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COMPUTABLE FRAMES IN COMPUTABLE BANACH SPACES

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Abstract. We develop some parts of the frame theory in Banach spaces from the point of view of Computable Analysis. We define computable M-basis and use it to construct a computable Banach space of scalar valued sequences. Computable X_d frames and computable Banach frames are also defined and computable versions of sufficient conditions for their existence are obtained.

1. Introduction

In functional analysis, a sequence (x_n) in a Banach space X is called a M-basis, if it is complete in X and there exists a total sequence of functionals $(f_n) \subseteq X^*$ such that (x_n, f_n) is a biorthogonal system. Not every separable Banach space has a Schauder basis but it has at least a bounded and norming M-basis. A Banach space with a M-basis is linearly isometric to the associated Banach space $X_d = \{(f_n(x)) : x \in X\}$ by a result in [17].
The notion of computable Banach spaces with computable basis has already been discussed in [7]. We define computable M-basis and prove the computable version of above result in the framework of computable analysis. The above mentioned computable Banach space X_d is generalized to a computable BK-space, which is further used to define computable X_d frame. A sufficient condition for the existence of a computable X_d frame is obtained. Computable version of a necessary and sufficient condition for the existence of X_d Bessel sequence [9], is also obtained. Finally, we define the concept of computable Banach frame and obtain some sufficient conditions for their existence.

2. Computable Analysis

In this section, we briefly summarize some notions from computable analysis as presented in [19]. Computable Analysis is the Turing machine based approach to computability in analysis. Pioneering work in this field has been done by Turing [18], Grzegorczyk [11], Lacombe [13], Banach and Mazur [1], Pour-El and Richards [16], Kreitz and Weihrauch [12] and many others. The basic idea of the representation based approach to computable analysis is to represent infinite objects like real numbers, functions or sets, by infinite strings over some alphabet Σ (which at least contains the symbols 0 and 1). Thus, a representation of a set X is a surjective function $\delta :\subseteq \Sigma^\omega \to X$ where Σ^ω denotes the set of infinite sequences over Σ and the inclusion symbol indicates that the mapping might be partial. Here, (X, δ) is called a represented space. Between two represented spaces, we define the notion of a computable function.

Definition 2.1. [6] Let (X, δ), (Y, δ') be represented spaces. A function $f :\subseteq X \to Y$ is called (δ, δ')-computable if there exists a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta' F(p) = f\delta(p)$ for all $p \in dom(f\delta)$.

A function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is said to be computable if there exists some Turing machine, which computes infinitely long and transforms each sequence p, written on the input tape, into the corresponding sequence $F(p)$, written on one way output tape. We simply call, a function f computable, if the represented spaces are clear from the context.
For comparing two representations, δ, δ' of a set X, we have the notion of reducibility of representations. δ is called reducible to δ', $\delta \leq \delta'$ (in symbols), if there exists a computable function

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$F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta(p) = \delta^{'} F(p)$ for all $p \in dom(\delta)$. This is equivalent to the fact that the identity $I : X \to X$ is $(\delta, \delta^{'})$- computable. If $\delta \leq \delta^{'}$ and $\delta^{'} \leq \delta$, then δ and $\delta^{'}$ are called computably equivalent. Analogously to the notion of computability ,we can define the notion of $(\delta, \delta^{'})$-continuity, by substituting a continuous function F in the definitions above. On Σ^ω, we use the Cantor topology, which is simply the product topology of the discrete topology on Σ.

Given a represented space (X, δ), a computable sequence is defined as a computable function $f : \mathbb{N} \to X$ where we assume that $\mathbb{N}$ is represented by $\delta_{\mathbb{N}}(1^n 0^\omega) = n$ and a point $x \in X$ is called computable if there is a constant computable sequence with value x. The notion of $(\delta, \delta^{'})$-continuity agrees with the ordinary topological notion of continuity, as long as, we are dealing with admissible representations.

A representation δ of a topological space X is called admissible if δ is continuous and if the identity $I : X \to X$ is (δ', δ)-continuous for any continuous representation δ' of X. If δ, δ', are admissible representation of topological spaces X, Y, then a function $f :\subseteq X \to Y$ is (δ, δ') continuous iff it is sequentially continuous [5].

