)+\beta\big(1-2\eta(1+\alpha)\big)}\Big\} < 1.$

Proof. Consider the Green's function $G_1(t,s)$ is given in (5).
Case (i): For $0 \le \max\{t,\eta\} \le s \le 1$

$$\frac{G_{13}(t,s)}{G_{13}(1,s)} = \frac{\frac{1}{d}\Big[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\Big]}{\frac{1}{d}\Big[\Big((1-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\Big]}$$

we have $G_{13}(t,s) \le G_{13}(1,s)$. And also from $(A2)$, we have

$$\begin{aligned}\frac{G_{13}(t,s)}{G_{13}(1,s)} &= \frac{\frac{1}{d}\Big[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\Big]}{\frac{1}{d}\Big[\Big((1-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\Big]}\\ &\ge \frac{\eta^2}{1-2\eta(1+\alpha)}\end{aligned}$$

Case (ii): For $0 \le \eta \le s \le t < 1$
From $(A2)$ and case (i),we have $G_{14}(t,s) \le G_{14}(1,s)$. And also, we have

$$\begin{aligned}\frac{G_{14}(t,s)}{G_{14}(1,s)} &= \frac{\frac{1}{d}\Big[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\Big]-\frac{(t-s)^{q_1-1}}{\Gamma(q_1)}}{\frac{1}{d}\Big[\Big((t-\eta)^2-\eta^2-2\alpha\eta\Big)\Big(\frac{\beta(1-s)^{q_1-1}}{\Gamma(q_1)}+\frac{\gamma(1-s)^{q_1-3}}{\Gamma(q_1-2)}\Big)\Big]-\frac{(t-s)^{q_1-1}}{\Gamma(q_1)}}\\ &\ge \frac{2\gamma\eta}{2\gamma(1+\eta)+\beta\big(1-2\eta(1+\alpha)\big)}\end{aligned}$$

Case (iii): For $0 \le t \le s \le \eta < 1$.
From $(A2)$ and case (i), we have $G_{11}(t,s) \le G_{11}(1,s)$. And also, from $(A2)$, we have

$$\begin{aligned}\frac{G_{11}(t,s)}{G_{11}(1,s)} &= \frac{G_{13}(t,s)-\frac{1}{d}(1+\alpha)\Big(\beta\big((t-\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}}{G_{13}(1,s)-\frac{1}{d}(1+\alpha)\Big(\beta\big((1-\eta)^2+(1+\eta)^2\big)+2\gamma-2\beta\eta(1+\alpha)\Big)\frac{(\eta-s)^{q_1-2}}{\Gamma(q_1-1)}}\\ &\ge \frac{2\alpha\eta\gamma}{\eta^2+\alpha\beta\eta+2\gamma}\end{aligned}$$

Case (iv): For $0 \le s \le \min\{t,\eta\} < 1$
From $(A2)$ and case (iii), we have $G_{12}(t,s) \le G_{12}(1,s)$. And also, from $(A2)$, we have

$$\begin{aligned}\frac{G_{12}(t,s)}{G_{12}(1,s)} &= \frac{G_{11}(t,s)-[\beta+2\gamma-2\eta\beta(1+\alpha)]\frac{(t-s)^{q_1-1}}{\Gamma(q_1)}}{G_{11}(1,s)-[\beta+2\gamma-2\eta\beta(1+\alpha)]\frac{(1-s)^{q_1-1}}{\Gamma(q_1)}}\\ &\ge \frac{2\gamma\eta}{2\gamma(1+\eta)+\beta\big(1-2\eta(1+\alpha)\big)}\end{aligned}$$

By above all cases, we get

$$m_1G_1(1,s) \le G_1(t,s) \le G_1(1,s), \text{ for all } (t,s) \in [0,1]\times[0,1],$$

where $0 < m_1 = \min\Big\{\frac{\eta^2}{1-2\eta(1+\alpha)}, \frac{2\alpha\eta\gamma}{\eta^2+\eta\alpha\beta+2\gamma}, \frac{2\gamma\eta}{2\gamma(1+\eta)+\beta\big(1-2\eta(1+\alpha)\big)}\Big\} < 1$.

