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STABILITY IN TOTALLY NONLINEAR NEUTRAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. Let $\mathbb{T}$ be a time scale which is unbounded above and below and such that $0 \in \mathbb{T}$. Let $id - \tau : [0, \infty) \cap \mathbb{T} \to \mathbb{T}$ be such that $(id - \tau)([0, \infty) \cap \mathbb{T})$ is a time scale. We use the Krasnoselskii-Burton's fixed point theorem to obtain stability results about the zero solution for the following totally nonlinear neutral dynamic equation with variable delay

$$x^{\triangle}(t) = -a(t)\, h(x^{\sigma}(t)) + c(t)\, x^{\widetilde{\triangle}}(t - \tau(t)) + b(t)\, G(x(t), x(t - \tau(t))),\ t \in [0, \infty) \cap \mathbb{T},$$

where $f^{\triangle}$ is the $\triangle$-derivative on $\mathbb{T}$ and $f^{\widetilde{\triangle}}$ is the $\triangle$-derivative on $(id - \tau)(\mathbb{T})$. The results obtained here extend the work of Ardjouni, Derrardjia and Djoudi [2].

1. Introduction

The concept of time scales analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [12]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area by Bohner and Peterson [6] and [7], more and more researchers were getting involved in this fast-growing field of mathematics.

The study of dynamic equations brings together the traditional research areas of differential and difference equations. It allows one to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [1, 3, 4, 13] and the references therein).

There is no doubt that the Lyapunov method have been used successfully to investigate stability properties of wide variety of ordinary, functional and partial equations. Nevertheless, the application of this method to problem of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded term. It has been noticed that some of theses difficulties vanish by using the fixed point technic. Other advantages of fixed point theory over Lyapunov's method is that the conditions of the former are average while those of the latter are pointwise (see [2, 5, 8, 9, 10, 11] and references therein).

In paper, we consider the following neutral nonlinear dynamic equations with variable delay given by

$$x^{\triangle}(t) = -a(t)\, h(x^{\sigma}(t)) + c(t)\, x^{\widetilde{\triangle}}(t - \tau(t)) + b(t)\, G(x(t), x(t - \tau(t))),\ t \in [0, \infty) \cap \mathbb{T}, \tag{1.1}$$

with an assumed initial function

$$x(t) = \psi(t),\ t \in [m_0, 0] \cap \mathbb{T},$$

where $\mathbb{T}$ is an unbounded above and below time scale and such that $0 \in \mathbb{T}$, $\psi : [m_0, 0] \cap \mathbb{T} \to \mathbb{R}$ is rd-continuous and $m_0 = \inf\{t - \tau(t) : t \in [0, \infty) \cap \mathbb{T}\}$. Throughout this paper, we assume that $a, b : [0, \infty) \cap \mathbb{T} \to \mathbb{R}$ are rd-continuous, $h : \mathbb{R} \to \mathbb{R}$ is continuous with $h(0) = 0$ and $c : [0, \infty) \cap \mathbb{T} \to \mathbb{R}$ is continuously delta-differentiable. In order for the function $x(t - \tau(t))$ to be well-defined and

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differentiable over $[0,\infty)\cap\mathbb{T}$, we assume that $\tau:[0,\infty)\cap\mathbb{T}\to\mathbb{T}$ is positive and twice continuously delta-differentiable, and that $id-\tau:[0,\infty)\cap\mathbb{T}\to\mathbb{T}$ is an increasing mapping such that $(id-\tau)([0,\infty)\cap\mathbb{T})$ is closed where id is the identity function.

Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due to Burton (see [8] Theorem 3) to show the asymptotic stability and the stability of the zero solution for equation (1.1). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [8], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we have to transform (1.1) into an integral equation written as a sum of two mapping; one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii fixed point theorem, to show the asymptotic stability and the stability of the zero for equation (1.1). In the special case $\mathbb{T}=\mathbb{R}$, Ardjouni, Derrardjia and Djoudi [2] show the zero solution of (1.1) is asymptotically stable by using Krasnoselskii-Burton's fixed point theorem.

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the modification of Krasnoselskii's fixed point theorem established by Burton (see ([8] Theorem 3) and [10]). For details on Krasnoselskii's theorem we refer the reader to [14]. We present our main results on stability in Section 3 and 4. The results presented in this paper extend the main results in [2].

2. Preliminaries

In this section, we consider some advanced topics in the theory of dynamic equations on a time scales. Again, we remind that for a review of this topic we direct the reader to the monographs of Bohner and Peterson [6] and [7].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t\in\mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ and $\rho(t)=\sup\{t\in\mathbb{T}:s<t\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t)>t$ and right dense if $\sigma(t)=t$. We say t is left scattered if $\rho(t)<t$ and left dense if $\rho(t)=t$. The graininess function $\mu:\mathbb{T}\to[0,\infty)$, is defined by $\mu(t)=\sigma(t)-t$ and gives the distance between an element and its successor. We set $\inf\emptyset=\sup\mathbb{T}$ and $\sup\emptyset=\inf\mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k=\mathbb{T}\diagdown\{M\}$. Otherwise, we define $\mathbb{T}^k=\mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k=\mathbb{T}\diagdown\{m\}$. Otherwise, we define $\mathbb{T}_k=\mathbb{T}$.