Given two represented spaces (X, δ), (Y, δ'), there is a canonical representation $[\delta \to \delta']$ of the set of (δ, δ')- continuous functions $f : X \to Y$. If δ and δ' are admissible representations of sequential topological spaces X and Y respectively, then $[\delta \to \delta']$ is actually a representation of the set $C(X, Y)$ of continuous functions $f : X \to Y$. The function space representation can be characterized by the fact that it admits evaluation and type conversion. See [19] for details.

If (X, δ), (Y, δ') are admissibly represented sequential topological spaces, then, in the following, we will always assume that $C(X, Y)$ is represented by $[\delta \to \delta']$. It follows by evaluation and Type conversion that the computable points in $(C(X, Y), [\delta \to \delta'])$ are just the (δ, δ') -computable functions $f :\subseteq X \to Y$ [19]. For a represented space (X, δ), we assume that the set of sequences $X^{\mathbb{N}}$ is represented by $\delta^{\mathbb{N}} = [\delta_{\mathbb{N}} \to \delta]$. The computable points in $(X^{\mathbb{N}}, \delta^{\mathbb{N}})$ are just the computable sequences in (X, δ).

The notion of computable metric space was introduced by Lacombe [14]. However, we state the following definition given by Brattka [6].

Definition 2.2. ([6]) A tuple (X, d, α) is called a *computable metric space* if

1. (X, d) is a metric space.
2. $\alpha : \mathbb{N} \to X$ is a sequence which is dense in X.
3. $do(\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence in $\mathbb{R}$.

Given a computable metric space (X, d, α), its Cauchy Representation $\delta_X :\subseteq \Sigma^\omega \to X$ is defined as

$$\delta_X(01^{n_0+1}01^{n_1+1}01^{n_2+1}...) := \lim_{i \to \infty} \alpha(n_i)$$

for all $n_i \in \mathbb{N}$ such that $(\alpha(n_i))_{i \in \mathbb{N}}$ converges and

$$d(\alpha(n_i), \alpha(n_j)) < 2^{-i}$$

for all $j > i$.

In the following, we assume that computable metric spaces are represented by their Cauchy representation. All Cauchy representations are admissible with respect to the corresponding metric topology.

An Example of a computable metric space is $(\mathbb{R}, d_{\mathbb{R}}, \alpha_{\mathbb{R}})$ with the Euclidean metric $d_{\mathbb{R}}(x, y) = \|x - y\|$ and a standard numbering of a dense subset $Q \subseteq \mathbb{R}$ as $\alpha_{\mathbb{R}} < i, j, k >= (i - j)/(k + 1)$. Here, the bijective Cantor pairing function $<, >: \mathbb{N}^2 \to \mathbb{N}$ is defined as $< i, j >= j + (i + j)(i + j + 1)/2$ and this definition can be extended inductively to finite tuples. It is known that the Cantor pairing function and the projections of its inverse are computable. In the following, we assume that $\mathbb{R}$ is endowed with the Cauchy representation $\delta_{\mathbb{R}}$ induced by the computable metric space given above.

Brattka [6] gave the following definition of a computable normed linear space.

Definition 2.3. ([6]) A space $(X, \|.\|, e)$ is called a *computable normed space* if:-

(1) $\|.\| : X \to \mathbb{R}$ is a norm on X.
(2) The linear span of $e : \mathbb{N} \to X$ is dense in X.
(3) (X, d, α_e) with $d(x, y) = \| x - y \|$ and

$$\alpha_e < k, < n_0, ..., n_k >> = \Sigma_{i=0}^{k} \alpha_F(n_i) e_i$$

is a computable metric space with Cauchy representation δ_X.