We can also formulate similar results as Lemma (3.1) - Lemma (3.3) above, for the fractional boundary value problem

$$^{c}D^{q_2}v(t)+y(t)=0,\ 0<t<1, \tag{7}$$

$$v(0)-\alpha v'(0)=v'(\eta)=\beta v(1)+\gamma v''(1)=0 \tag{8}$$

where ${}^cD^{q_2}$ is the Caputo fractional derivative of order q_2 with $2 < q_2 \le 3$, $\alpha > 0, \beta > 0, \gamma > 0, \eta \in (0,1)$. We denote by G_2 and m_2 the corresponding Green's function and constant for the problem (7)-(8) defined in a similar manner as G_1 and m_1 respectively.

By using Green functions G_1 and G_2 our problem (1)-(2) can be written equivalently as the following nonlinear system of integral equations

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s) f_1(s, u(s), v(s)) ds, \ t \in [0,1], \\ v(t) = \displaystyle\int_0^1 G_2(t,s) f_2(s, u(s), v(s)) ds, \ t \in [0,1]. \end{cases}$$

We consider the Banach space $E = C([0,1])$ with supremum norm $\| \cdot \|$, and the Banach space $B = E \times E$ with the norm $\| (u,v) \| = \| u \| + \| v \|$. We define the cone $P \subset B$ by

$$P = \left\{ (u,v) \in B; u(t) \ge 0, \ v(t) \ge 0, \forall t \in [0,1], \text{ and } \inf_{t \in \ [\eta,1]} (u(t) + v(t)) \ge m \| (u,v) \| \right\},$$

where $m = \min\{m_1, m_2\}$.

We introduce the operators $T_1, \ T_2 : P \to B$ and $T : P \to B$ defined by

$$T_1(u,v)(t) = \int_0^1 G_1(t,s) f_1(s, u(s), v(s)) ds, \ t \in [0,1],$$
$$T_2(u,v)(t) = \int_0^1 G_2(t,s) f_2(s, u(s), v(s)) ds, \ t \in [0,1].$$

$$T(u,v) = \big(T_1(u,v), T_2(u,v)\big), \ (u,v) \in P. \tag{9}$$

The solutions of our problem (1)-(2) are the fixed points of the operator T.

Lemma 3.4 If $(A1) - (A2)$ hold, then $T : P \to P$ is a completely continuous operator.

Proof. Let $(u,v) \in P$ be an arbitrary element. Because $T_1(u,v)$ and $T_2(u,v)$ satisfy the problem (3)-(4) for $h(t) = f_1(t, u(t), v(t)), t \in [0,1]$, and the problem (7)-(8) for $y(t) = f_2(t, u(t), v(t)), t \in [0,1]$ respectively, then by Lemma 3, we obtain

$$\inf_{t \in [\eta,1]} T_1(u,v)(t) \ge m_1 \max_{t \in [0,1]} T_1(u,v)(t) = m_1 \| T_1(u,v) \|,$$
$$\inf_{t \in [\eta,1]} T_2(u,v)(t) \ge m_2 \max_{t \in [0,1]} T_2(u,v)(t) = m_2 \| T_2(u,v) \| .$$

Hence, we conclude

$$\begin{aligned} \inf_{t \in [\eta,1]} \Big[T_1(u,v)(t) + T_2(u,v)(t) \Big] & \\ &\ge \inf_{t \in [\eta,1]} T_1(u,v)(t) + \inf_{t \in [\eta,1]} T_2(u,v)(t) \\ &\ge m_1 \| T_1(u,v) \| + m_2 \| T_2(u,v) \| \\ &\ge m \| (T_1(u,v), T_2(u,v)) \| = m \| T(u,v) \| . \end{aligned}$$

Clearly, we obtain $T_1(u,v)(t) \ge 0, \ T_2(u,v)(t) \ge 0$ for all $t \in [0,1]$, and so, we deduce that $T(u,v) \in P$. Hence, we get $T(P) \subset P$. By using standard arguments involving the Arzela-Ascoli theorem, we can easily show that T_1 and T_2 are completely continuous, and then T is a completely continuous operator from P to P.