Let $t\in\mathbb{T}^k$ and let $f:\mathbb{T}\to\mathbb{R}$. The delta derivative of $f(t)$, denoted $f^{\triangle}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon>0$, there is a neighborhood U of t such that

$$\left|f(\sigma(t))-f(s)-f^{\triangle}(t)[\sigma(t)-s]\right|\le\epsilon\left|\sigma(t)-s\right|,$$

for all $s\in U$. If $\mathbb{T}=\mathbb{R}$ then $f^{\triangle}(t)=f'(t)$ is the usual derivative. If $\mathbb{T}=\mathbb{Z}$ then $f^{\triangle}(t)=\triangle f(t)=f(t+1)-f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f\in C_{rd}=C_{rd}(\mathbb{T},\mathbb{R})$, if it is continuous at every right dense point $t\in\mathbb{T}$ and its left-hand limits exist at each left dense point $t\in\mathbb{T}$. The function $f:\mathbb{T}\to\mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t\in\mathbb{T}^k$.

We are now ready to state some properties of the delta-derivative of f. Note $f^{\sigma}(t)=f(\sigma(t))$.

Theorem 1 ([6, Theorem 1.20]). *Assume $f,g:\mathbb{T}\to\mathbb{R}$ are differentiable at $t\in\mathbb{T}^k$ and let α be a scalar.*

(i) $(f+g)^{\triangle}(t)=g^{\triangle}(t)+f^{\triangle}(t)$.

(ii) $(\alpha f)^{\triangle}(t)=\alpha f^{\triangle}(t)$.

(iii) The product rules

$$\begin{aligned}(fg)^{\triangle}(t)&=f^{\triangle}(t)g(t)+f^{\sigma}(t)g^{\triangle}(t),\\(fg)^{\triangle}(t)&=f(t)g^{\triangle}(t)+f^{\triangle}(t)g^{\sigma}(t).\end{aligned}$$

(iv) If $g(t)\,g^{\sigma}(t) \neq 0$ *then*

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t)\,g(t) - f(t)\,g^{\triangle}(t)}{g(t)\,g^{\sigma}(t)}.$$

The next theorem is the chain rule on time scales ([6, Theorem 1.93], Theorem 1.93).

Theorem 2 (Chain Rule)**.** *Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. Let* $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$*. If* $\nu^{\triangle}(t)$ *and* $\omega^{\widetilde{\triangle}}(\nu(t))$ *exist for* $t \in \mathbb{T}^k$*, then* $(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right)\nu^{\triangle}$.

In the sequel we will need to differentiate and integrate functions of the form $f(t - \tau(t)) = f(\nu(t))$ where, $\nu(t) := t - \tau(t)$. Our next theorem is the substitution rule ([6, Theorem 1.98], Theorem 1.98).

Theorem 3 (Substitution)**.** *Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(T)$ *is a time scale. If* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous function and* ν *is differentiable with rd-continuous derivative, then for* $a, b \in T$,

$$\int_a^b f(t)\,\nu^{\triangle}(t)\,\triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\,\widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)\,p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$, is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)\,f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log\left(1 + \mu(z)\,p(z)\right)\Delta z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^{\triangle} = p(t)\,y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

Lemma 1 ([6, Theorem 2.36])**.** *Let* $p, q \in \mathcal{R}$*. Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)\,p(t))\,e_p(t,s)$,

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

(v) $e_p(t,s)\,e_p(s,r) = e_p(t,r)$,

(vi) $e_p^{\triangle}(.,s) = p e_p(.,s)$ *and* $\left(\frac{1}{e_p(.,s)}\right)^{\triangle} = -\frac{p(t)}{e_p^{\sigma}(.,s)}$.

Lemma 2 ([1])**.** *If* $p \in \mathcal{R}^+$*, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\,\triangle u\right),\ \forall t \in \mathbb{T}.$$

Corollary 1 ([1])**.** *If* $p \in \mathcal{R}^+$ *and* $p(t) < 0$ *for all* $t \in \mathbb{T}$*, then for all* $s \in \mathbb{T}$ *with* $s \leq t$ *we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\,\triangle u\right) < 1.$$

3. The inversion and the fixed point theorem

In addition to the conditions mentioned in Section 1, we assume that $a \in \mathcal{R}^+$ is rd-continuous and $a(t) > 0$ for all $t \in [0,\infty) \cap \mathbb{T}$, c is continuously delta-differentiable and τ is twice continuously delta-differentiable with

$$\tau^{\triangle}(t) \neq 1,\ t \in [0,\infty) \cap \mathbb{T}. \tag{3.1}$$

We also assume that $G(x,y)$ is locally Lipschitz continuous in x and y. That is, there are positive constants N_1 and N_2 so that if $|x|, |y| \leq \sqrt{3}/3$, then

$$|G(x,y) - G(z,w)| \leq N_1 \|x - z\| + N_2 \|y - w\| \text{ and } G(0,0) = 0. \tag{3.2}$$

One crucial step in the investigation of an equation using fixed point theory involves the construction of a suitable fixed point mapping. For that end we must invert (1.1) to obtain an equivalent integral equation from which we derive the needed mapping. During the process, an integration by parts has to be performed on the neutral term $x^{\widetilde{\triangle}}(t-\tau(t))$. Unfortunately, when doing this, a derivative $\tau^{\triangle}(t)$ of the delay appears on the way, and so we have to support it.