It was observed that computable normed space is automatically a computable vector space, that is, the linear operations are all computable. If the underlying space $(X, \|.\|)$ is a Banach space then $(X, \|.\|, e)$ is called a computable Banach space.
We always assume that computable normed spaces are represented by their Cauchy representations, which are admissible with respect to norm topology. Two computable Banach space with the same underlying set are called computably equivalent if the corresponding Cauchy representations are computably equivalent.
A sequence $(e_i)_{i \in \mathbb{N}}$ in a Banach space X, is called a Schauder basis of X if every $x \in X$ can be uniquely represented as $x = \Sigma_{i=0}^{\infty} x_i e_i$ with $x_i \in F$. If X is a computable Banach space, then a sequence $(e_i)_{i \in \mathbb{N}}$ is called a computable basis, if it is a Schauder basis of X that is computable in X.
A sequence space S is called a BK space if it is a Banach space and the coordinate functionals are continuous on S. Here, a sequence space is a set S of sequences of scalars which is closed under co-ordinatewise addition and scalar multiplication.
Several representations for the operator space $B(X, Y)$ of bounded linear operators between two computable normed spaces are defined in [3]. In the following, we state some of the representations for the operator space $B(X, Y)$, as they are used in consequent results of this paper.

Definition 2.4. ([3]) Let $(X, \|.\|, e)$ and Y be computable normed spaces. Define representations of $B(X, Y)$:
1) $\delta_{ev}(p) = T \Leftrightarrow [\delta_X \to \delta_Y](p) = T$
2) $\delta_{seq}(p) = T \Leftrightarrow \delta_Y^{\mathbb{N}}(p) = (Te_i)_{i \in \mathbb{N}}$
3) $\delta_{seq}^{\geq} < p, q >= T \Leftrightarrow \delta_{seq}(p) = T$ and $\delta_{\mathbb{R}}(q) \geq \|T\|$.

3. Main Results

In order to develop a systematic computable frame theory on Banach spaces, we first extend the notion of computability to M-basis. M-basis were introduced by A.I. Markusevic, who regarded them as a natural generalization of the trigonometric system in $C[0, 2\pi]$ and hence, as a natural replacement for basis. M-basis exists in every separable Banach space. We begin with the following definition of computable M-basis.

Definition 3.1. A sequence (x_n) in a computable Banach space $(X, \|.\|, e)$ is a *Computable M-Basis* of X if :-

(1) (x_n) is a computable complete sequence in X.
(2) There exists a total computable sequence of functionals $(f_n) \subseteq X^*$, with respect to $[\delta_X \to \delta_F]$ representation, such that (x_n, f_n) is a biorthogonal system.

Let $(X, \|.\|, e)$ be a computable Banach space with a computable M-basis (x_n). Since (x_n) is a computable complete sequence in X, $(X, \|.\|, (x_n))$ is a computable Banach space that is computably equivalent to $(X, \|.\|, e)$.

Remark 3.2. Let $(X, \|.\|, (e_n))$ be a computable Banach space with computable basis (e_n), then (e_n) is a computable complete sequence in X. The sequence (e'_n) of co-ordinate functionals is a computable sequence in $C(X, F)$ by Proposition 3.3 in [7]. Also, (e'_n) is a total sequence of functionals such that (e_n, e'_n) is a biorthogonal system. Hence, a computable basis in a computable Banach space is a computable M-Basis.

Next, we prove that an associated Banach space of scalar valued sequences with respect to a computable M-basis is a computable Banach space.

Theorem 3.3. *Let (x_n) be a computable M-basis for a computable Banach space $(X, \|.\|, (x_n))$, with associated sequence of functionals (f_n). Let $E_d = \{(f_n(x)) : x \in X\}$ be the associated Banach space with norm $\|(f_n(x))\|_{E_d} = \|x\|_X$, $x \in X$. Then $(E_d, \|.\|, (e_i))$, (e_i) being the sequence of canonical unit vectors, forms a computable Banach space.*