4. Existence of Multiple Positive Solutions

In this section, we establish the existence of at least one and two positive solutions for the BVP (1)-(2) by using abstract fixed point theorems [6, 9, 14].

Theorem 4.1 Assume that $(A1) - (A4)$ are hold, then the BVP (1)-(2) has at least one positive solution $(u(t), v(t)), \ t \in [0,1]$.

Proof. From assumption $(A3)$ we deduce that there exists $H_1 > 0$ such that for all $t \in [0,1], u, v \in R^+$ with $0 \leq u + v \leq H_1$, we have $f_1(t,u,v) \leq \eta(u+v),\ f_2(t,u,v) \leq \eta'(u+v)$, where η and η' are satisfy

$$\eta \int_0^1 G_1(1,t)dt \leq \frac{1}{2} \text{ and } \eta' \int_0^1 G_2(1,t)dt \leq \frac{1}{2}.$$

We define the set $\Omega_1 = \{(u,v) \in B : \| (u,v) \| < H_1\}$. Now let $(u,v) \in P \cap \partial\Omega_1$, that is $(u,v) \in P$ with $\| (u,v) \| = H_1$ or equivalently $\| u \| + \| v \| = H_1$. Then $u(t) + v(t) \leq H_1$, thus we have

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 G_1(t,s) f_1(s,u(s),v(s))ds \\ &\leq \eta \int_0^1 G_1(1,s)(u(s)+v(s))ds \\ &\leq \eta \int_0^1 G_1(1,s)\big[\, \| u \| + \| v \| \,\big]ds \\ &\leq \frac{1}{2}\big[\, \| u \| + \| v \| \,\big] = \frac{1}{2} \| (u,v) \| \end{aligned}$$

and so, $\| T_1(u,v) \| \leq \frac{1}{2} \| (u,v) \|$. Similarly, we may take

$$\begin{aligned} T_2(u,v)(t) &= \int_0^1 G_2(t,s) f_2(s,u(s),v(s))ds \\ &\leq \eta' \int_0^1 G_2(1,s)(u(s)+v(s))ds \\ &\leq \eta' \int_0^1 G_2(1,s)\big[\, \| u \| + \| v \| \,\big]ds \\ &\leq \frac{1}{2}\big[\, \| u \| + \| v \| \,\big] = \frac{1}{2} \| (u,v) \| \end{aligned}$$

and so, $\| T_2(u,v) \| \leq \frac{1}{2} \| (u,v) \|$. Thus, for $(u,v) \in P \cap \partial\Omega_1$ it follows that

$$\begin{aligned} \| T(u,v) \| &= \| \big(T_1(u,v), T_2(u,v)\big) \| = \| T_1(u,v) \| + \| T_2(u,v) \| \\ &\leq \frac{1}{2} \| (u,v) \| + \frac{1}{2} \| (u,v) \| = \| (u,v) \| . \end{aligned}$$

Therefore,

$$\| T(u,v) \| \leq \| (u,v) \|, \text{ for all } (u,v) \in P \cap \partial\Omega_1. \tag{10}$$

On the other hand, from $(A4)$ there exist four positive constants μ, μ', C_1 and C_2 such that $f_1(t,u,v) \geq \mu(u+v) - C_1,\ f_2(t,u,v) \geq \mu^{'}(u+v) - C_2,\ \forall (u,v) \in R^+ \times R^+$, where μ and μ' satisfy

$$\mu m^2 \int_\eta^1 G_1(1,s)ds \geq 1,\ \mu' m^2 \int_\eta^1 G_2(1,s)ds \geq 1.$$