Lemma 3. *Suppose (3.1) holds. x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) &= \left[\psi(0) - \frac{c(0)}{1-\tau^{\triangle}(0)}\psi(-\tau(0))\right] e_{\ominus a}(t,0) + \int_0^t a(s)(Hx)(s)\, e_{\ominus a}(t,s)\,\Delta s \\ &\quad + \frac{c(t)}{1-\tau^{\triangle}(t)} x(t-\tau(t)) - \int_0^t \mu(s)\, x^{\sigma}(s-\tau(s))\, e_{\ominus a}(t,s)\,\Delta s \\ &\quad + \int_0^t b(s)\, G(x(s), x(s-\tau(s)))\, e_{\ominus a}(t,s)\,\Delta s, \end{aligned} \tag{3.3}$$

where

$$\mu(t) = \frac{\left(c^{\triangle}(t) + a(t)c^{\sigma}(t)\right)\left(1-\tau^{\triangle}(t)\right) + c(t)\tau^{\triangle\triangle}(t)}{\left(1-\tau^{\triangle}(t)\right)\left(1-\tau^{\triangle}(\sigma(t))\right)}, \tag{3.4}$$

and

$$(Hx)(t) = x^{\sigma}(t) - h(x^{\sigma}(t)). \tag{3.5}$$

Proof. Let x be a solution of equation (1.1). Rewrite (1.1) as

$$\begin{aligned} &x^{\triangle}(t) + a(t)x^{\sigma}(t) \\ &= a(t)x^{\sigma}(t) - a(t)h(x^{\sigma}(t)) + c(t)x^{\widetilde{\triangle}}(t-\tau(t)) + b(t)G(t, x(t), x(t-\tau(t))). \end{aligned}$$

Multiply both sides of the above equation by $e_a(t,0)$ and integrate from 0 to t to obtain

$$\begin{aligned} x(t) &= \psi(0)\, e_{\ominus a}(t,0) \\ &\quad + \int_0^t a(s)(Hx)(s)\, e_{\ominus a}(t,s)\,\Delta s + \int_0^t c(s)\, x^{\widetilde{\triangle}}(s-\tau(s))\, e_{\ominus a}(t,s)\,\Delta s \\ &\quad + \int_0^t b(s)\, G(x(s), x(s-\tau(s)))\, e_{\ominus a}(t,s)\,\Delta s, \end{aligned} \tag{3.6}$$

letting

$$\begin{aligned} &\int_0^t c(s)\, x^{\widetilde{\triangle}}(s-\tau(s))\, e_{\ominus a}(t,s)\,\Delta s \\ &= \int_0^t \frac{c(s)}{\left(1-\tau^{\triangle}(s)\right)}\left(1-\tau^{\triangle}(s)\right) x^{\widetilde{\triangle}}(s-\tau(s))\, e_{\ominus a}(t,s)\,\Delta s. \end{aligned}$$

By performing an integration by parts, we obtain

$$\begin{aligned} &\int_0^t c(s)\, x^{\widetilde{\triangle}}(s-\tau(s))\, e_{\ominus a}(t,s)\,\Delta s \\ &= \frac{c(t)}{1-\tau^{\triangle}(t)} x(t-\tau(t)) - \frac{c(0)}{1-\tau^{\triangle}(0)}\psi(-\tau(0))\, e_{\ominus a}(t,0) \\ &\quad - \int_0^t \mu(s)\, x^{\sigma}(s-\tau(s))\, e_{\ominus a}(t,s)\,\Delta s, \end{aligned} \tag{3.7}$$

where $\mu(s)$ is given by (3.4). We obtain (3.3) by substituting (3.7) in (3.6). Since each step is reversible, the converse follows easily. This completes the proof. □

Burton [8] observed that Krasnoselskii result can be more interesting in applications with certain changes and formulated in Theorem 5 below (see [8] for the proof).

Definition 1. *Let (M,d) be a metric space and $F:M\to M$. F is said to be a large contraction if $\varphi,\psi\in M$ with $\varphi\neq\psi$, then $d(F\varphi,F\psi)<d(\varphi,\psi)$, and if for all $\epsilon>0$, there exists $\eta<1$ such that*

$$[\varphi,\psi\in M,\ d(\varphi,\psi)\geq\epsilon]\Longrightarrow d(F\varphi,F\psi)\leq\eta d(\varphi,\psi).$$

Theorem 4 (Burton)**.** *Let (M,d) be a complete metric space and F be a large contraction. Suppose there is $x\in M$ and $\rho>0$ such that $d(x,F^n x)\leq\rho$ for all $n\geq1$. Then F has a unique fixed point in M.*

Below, we state Krasnoselskii-Burton's hybrid fixed point theorem which enables us to establish a stability result of the trivial solution of (1.1). For more details on Krasnoselskii's captivating theorem we refer to Smart [14] or [10].

Theorem 5 (Krasnoselskii-Burton)**.** *Let M be a closed bounded convex non-empty subset of a Banach space $(S,\|.\|)$. Suppose that A, B map M into M and that*
(i) for all $x,y\in M\Longrightarrow Ax+By\in M$,
(ii) A is continuous and AM is contained in a compact subset of M,
(iii) B is a large contraction.
Then there is $z\in M$ with $z=Az+Bz$.

Here we manipulate function spaces defined on infinite t-intervals. So for compactness, we need an extension of Arzela-Ascoli theorem. This extension is taken from [10, Theorem 1.2.2, p. 20] and is as follows.