Proof: Let d and d' be the metric induced by the norm of E_d and X, respectively.
Note that, for $h =< k, < n_0, ...n_k >>$ and $h' =< p, < m_0, ...m_p >> \in \mathbb{N}$,

$$\begin{aligned} d(\alpha_e(h), \alpha_e(h')) &= \|\Sigma_{i=0}^{k} \alpha_F(n_i) e_i - \Sigma_{i=0}^{p} \alpha_F(m_i) e_i\| \\ &= \|(f_n(\Sigma_{i=0}^{k} \alpha_F(n_i) x_i - \Sigma_{i=0}^{p} \alpha_F(m_i) x_i))\|_{E_d} \\ &= \|\Sigma_{i=0}^{k} \alpha_F(n_i) x_i - \Sigma_{i=0}^{p} \alpha_F(m_i) x_i\|_X \\ &= d'(\alpha_x(h), \alpha_x(h')) \end{aligned}$$

Now, the result follows from the fact that (X, d', α_x) is a computable metric space.

In the following result, we prove that the computable Banach space $(X, \|.\|, (x_n))$ with computable M-basis (x_n) and the associated computable Banach space $(E_d, \|.\|, (e_i))$ are computably isomorphic. Here, a computable isomorphism T is an isomorphism such that T as well as T^{-1} are computable.

Theorem 3.4. *Let $(X, \|.\|, (x_n))$ be a computable Banach space, (x_n) be a computable M-basis and $(E_d, \|.\|, (e_i))$ be the associated computable Banach space. Then the mapping $T : X \to E_d$ given by $T(x) = (f_n(x)), x \in X$ is a computable isometrical isomorphism.*

Proof: The map T is a bounded linear operator satisfying $\|Tx\| = \|x\|$, for all $x \in X$ and therefore, $\|T\| \leq 1$. Also, $(T(x_n)) = (e_n)$ is a computable sequence in E_d. Thus, we can get a $\delta_{seq}^{\geq}$ name of T and so a δ_{ev} name. Therefore, T is $[\delta_X \to \delta_{E_d}]$ computable. Hence, by computable Banach Inverse Mapping Theorem in [6], T is a computable isometrical isomorphism.

Next, we generalize the notion of associated computable Banach space of scalar valued sequences.

Definition 3.5. A BK space X_d is said to be a *computable BK-space* if it is a computable Banach space such that the sequence of co-ordinate functionals $\tau_j : X_d \to F$ given by $\tau_j((x_i)) = x_j$, $j \in \mathbb{N}$ is computable with respect to $[\delta_{X_d} \to \delta_F]$ representation.

Example 3.6. Let (x_n) be a computable M-basis for a computable Banach space $(X, \|.\|, (x_n))$, with associated total sequence of functionals $(f_n) \subseteq X^*$. Then $E_d = \{(f_n(x)) : x \in X\}$, the associated Banach space is a BK space. Also, $(E_d, \|.\|, (e_i))$ is a computable Banach space as proved above. The sequence of co-ordinate functionals $\tau_j : E_d \to F$ given by $\tau_j((f_n(x))) = f_j(x), x \in X$ is computable in $C(E_d, F)$ as $\|\tau_n\| \leq \|f_n\|$ and (f_n) is a computable sequence with respect to $[\delta_X \to \delta_F]$ representation. Also, $\tau_n(e_k) = \delta_{kn}$, that is, given n and k, $\tau_n(e_k)$ can be computed. Hence, $(E_d, \|.\|, (e_i))$ is a computable BK-space.

We define computable X_d frame for a computable Banach space in the following.

Definition 3.7. Let X be a computable Banach space, X_d be a computable BK-space. A computable sequence $(g_i) \subseteq X^*$, with respect to $[\delta_X \to \delta_F]$ representation is called a *computable X_d frame* for X if:

(1) $(g_i(f)) \in X_d \ \forall f \in X$.
(2) $\|f\|_X$ and $\|(g_i(f))\|_{X_d}$ are equivalent, that is, there exists constants A, $B > 0$ such that

$$A\|f\|_X \leq \|(g_i(f))\|_{X_d} \leq B\|f\|_X$$

for all $f \in X$.

If, only the (1) and the upper condition in (2) are satisfied, (g_i) is called a *computable X_d Bessel sequence* for X.