For $(u,v) \in P,\ \tau \in (0,1)$, we have

$$\begin{aligned} T_1(u,v)(\tau) &= \int_0^1 G_1(\tau,s) f_1(s,u(s),v(s))ds \\ &\geq \int_0^1 G_1(\tau,s)\big[\mu(u+v) - C_1\big]ds \\ &\geq \mu \int_\eta^1 G_1(\tau,s)\big(u(s)+v(s)\big)ds - C_1 \int_\eta^1 G_1(\tau,s)ds \\ &\geq \mu m^2 \int_\eta^1 G_1(1,s)ds\big(\, \| u \| + \| v \| \,\big) - C_1 \int_\eta^1 G_1(\tau,s)ds \\ &\geq \big(\, \| u \| + \| v \| \,\big) - C_1 \int_\eta^1 G_1(\tau,s)ds. \end{aligned}$$

In a similar manner, we deduce

$$
\begin{aligned}
T_2(u,v)(\tau) &= \int_0^1 G_2(\tau,s) f_2(s,u(s),v(s))ds \\
&\geq \int_0^1 G_2(\tau,s)\big[\mu'(u+v) - C_2\big]ds \\
&\geq \mu' \int_\eta^1 G_1(\tau,s)\big(u(s)+v(s)\big)ds - C_2 \int_\eta^1 G_2(\tau,s)ds \\
&\geq \mu' m^2 \int_\eta^1 G_1(1,s)\big(\parallel u \parallel + \parallel v \parallel \big)ds - C_2 \int_\eta^1 G_2(\tau,s)ds \\
&\geq \big(\parallel u \parallel + \parallel v \parallel \big) - C_2 \int_\eta^1 G_2(\tau,s)ds.
\end{aligned}
$$

Therefore $T(u,v)(\tau) \geq 2 \parallel (u,v) \parallel - C_3$, where $C_3 = C_1 \int_\eta^1 G_1(\tau,s)ds + C_2 \int_\eta^1 G_2(\tau,s)ds$. From which it follows that $\parallel T(u,v) \parallel \geq T(u,v)(\tau) \geq \parallel (u,v) \parallel$ as $\parallel (u,v) \parallel \to \infty$.
Let $\Omega_2 = \{(u,v) \in B : \parallel (u,v) \parallel < H_2\}$. Then for $(u,v) \in P$ and $\parallel (u,v) \parallel = H_2 > 0$ sufficiently large, we have

$$
\parallel T(u,v) \parallel \geq \parallel (u,v) \parallel, \text{ for all } (u,v) \in P \cap \partial\Omega_2. \tag{11}
$$

Thus, from (10), (11) and Theorem (2.1), we know that the operator T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 4.2 Assume that $(A1)$, $(A2)$, $(A5)$ and $(A6)$ are hold, then (1)-(2) has at least one positive solution $(u(t), v(t))$, $t \in [0,1]$

Proof. From $(A5)$ there is a number $\hat{H}_3 \in (0,1)$ such that for each $(t,u,v) \in [0,1] \times (0,\hat{H}_3) \times (0,\hat{H}_3)$. One has $f_1(t,u,v) \geq \lambda(u+v)$, where λ satisfy $\lambda m^2 \int_\eta^1 G_1(1,s)ds \geq 1$. From $(A1)$ that implies $f_1(t,0,0) \equiv 0$ and the continuity of $f_1(t,u,v)$, we know that there exists a number $H_3 \in (0,\hat{H}_3)$ small enough such that

$$
f_1(t,u,v) \leq \frac{\hat{H}_3}{\int_0^1 G_1(1,t)dt} \text{ whenever } (t,u,v) \in [0,1] \times (0,H_3) \times (0,H_3).
$$

For every $(u,v) \in P$ and $\parallel (u,v) \parallel = H_3$, note that

$$
\int_0^1 G_1(1,\tau) f_1(\tau,u(\tau),v(\tau))d\tau \leq \int_0^1 G_1(1,\tau) \frac{\hat{H}_3}{\int_0^1 G_1(1,t)dt} d\tau \leq \hat{H}_3.
$$