Theorem 6. *Let $q:[0,\infty)\cap\mathbb{T}\longrightarrow\mathbb{R}_+$ be a rd-continuous function such that $q(t)\longrightarrow0$ as $t\longrightarrow\infty$. If $\{\varphi_n(t)\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $[0,\infty)\cap\mathbb{T}$ with $|\varphi_n(t)|\leq q(t)$ for $t\in[0,\infty)\cap\mathbb{T}$, then there is a subsequence that converges uniformly on $[0,\infty)\cap\mathbb{T}$ to a rd-continuous function $\varphi(t)$ with $|\varphi(t)|\leq q(t)$ for $t\in[0,\infty)\cap\mathbb{T}$, where $|.|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

4. Stability by Krasnoselskii-Burton's theorem

From the existence theory which can be found in [10], we conclude that for each continuous initial function $\psi:[m_0,0]\cap\mathbb{T}\to\mathbb{R}$, there exists a continuous solution $x(t,0,\psi)$ which satisfies (1.1) on an interval $[0,\sigma)\cap\mathbb{T}$ for some $\sigma>0$ and $x(t,0,\psi)=\psi(t)$, $t\in[m_0,0]\cap\mathbb{T}$.

We need the following stability definitions taken from [10].

Definition 2. *The zero solution of (1.1) is said to be stable at $t=0$ if for each $\epsilon>0$, there exists $\delta>0$ such that $\psi:[m_0,0]\cap\mathbb{T}\to(-\delta,\delta)$ implies that $|x(t)|<\epsilon$ for $t\geq m_0$.*

Definition 3. *The zero solution of (1.1) is said to be asymptotically stable if it is stable at $t=0$ and $\delta>0$ exists such that for any rd-continuous function $\psi:[m_0,0]\cap\mathbb{T}\to(-\delta,\delta)$, the solution x with $x(t)=\psi(t)$ on $[m_0,0]\cap\mathbb{T}$ tends to zero as $t\to\infty$.*

To apply Theorem 5, we have to choose carefully a Banach space depending on the initial function ψ and construct two mappings, a large contraction and a compact operator which obey the conditions of the theorem. So let S be the Banach space of rd-continuous bounded functions $\varphi:[m_0,\infty)\cap\mathbb{T}\to\mathbb{R}$ with the supremum norm $\|.\|$. Let $L>0$ and define the set

$$\begin{aligned}S_\psi=\{\varphi\in S\mid\varphi\text{ is Lipschitzian }&|\varphi(t)|\leq L,\ t\in[m_0,\infty)\cap\mathbb{T},\\ &\varphi(t)=\psi(t)\ \text{ if } t\in[m_0,0]\cap\mathbb{T}\text{ and }\varphi(t)\to0\text{ as }t\to\infty\}.\end{aligned}$$

Clearly, if $\{\varphi_n\}$ is a sequence of k-Lipschitzian functions converging to a function φ, then

$$\begin{aligned}|\varphi(u)-\varphi(v)|&\leq|\varphi(u)-\varphi_n(u)|+|\varphi_n(u)-\varphi_n(v)|+|\varphi_n(v)-\varphi(v)|\\&\leq\|\varphi-\varphi_n\|+k|u-v|+\|\varphi-\varphi_n\|.\end{aligned}$$

Consequently, as $n\to\infty$, we see that φ is k-Lipschitzian. It is clear that S_ψ is convex, bounded and complete endowed with $\|.\|$.

For $\varphi \in S_\psi$ and $t \in [0,\infty) \cap \mathbb{T}$, define the maps A, B and C on S_ψ as follows

$$(A\varphi)(t) := \frac{c(t)}{1-\tau^{\triangle}(t)}\varphi(t-\tau(t)) + \int_0^t b(s)\,G(\varphi(s),\varphi(s-\tau(t)))\,e_{\ominus a}(t,s)\,\Delta s$$
$$- \int_0^t \mu(s)\,\varphi^{\sigma}(s-\tau(s))\,e_{\ominus a}(t,s)\,\Delta s, \tag{4.1}$$

$$(B\varphi)(t) := \left[\psi(0) - \frac{c(0)}{1-\tau^{\triangle}(0)}\psi(-\tau(0))\right] e_{\ominus a}(t,0)$$
$$+ \int_0^t a(s)\,(Hx)(s)\,e_{\ominus a}(t,s)\Delta s, \tag{4.2}$$

and

$$(C\varphi)(t) := (A\varphi)(t) + (B\varphi)(t). \tag{4.3}$$

If we are able to prove that C possesses a fixed point φ on the set S_ψ, then $x(t,0,\psi) = \varphi(t)$ for $t \in [0,\infty) \cap \mathbb{T}$, $x(t,0,\psi) = \psi(t)$ on $[m_0,0] \cap \mathbb{T}$, $x(t,0,\psi)$ satisfies (1.1) when its derivative exists and $x(t,0,\psi) \to 0$ as $t \to \infty$.