Example 3.8. Let $(x_n) \subseteq X$, $(f_n) \subseteq X^*$ and $X_d = E_d$ be as in Example 3.6. Then $(X_d, \|.\|, (e_i))$ is a computable BK space. Also, since $(f_n) \subseteq X^*$ is a computable sequence with respect to $[\delta_X \to \delta_F]$ representation such that $(f_n(x)) \in X_d$ for all $x \in X$ and $\|(f_n(x))\|_{X_d} = \|x\|_X$, $x \in X$, (f_n) forms a computable X_d frame for X.

In the following, we present a computable version of a sufficient condition for the existence of a X_d frame (Theorem 2.1 in [9]).

Theorem 3.9. *Let X be any computable Banach space and X_d be any computable BK space. If X is isometrically isomorphic to a subspace of X_d by a computable map, then there exists a computable X_d frame for X.*

Proof: Let $T : X \to X_d$ be a computable map such that X is isometrically isomorphic to $T(X)$. Let (τ_i) be the computable sequence of co-ordinate functionals of X_d, given by $\tau_i((x_j)) = x_i, i \in \mathbb{N}$, $(x_j) \in X_d$. For each $i \in \mathbb{N}$, define $g_i = \tau_i \circ T$. Then (g_i) is a computable sequence of functionals such that $(g_i(f))= (\tau_i(T(f))) = Tf \in X_d$ $, f \in X$ and $\|(g_i(f))\|_{X_d} = \|Tf\|_{X_d}= \|f\|_X$. Hence, $(g_i) \subseteq X^*$ is a computable X_d frame for X.

For the converse, Theorem 2.1 in [9] states that if (g_i) be an X_d frame for a Banach space X, then the map $U : X \to X_d$ given by $U(f) = (g_i(f))$ is an isomorphism of X into X_d. Using the techniques from [8], we show that the computable version of this result does not hold.

Example 3.10. Consider the computable Banach space $(l_2, \|.\|, (e_i))$ with the sequence of canonical unit vectors (e_i) as computable basis. Let (a_i) be a computable sequence of positive reals such that $\|(a_i)\|_{l_2}$ exists but is not computable. We assume $a_0 = 1$. Define a linear bounded operator $T : l_2 \to l_2$ as

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

Then, $(f_i) = (Te_i)$ is a frame for l_2. Define $g_i : l_2 \to \mathbb{R}$ by $g_i(f) =< f, f_i >$, $i \in \mathbb{N}$. Then, (g_i) forms a computable X_d frame for l_2 where X_d is the computable BK space $(l_2, \|.\|, (e_i))$. But the operator $U : l_2 \to l_2$, $U(f) = (< f, f_i >)$ is not computable as $U(e_0) = (a_i)$ is not computable in l_2.

The next result shows that the map U is computable with respect to $[\delta_X \to \delta_F^{\mathbb{N}}]$ representation.

Theorem 3.11. *Let X be a computable Banach space and X_d be a computable BK space. If $(g_i) \subseteq X^*$ be a computable X_d frame for X then the map*

$$U : X \to X_d$$
$$f \to (g_i(f))$$

is $(\delta_X, [\delta_{\mathbb{N}} \to \delta_F])$ computable isomorphism of X into X_d.

Proof: The map U is an isomorphism of X into X_d by Theorem 2.1 in [9]. By the computability of the sequence (g_i) and the Evaluation property , we get that the map

$$U^{'} : \mathbb{N} \times X \to F$$
$$(i, f) \to g_i(f)$$

is computable with respect to $([\delta_{\mathbb{N}}, \delta_X], \delta_F)$ representation. By Type Conversion, the map U is computable with respect to $(\delta_X, [\delta_{\mathbb{N}} \to \delta_F])$ representation.

Now, we give a computable version of a necessary and sufficient condition for the existence of a X_d Bessel sequence (Corollary 3.3 in [9]).
First, we recall, the Dual space representation δ_{X^*} of the dual space X^* as given in [7].

Definition 3.12. ([7]) For a separable Banach space X, define a representation δ_{X^*} of the dual space X^* by

$$\delta_{X^*} < p, q >= f \Leftrightarrow [\delta_X \to \delta_F](p) = f \ \ and \ \ \delta_{\mathbb{R}}(q) = \|f\|$$

.