Thus

$$
\begin{aligned}
T_1(u,v)(\tau) &= \int_0^1 G_1(\tau,s) f_1(s,u,v)ds \\
&\geq m\lambda \int_\eta^1 G_1(1,s)\big(u(s)+v(s)\big)ds \\
&\geq \lambda m^2 \int_\eta^1 G_1(1,s)ds(\parallel u \parallel + \parallel v \parallel) \\
&\geq \big(\parallel u \parallel + \parallel v \parallel \big) = \parallel (u,v) \parallel
\end{aligned}
$$

that is

$$
T_1(u,v)(t) \geq \parallel (u,v) \parallel \text{ for all } t \geq \tau.
$$

So, $\parallel T(u,v) \parallel \geq \parallel T_1(u,v) \parallel \geq \parallel (u,v) \parallel$. If set $\Omega_3 = \{(u,v) \in B : \parallel (u,v) \parallel < H_3\}$, then

$$
\parallel T(u,v) \parallel \geq \parallel (u,v) \parallel, \text{ for all } (u,v) \in P \cap \partial\Omega_3. \tag{12}
$$

On the other hand, we know from $(A6)$ that there exist four positive numbers η, η', C_4 and C_5 such that for every $(t,u,v) \in [0,1] \times R^+ \times R^+$, we have $f_1(t,u,v) \leq \eta(u+v) + C_4$, $f_2(t,u,v) \leq \eta'(u+v) + C_5$, where η and $\eta^{'}$ satisfy

$$\eta\int_0^1 G_1(1,s)ds \leq \frac{1}{2} \text{ and } \eta^{'}\int_0^1 G_2(1,s)ds \leq \frac{1}{2}.$$

Thus we have

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 G_1(t,s)f_1(s,u,v)ds \\ &\leq \int_0^1 G_1(1,s)(\eta(u+v)+C_4)ds \\ &\leq \eta\int_0^1 G_1(1,s)\big(\parallel u\parallel + \parallel v\parallel\big)ds + C_4\int_0^1 G_1(1,s)ds \\ &\leq \frac{1}{2}\parallel (u,v)\parallel + C_4\int_0^1 G_1(1,s)ds. \end{aligned}$$

Similarly, we deduce

$$\begin{aligned} T_2(u,v)(t) &= \int_0^1 G_2(t,s)f_2(s,u,v)ds \\ &\leq \int_0^1 G_2(1,s)(\eta'(u+v)+C_5)ds \\ &\leq \eta'\int_0^1 G_2(1,s)\big(\parallel u\parallel + \parallel v\parallel\big)ds + C_5\int_0^1 G_2(1,s)ds \\ &\leq \frac{1}{2}\parallel (u,v)\parallel + C_5\int_0^1 G_2(1,s)ds. \end{aligned}$$

Therefore $T(u,v)(t) \leq\parallel (u,v)\parallel + C_6$, where $C_6 = C_4\int_0^1 G_1(1,s)ds + C_5\int_0^1 G_2(1,s)ds$, from which it follows that $T(u,v)(t) \leq\parallel (u,v)\parallel$ as $\parallel (u,v)\parallel\to\infty$. Let $\Omega_4 = \{(u,v)\in B :\parallel (u,v)\parallel < H_4\}$. For each $(u,v)\in P$ and $\parallel (u,v)\parallel = H_4 > 0$ large enough, we have

(13) $$\parallel T(u,v)\parallel\leq\parallel (u,v)\parallel, \text{ for all } (u,v)\in P\cap\partial\Omega_4.$$

From (12),(13) and Theorem (2.1), we know that the operator T has a fixed point in $P\cap(\overline{\Omega}_4\setminus\Omega_3)$.

Theorem 4.3 Assume that $(A1),(A2),(A4),(A5)$ and $(A7)$ are satisfied, then (1)-(2) has at least two positive solutions $(u_1(t),v_1(t)),(u_2(t),v_2(t)), t\in[0,1]$.