Let $\alpha(t) = \frac{c(t)}{1-\tau^{\triangle}(t)}$ and assume that there are constants $k_1, k_2, k_3 > 0$ such that for $0 \le t_1 < t_2$,

$$\left|\int_{t_1}^{t_2} a(u)\,\Delta u\right| \le k_1 |t_2 - t_1|, \tag{4.4}$$

$$|\tau(t_2) - \tau(t_1)| \le k_2 |t_2 - t_1|, \tag{4.5}$$

and

$$|\alpha(t_2) - \alpha(t_1)| \le k_3 |t_2 - t_1|. \tag{4.6}$$

Suppose that for $t \in [0,\infty) \cap \mathbb{T}$,

$$|\mu(t)| \le \delta a(t), \tag{4.7}$$

$$(N_1 + N_2)\,|b(t)| \le \beta a(t), \tag{4.8}$$

$$\sup_{t \ge 0} |\alpha(t)| = \alpha_0, \tag{4.9}$$

and that

$$J(\alpha_0 + \beta + \delta) < 1, \tag{4.10}$$

$$\max(|H(-L)|, |H(L)|) \le \frac{2L}{J}, \tag{4.11}$$

where α_0, β, δ and J are constants with $J > 3$.

Choose $\gamma > 0$ small enough and such that

$$\left(1 + \left|\frac{c(0)}{1-\tau^{\triangle}(0)}\right|\right)\gamma + \frac{3L}{J} \le L. \tag{4.12}$$

The chosen γ in the relation (4.12) is used below in Lemma 5 to show that if $\epsilon = L$ and if $\|\psi\| < \gamma$, then the solutions satisfy $|x(t,0,\psi)| < \epsilon$.

Assume further that

$$t - \tau(t) \to \infty \text{ as } t \to \infty \text{ and } \int_0^t a(u)\,\Delta u \to \infty \text{ as } t \to \infty, \tag{4.13}$$

$$\alpha(t) \to 0 \text{ as } t \to \infty, \tag{4.14}$$

$$\frac{\mu(t)}{a(t)} \to 0 \text{ as } t \to \infty, \tag{4.15}$$

and

$$\frac{b(t)}{a(t)} \to 0 \text{ as } t \to \infty. \tag{4.16}$$

We begin with the following theorem (see [1]) and for convenience we present its proof below. In the next theorem, we prove that for a well chosen function h, the mapping H given by (3.5) is a large contraction on the set S_ψ. So let us make the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$.

(H1) $h : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,
(H2) the function h is strictly increasing on $[-L, L]$,
(H3) $\sup_{t\in(-L,L)} h'(t) \leq 1$.

Theorem 7. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1) - (H3)$. Then the mapping H in (3.5) is a large contraction on the set S_ψ.*

Proof. Let $\phi, \varphi \in S_\psi$ with $\phi^\sigma \neq \varphi^\sigma$. Then $\phi^\sigma(t) \neq \varphi^\sigma(t)$ for some $t \in \mathbb{T}$. Let us denote the set of all such t by $D(\phi, \varphi)$, i.e.,

$$D(\phi, \varphi) = \{t \in \mathbb{T} : \phi^\sigma(t) \neq \varphi^\sigma(t)\}.$$

For all $t \in D(\phi, \varphi)$, we have

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &= |\phi^\sigma(t) - h(\phi^\sigma(t)) - \varphi^\sigma(t) + h(\varphi^\sigma(t))| \\ &= |\phi^\sigma(t) - \varphi^\sigma(t)| \left| 1 - \left(\frac{h(\phi^\sigma(t)) - h(\varphi^\sigma(t))}{\phi^\sigma(t) - \varphi^\sigma(t)} \right) \right|. \end{aligned} \tag{4.17}$$

Since h is a strictly increasing function, we have

$$\frac{h(\phi^\sigma(t)) - h(\varphi^\sigma(t))}{\phi^\sigma(t) - \varphi^\sigma(t)} > 0 \text{ for all } t \in D(\phi, \varphi). \tag{4.18}$$

For each fixed $t \in D(\phi, \varphi)$, define the interval $U_t \subset [-L, L]$ by

$$U_t = \begin{cases} (\varphi^\sigma(t), \phi^\sigma(t)) & \text{if } \phi^\sigma(t) > \varphi^\sigma(t), \\ (\phi^\sigma(t), \varphi^\sigma(t)) & \text{if } \varphi^\sigma(t) > \phi^\sigma(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\phi, \varphi)$, there exists a real number $c_t \in U_t$ such that

$$\frac{h(\phi^\sigma(t)) - h(\varphi^\sigma(t))}{\phi^\sigma(t) - \varphi^\sigma(t)} = h'(c_t).$$

By $(H2)$ and $(H3)$, we have

$$0 \leq \inf_{u\in(-L,L)} h'(u) \leq \inf_{u\in U_t} h'(u) \leq h'(c_t) \leq \sup_{u\in U_t} h'(u) \leq \sup_{u\in(-L,L)} h'(u) \leq 1. \tag{4.19}$$

Hence, by (4.17)-(4.19), we obtain

$$|(H\phi)(t) - (H\varphi)(t)| \leq \left| 1 - \inf_{u\in(-L,L)} h'(u) \right| |\phi^\sigma(t) - \varphi^\sigma(t)|, \tag{4.20}$$

for all $t \in D(\phi, \varphi)$. Then by $(H3)$, we have

$$\|H\phi - H\varphi\| \leq \|\phi - \varphi\|.$$

Now, choose a fixed $\epsilon \in (0, 1)$ and assume that ϕ and φ are two functions in S_ψ satisfying

$$\epsilon \leq \sup_{t\in D(\phi,\varphi)} |\phi(t) - \varphi(t)| = \|\phi - \varphi\|.$$

If $|\phi^\sigma(t) - \varphi^\sigma(t)| \leq \frac{\epsilon}{2}$ for some $t \in D(\phi, \varphi)$, then by (4.19) and (4.20), we get

$$|(H\phi)(t) - (H\varphi)(t)| \leq |\phi^\sigma(t) - \varphi^\sigma(t)| \leq \frac{1}{2} \|\phi - \varphi\|. \tag{4.21}$$