Next, we give the definition of computable dual space given in [7], as it is required in the subsequent result.

Definition 3.13. ([7]) Let X be a computable Banach space. X is said to have a *computable dual space* X^* if there exists a sequence $e^* : \mathbb{N} \to X^*$ such that:

1. $(X^*, \|.\|, e^*)$ is a computable Banach space.
2. δ_{X^*} is computably equivalent to the Cauchy representation $\delta^c_{X^*}$ of $(X^*, \|.\|, e^*)$.

Theorem 3.14. *Let X be a computable Banach space with computable dual space. Let $(X_d, \|.\|, (e_i))$ be a computable BK-space and $(X^*_d, \|.\|, (E_i))$ be a computable Banach space, (e_i) and (E_i) being the sequences of standard unit vectors as basis.Then if, $(g_i) \subseteq X^*$ be a computable X_d-Bessel sequence for X with bound B with $(\|g_i\|)$ being a computable sequence then $T : X^*_d \to X^*$ given by $T((d_i)) = \Sigma d_i g_i$ is a well defined computable operator from $\|T\| \leq B$. Converse holds if X_d is reflexive space.*

Proof: Define $R : X \to X_d$ by $R(x) = (g_i(x))$,$x \in X$. Since (g_i) is an X_d- Bessel sequence, $(g_i(x)) \in X_d$, $x \in X$ and $\|R(x)\| = \|(g_i(x))\|_{X_d} \leq B\|x\|_X$, $x \in X$. Consider $R^* : X^*_d \to X^*$. Then $R^*(E_j) : X \to F$ is such that $R^*(E_j)(x) = E_j(R(x)) = g_j(x), x \in X$. Therefore,$R^*(E_j) = g_j$, for all $j \in \mathbb{N}$ such that $R^*(\Sigma_i d_i E_i) = \Sigma_i d_i g_i$.
Thus, $T = R^*$ is a well defined operator given by $T((d_i)) = \Sigma_i d_i g_i$ satisfying $\|T\| \leq B$. As, $(g_i) \subseteq X^*$ is a computable sequence with respect to $[\delta_X \to \delta_F]$ representation and $(\|g_i\|)$ is assumed to be a computable sequence, therefore, (g_i) is a computable sequence in X^* with respect to Cauchy representation. Therefore, T is $[\delta^c_{X^*_d} \to \delta^c_{X^*}]$ computable.
Conversely, let $T : X^*_d \to X^*$ be a computable operator given by $T((d_i)) = \Sigma_i d_i g_i$. Then $T(E_i) = g_i$, for all i and since, a computable operator maps computable sequences to computable sequences, therefore, (g_i) is a computable sequence in X^* with respect to $[\delta_X \to \delta_F]$ representation and $(\|g_i\|)$ is a computable sequence. Consider $T^* : X^{**} \to X^{**}_d$ which satisfies $T^*(f)(E_j) = f(g_j)$, for all $f \in X^{**}$. Therefore, $(g_i(x)) = (T^*(i(x))(E_i)) \in X^{**}_d$, identified with $T^*(i(x))$, where i is the natural embedding of X into X^{**}. Since X_d is reflexive, therefore, $(g_i(x)) \in X_d$ and satisfies

$$\|(g_i(x))\|_{X_d} = \|T^*(i(x))\| = \|T\|\|x\| \leq B\|x\|_X$$

Hence, $(g_i) \subseteq X^*$ is a computable X_d Bessel sequence for X with bound B and computable sequence of norms.

Banach frames were introduced by Grochenig [10] as a generalization of the notion of frames in Hilbert spaces. In the following definition, we introduce the notion of computable Banach frame.

Definition 3.15. Let X be a computable Banach space, X_d be a computable BK-space. Given a computable linear operator $S : X_d \to X$ and a computable X_d frame $(g_i) \subseteq X^*$, we say that $((g_i), S)$ is a *computable Banach frame* for X with respect to X_d if $S((g_i(x))) = x$ for all $x \in X$.