Proof. Note that we have $G_i(t,s) \leq \frac{(1-2\eta(1+\alpha))(\beta+2\gamma)}{2d} = N'$ for $i=1,2$ for all $(t,s)\in[0,1]\times[0,1]$. Let $B_N = \{(u,v)\in P :\parallel (u,v)\parallel < N\}$. By using $(A7)$, for any $(u,v)\in\partial B_N\cap P$, we obtain

$$T_1(u,v)(t) = \int_0^1 G_1(t,s)f_1(s,u,v)ds < N'\frac{N}{2N'} = \frac{N}{2}$$

which implies $\parallel T_1(u,v)\parallel\leq\frac{N}{2}$. In a similar manner, we may take $\parallel T_2(u,v)\parallel\leq\frac{N}{2}$. Therefore

$$\parallel T(u,v)\parallel = \parallel T_1(u,v)\parallel + \parallel T_2(u,v)\parallel\leq\frac{N}{2}+\frac{N}{2} = N.$$

Thus

(14) $$\parallel T(u,v)\parallel\leq\parallel (u,v)\parallel, \text{ for all } (u,v)\in P\cap\partial B_N.$$

And from $(A4)$ and $(A5)$ we have

(15) $$\parallel T(u,v)\parallel\geq\parallel (u,v)\parallel, \text{ for all } (u,v)\in P\cap\partial\Omega_2,$$

(16) $$\parallel T(u,v)\parallel\geq\parallel (u,v)\parallel, \text{ for all } (u,v)\in P\cap\partial\Omega_3.$$

We have choose H_2, H_3 and N such that $H_3\leq N\leq H_2$ and (14)-(16) are satisfied. From Theorem (2.2), T has at least two fixed points in $P\cap(\overline{\Omega}_2\setminus B_N)$ and $P\cap(\overline{B}_N\setminus\Omega_3)$, respectively.

5. Example

In this section, we demonstrate our results with some examples.

We consider the system of fractional order differential equations

$$\begin{aligned} {}^cD^{2.5}u(t) + f_1(t,u(t),v(t)) = 0,\ t \in (0,1) \\ {}^cD^{2.5}v(t) + f_2(t,u(t),v(t)) = 0,\ t \in (0,1) \end{aligned} \tag{17}$$

with the three-point boundary conditions

$$\begin{aligned} u(0) - u'(0) = u'\Big(\frac{1}{8}\Big) = 2u(1) + \frac{1}{2}u''(1) = 0, \\ v(0) - v'(0) = v'\Big(\frac{1}{8}\Big) = 2v(1) + \frac{1}{2}v''(1) = 0. \end{aligned} \tag{18}$$

Here $q_1 = q_2 = \frac{5}{2}, \alpha = 1,\ \beta = 2,\ \eta = \frac{1}{8},\ \gamma = \frac{1}{2}$ and we deduce that $m = \min\{m_1, m_2\} = 0.03125$

Example 5.1: Let $f_1(t,u,v) = \frac{t}{4}(u+v) + t^2 + 4,\ f_2(t,u,v) = \frac{t^4}{2}(u+v) + e^{-(u+v)}$, then conditions of Theorem (4.1) are satisfied. From Theorem (4.1), the BVP (17)-(18) has at least one positive solution.

Example 5.2: Let $f_1(t,u,v) = (1-t)\big[e^{-(u+v)}(u+v)\big],\ f_2(t,u,v) = \frac{4}{1+t}(u^2+v^2)$ then $N' = \frac{9}{8}$. We can choose $N = 1$ and conditions of Theorem (4.3) are satisfied. From Theorem (4.3), the BVP (17)-(18) has at least two positive solutions.

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Department of Mathematics, Jazan University, Jazan, Kingdom of Saudi Arabia

*Corresponding author: sabbavarapu_nag@yahoo.co.in; snrao@jazanu.edu.sa