Since h is continuous and strictly increasing, the function $h\left(u + \frac{\epsilon}{2}\right) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\epsilon}{2} \leq |\phi^\sigma(t) - \varphi^\sigma(t)|$ for some $t \in D(\phi, \varphi)$, then by $(H2)$ and $(H3)$, we conclude that

$$1 \geq \frac{h(\phi^\sigma(t)) - h(\varphi^\sigma(t))}{\phi^\sigma(t) - \varphi^\sigma(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2L} \min \left\{ h \left(u + \frac{\epsilon}{2} \right) - h(t), \ u \in [-L, L] \right\} > 0.$$

Hence, (4.17) implies

$$|(H\phi)(t) - (H\varphi)(t)| \leq (1 - \lambda) \|\phi - \varphi\| . \tag{4.22}$$

Consequently, combining (4.21) and (4.22), we obtain

$$|(H\phi)(t) - (H\varphi)(t)| \leq \eta \|\phi - \varphi\| ,$$

where

$$\eta = \max \left\{ \frac{1}{2}, 1 - \lambda \right\} < 1.$$

The proof is complete. □

By step we will prove the fulfillment of (i), (ii) and (iii) in Theorem 5.

Lemma 4. *Suppose that (3.1), (3.2), (4.7)-(4.10) and (4.13) are true. For A defined in (4.1), if $\varphi \in S_\psi$, then $|(A\varphi)(t)| \leq L/J < L$. Moreover, $(A\varphi)(t) \to 0$ as $t \to \infty$.*

Proof. Using the conditions (4.7)-(4.10) and the expression (4.1) of the map A, we get

$$\begin{aligned}
|(A\varphi)(t)| &\leq \left| \frac{c(t)}{1 - \tau^{\triangle}(t)} \varphi(t - \tau(t)) \right| + \int_0^t |b(t)| \, |G(\varphi(s), \varphi(s - \tau(s)))| \, e_{\ominus a}(t, s) \, \Delta s \\
&\quad + \int_0^t |\mu(s)| \, |\varphi^\sigma(s - \tau(s))| \, e_{\ominus a}(t, s) \, \Delta s \\
&\leq \alpha_0 L + \int_0^t (N_1 + N_2) |b(t)| \, L e_{\ominus a}(t, s) \, \Delta s + L \int_0^t |\mu(s)| \, e_{\ominus a}(t, s) \, \Delta s \\
&\leq L \left\{ \alpha_0 + \int_0^t \beta a(s) \, e_{\ominus a}(t, s) \, \Delta s + \int_0^t \delta a(s) \, e_{\ominus a}(t, s) \, \Delta s \right\} \\
&\leq L(\alpha_0 + \beta + \delta) \leq \frac{L}{J} < L.
\end{aligned}$$

So AS_ψ is bounded by L as required.

Let $\varphi \in S_\psi$ be fixed. We will prove that $(A\varphi)(t) \to 0$ as $t \to 0$. Due to the conditions $t - \tau(t) \to \infty$ as $t \to \infty$ in (4.13) and (4.9), it is obvious that the first term on the right hand side of A tends to 0 as $t \to \infty$. That is

$$\left| \frac{c(t)}{1 - \tau^{\triangle}(t)} \varphi(t - \tau(t)) \right| \leq \alpha_0 |\varphi(t - \tau(t))| \to 0 \text{ as } t \to \infty.$$

It is left to show that the two remaining integral terms of A go to zero as $t \to \infty$. Let $\epsilon > 0$ be given. Find T such that $|\varphi^\sigma(t - \tau(t))| < \epsilon$ for $t \geq T$. Then we have

$$\begin{aligned}
&\left| \int_0^t \mu(s) \, \varphi^\sigma(s - \tau(s)) \, e_{\ominus a}(t, s) \, \Delta s \right| \\
&\leq \int_0^T |\mu(s) \, \varphi^\sigma(s - \tau(s))| \, e_{\ominus a}(t, s) \, \Delta s + \int_T^t |\mu(s)| \, |\varphi^\sigma(s - \tau(s))| \, e_{\ominus a}(t, s) \, \Delta s \\
&\leq L e_{\ominus a}(t, T) \int_0^T |\mu(s)| \, e_{\ominus a}(T, s) \, \Delta s + \epsilon \int_T^t |\mu(s)| \, e_{\ominus a}(t, s) \, \Delta s \\
&\leq L \delta e_{\ominus a}(t, T) + \epsilon \delta.
\end{aligned}$$

The term $L\delta e_{\ominus a}(t, T)$ is arbitrarily small as $t \to \infty$, because of (4.13). The remaining integral term in A goes to zero by just a similar argument. This ends the proof. □

Lemma 5. *Let $(H1) - (H3)$, (3.1), (3.2), (4.7)-(4.11) and (4.13) hold. For A, B defined in (4.1) and (4.2), if $\phi, \varphi \in S_\psi$ are arbitrary, then*

$$\|B\varphi + A\phi\| \leq L.$$

Moreover, B is a large contraction on S_ψ with a unique fixed point in S_ψ and $B\varphi(t) \to 0$ as $t \to \infty$.