Example 3.16. Let (x_n) be a computable M-Basis and $X_d = \{(f_n(x)) : x \in X\}$ be a computable BK-space and $(f_n) \subseteq X^*$ is a computable X_d frame. By Theorem 3.4, the map $T : X \to X_d$ given by $x \to (f_n(x))$, $x \in X$ is a computable isometrical isomorphism. Thus, $((f_i), T^{-1})$ is a computable Banach frame for X with respect to X_d.

Next, we give a sufficient condition for the existence of a computable Banach frame. In the following result, we call a closed subspace of a computable Banach space to be computably complemented if it is the range of a computable linear projection in the space. Clearly, a computably complemented subspace is a computable subspace as defined in [7].

Theorem 3.17. *A computable Banach space X has a computable Banach frame with respect to a given computable BK space X_d if X is isometrically isomorphic to a computably complemented subspace of X_d by a computable map.*

Proof: Suppose $T : X \to X_d$ be a computable map which is an isometric isomorphism of X into X_d and $T(X)$ be the computably complemented subspace of X_d. Then, there exists a computable projection $P : X_d \to X_d$ such that $P(X_d) = T(X)$ and $P^2 = P$. Define $S : X_d \to X$ by $Sx = T^{-1}Px$, $x \in X_d$. Then, by computable Banach Inverse Mapping theorem, S is a computable linear operator. Let (τ_j) be the computable sequence of co-ordinate functionals of X_d and $g_i = T^*(\tau_i)$, for all $i \in \mathbb{N}$. Then, for $x \in X$, we have $g_i(x) = T^*\tau_i(x) = \tau_i T(x)$, for all $i \in \mathbb{N}$. Therefore, $g_i = \tau_i T$ and hence, (g_i) is a computable sequence in X^* with respect to $[\delta_X \to \delta_F]$ representation.
Also, $Tx = (g_i(x)) \in X_d$, for all $x \in X$ and $\|(g_i(x))\|_{X_d} = \|Tx\|_{X_d} = \|x\|_X$. Also $S((g_i(x))) = T^{-1}P((g_i(x))) = T^{-1}((g_i(x))) = x$. Thus, $((g_i), S)$ is a computable Banach frame for X with respect to X_d.

Finally, we give sufficient condition under which a computable X_d frame for X is a computable Banach frame for X.

Theorem 3.18. *Let X be a computable Banach space, $(X_d, \|.\|, (e_i))$ be a computable BK space with sequence (e_i) of canonical unit vectors. Let $(g_i) \subseteq X^*$ be a computable X_d frame for X. If there exists a computable sequence $(f_i) \subseteq X$ such that $\Sigma_i c_i f_i$ is convergent for all $(c_i) \in X_d$ and $f = \Sigma_i g_i(f) f_i$, for all $f \in X$. Then, there exists a computable linear operator $T : X_d \to X$ such that $((g_i), T)$ is a computable Banach frame for X with respect to X_d.*

Proof: Define $T_n : X_d \to X$ as $T_n((c_i)) = \Sigma_i^n c_i f_i$. Then, $T_n(e_i) = f_i$, for all $i \le n$ and 0 otherwise, that is, given n and k, we can compute $T_n(e_k)$. Since, $(T_n((c_i)))_n$ is convergent and hence a bounded sequence, by Uniform Boundedness Principle, $Sup_n \|T_n\| < \infty$. Therefore, (T_n) is a computable sequence of linear and computable operators.
Define $m : \mathbb{N} \to \mathbb{N}$ as $m(< i, j >) = j$. Then m is a computable function such that $\|T_{m<i,j>} e_j - lim_{n\to\infty} T_n e_j\| \le 2^{-i}$, for all $i, j \in \mathbb{N}$. By computable Banach Steinhaus Theorem in [2], $T : X_d \to X$ defined as $T((c_i)) = lim_{n\to\infty} T_n((c_i)) = \Sigma_i c_i f_i$ is a linear computable operator satisfying $T((g_i(f))) = f$ for all $f \in X$. Hence, $((g_i), T)$ is a computable Banach frame for X with respect to X_d.

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