Proof. Using the definitions (4.1), (4.2) of A and B and applying (4.7)-(4.11), we obtain

$$\begin{aligned}
&\left|(B\varphi)(t)+(A\phi)(t)\right| \\
&\leq \left(1+\left|\frac{c(0)}{1-\tau^{\triangle}(0)}\right|\right)\|\psi\| e_{\ominus a}(t,0)+\alpha_0 L+L\int_T^t |\mu(s)| e_{\ominus a}(t,s)\Delta s \\
&+\int_0^t (N_1+N_2)|b(s)| L e_{\ominus a}(t,s)\Delta s+\frac{2L}{J}\int_0^t a(s) e_{\ominus a}(t,s)\Delta s \\
&\leq \left(1+\left|\frac{c(0)}{1-\tau^{\triangle}(0)}\right|\right)\|\psi\|+(\alpha_0+\beta+\delta)L+\frac{2L}{J} \\
&\leq \left(1+\left|\frac{c(0)}{1-\tau^{\triangle}(0)}\right|\right)\|\psi\|+\frac{L}{J}+\frac{2L}{J},
\end{aligned}$$

by the monotonicity of the mapping H. So from the above inequality, by choosing the initial function ψ having small norm, say $\|\psi\|<\gamma$, then, and referring to (4.12), we obtain

$$\left|(B\varphi)(t)+(A\phi)(t)\right| \leq \left(1+\left|\frac{c(0)}{1-\tau^{\triangle}(0)}\right|\right)\gamma+\frac{3L}{J}\leq L.$$

Since $0\in S_\psi$, we have also proved that $|(B\varphi)(t)|\leq L$. The proof that $B\varphi$ is Lipschitzian is similar to that of the map $A\varphi$ below. To see that B is a large contraction on S_ψ with a unique fixed point, we know from Theorem 7 that $H(\varphi)=\varphi^\sigma-h(\varphi^\sigma)$ is a large contraction within the integrand. Thus, for any ϵ, from the proof of that Theorem 7, we have found $\eta<1$ such that

$$\begin{aligned}
\left|(B\varphi)(t)-(A\phi)(t)\right| &\leq \int_0^t a(s)\left|(H\phi)(s)-(H\varphi)(s)\right| e_{\ominus a}(t,s)\Delta s \\
&\leq \eta\int_0^t a(s)\|\varphi-\phi\| e_{\ominus a}(t,s)\Delta s \\
&\leq \eta\|\varphi-\phi\|.
\end{aligned}$$

To prove that $(B\varphi)(t)\to 0$ as $t\to\infty$, we use (4.13) for the first term, and for the second term, we argue as above for the map A. □

Lemma 6. *Suppose (3.1), (3.2), (4.7)-(4.10) hold. Then the mapping A is continuous on S_ψ.*

Proof. Let $\varphi,\phi\in S_\psi$, then

$$\begin{aligned}
&\left|(A\varphi)(t)-(A\phi)(t)\right| \\
&\leq \{\alpha_0\left|\varphi(t-\tau(t))-\phi(t-\tau(t))\right| \\
&+\left|\int_0^t b(s)\left[G(\varphi(s),\varphi(s-\tau(s)))-G(\phi(s),\phi(s-\tau(s)))\right] e_{\ominus a}(t,s)\Delta s\right| \\
&+\left|\int_0^t \mu(s)\left[\varphi^\sigma(s-\tau(s))-\phi^\sigma(s-\tau(s))\right] e_{\ominus a}(t,s)\Delta s\right|\Big\} \\
&\leq \alpha_0\|\varphi-\phi\|+\int_0^t (N_1+N_2)|b(s)|\|\varphi-\phi\| e_{\ominus a}(t,s)\Delta s \\
&+\|\varphi-\phi\|\int_0^t |\mu(s)| e_{\ominus a}(t,s)\Delta s \\
&\leq (\alpha_0+\beta+\delta)\|\varphi-\phi\|\int_0^t a(s) e_{\ominus a}(t,s)\Delta s \\
&\leq (\alpha_0+\beta+\delta)\|\varphi-\phi\|\leq (1/J)\|\varphi-\phi\|.
\end{aligned}$$

Let $\epsilon>0$ be arbitrary. Define $\eta=\epsilon J$. Then for $\|\varphi-\phi\|\leq\eta$, we obtain

$$\|A\varphi-A\phi\|\leq\frac{1}{J}\|\varphi-\phi\|\leq\epsilon.$$

Therefore, A is continuous. □

Lemma 7. *Let (3.1), (3.2), (4.4)-(4.9) and (4.14)-(4.16) hold. The function $A\varphi$ is Lipschitzian and the operator A maps S_ψ into a compact subset of S_ψ*

Proof. Let $\varphi \in S_\psi$ and let $0 \le t_1 < t_2$. Then

$$\begin{aligned}
&\left|(A\varphi)(t_2) - (A\varphi)(t_1)\right| \\
&\le \left|\frac{c(t_2)}{1-\tau^{\triangle}(t_2)}\varphi(t_2-\tau(t_2)) - \frac{c(t_1)}{1-\tau^{\triangle}(t_1)}\varphi(t_1-\tau(t_1))\right| \\
&+ \left|\int_0^{t_2}\mu(s)\varphi^\sigma(s-\tau(s))e_{\ominus a}(t_2,s)\Delta s - \int_0^{t_1}\mu(s)\varphi^\sigma(s-\tau(s))e_{\ominus a}(t_1,s)\Delta s\right| \\
&+ \left|\int_0^{t_2} b(s)G(\varphi(s),\varphi(s-\tau(s)))e_{\ominus a}(t_2,s)\Delta s\right. \\
&\left. - \int_0^{t_1} b(s)G(\varphi(s),\varphi(s-\tau(s)))e_{\ominus a}(t_1,s)\Delta s\right|. \qquad (4.23)
\end{aligned}$$

By hypotheses (4.5)-(4.6), we have

$$\begin{aligned}
&\left|\alpha(t_2)\varphi(t_2-\tau(t_2)) - \alpha(t_1)\varphi(t_1-\tau(t_1))\right| \\
&\le |\alpha(t_2)|\left|\varphi(t_2-\tau(t_2)) - \varphi(t_1-\tau(t_1))\right| + \left|\varphi(t_1-\tau(t_1))\right|\left|\alpha(t_2)-\alpha(t_1)\right| \\
&\le \alpha_0 k\left|(t_2-t_1) - (\tau(t_2)-\tau(t_1))\right| + Lk_3|t_2-t_1| \\
&\le (\alpha_0 k + \alpha_0 k k_2 + Lk_3)|t_2-t_1|, \qquad (4.24)
\end{aligned}$$

where k is the Lipschitz constant of φ. By hypotheses (4.4) and (4.7), we have

$$\begin{aligned}
&\left|\int_0^{t_2}\mu(s)\varphi^\sigma(s-\tau(s))e_{\ominus a}(t_2,s)\Delta s - \int_0^{t_1}\mu(s)\varphi^\sigma(s-\tau(s))e_{\ominus a}(t_1,s)\Delta s\right| \\
&= \left|\int_0^{t_1}\mu(s)\varphi^\sigma(s-\tau(s))e_{\ominus a}(t_1,s)\left(e_{\ominus a}(t_2,t_1)-1\right)\Delta s\right. \\
&\left. + \int_{t_1}^{t_2}\mu(s)\varphi^\sigma(s-\tau(s))e_{\ominus a}(t_2,s)\Delta s\right| \\
&\le L\left|e_{\ominus a}(t_2,t_1)-1\right|\int_0^{t_1}\delta a(s)e_{\ominus a}(t_1,s)\Delta s + L\int_{t_1}^{t_2}|\mu(s)|e_{\ominus a}(t_2,s)\Delta s \\
&\le L\delta\int_{t_1}^{t_2}a(s)\Delta s + L\int_{t_1}^{t_2}e_{\ominus a}(t_2,s)\left(\int_{t_1}^{s}|\mu(v)|\Delta v\right)^{\triangle}\triangle s \\
&\le L\delta\int_{t_1}^{t_2}a(s)\Delta s + L\left\{\left[e_{\ominus a}(t_2,s)\int_{t_1}^{s}|\mu(v)|\Delta v\right]_{t_1}^{t_2}\right. \\
&\left. + \int_{t_1}^{t_2}a(s)e_{\ominus a}(t_2,s)\int_{t_1}^{s}|\mu(v)|\Delta v\Delta s\right\} \\
&\le L\delta\int_{t_1}^{t_2}a(s)\Delta s + L\int_{t_1}^{t_2}|\mu(s)|\Delta s\left(1+\int_{t_1}^{t_2}a(s)e_{\ominus a}(t_2,s)\Delta s\right) \\
&\le L\delta\int_{t_1}^{t_2}a(s)\Delta s + 2L\int_{t_1}^{t_2}|\mu(s)|\Delta s \\
&\le L\delta\int_{t_1}^{t_2}a(s)\Delta s + 2L\delta\int_{t_1}^{t_2}a(s)\Delta s \\
&\le 3L\delta k_1|t_2-t_1|. \qquad (4.25)
\end{aligned}$$

Similarly, by (4.4) and (4.8), we deduce

$$\begin{aligned}
&\left|\int_0^{t_2} b(s)\, G(\varphi(s), \varphi(s-\tau(s)))\, e_{\ominus a}(t_2, s)\,\Delta s\right. \\
&\left.-\int_0^{t_1} b(s)\, G(\varphi(s), \varphi(s-\tau(s)))\, e_{\ominus a}(t_1, s)\,\Delta s\right| \\
&=\left|\int_0^{t_1} b(s)\, G(\varphi(s), \varphi(s-\tau(s)))\, e_{\ominus a}(t_1, s)\,(e_{\ominus a}(t_2, t_1)-1)\,\Delta s\right. \\
&\left.+\int_{t_1}^{t_2} b(s)\, G(\varphi(s), \varphi(s-\tau(s)))\, e_{\ominus a}(t_2, s)\,\Delta s\right| \\
&\leq L\left|e_{\ominus a}(t_2, t_1)-1\right| \int_0^{t_1} \beta a(s)\, e_{\ominus a}(t_1, s)\,\Delta s+(N_1+N_2)\, L \int_{t_1}^{t_2}|b(s)|\, e_{\ominus a}(t_2, s)\,\Delta s \\
&\leq L\beta \int_{t_1}^{t_2} a(u)\,\Delta u+(N_1+N_2)\, L \int_{t_1}^{t_2} e_{\ominus a}(t_2, s)\left(\int_{t_1}^{s}|b(v)|\,\Delta v\right)^{\triangle} \triangle s \\
&\leq L\beta \int_{t_1}^{t_2} a(u)\,\Delta u+(N_1+N_2)\, L\left\{\left[e_{\ominus a}(t_2, s) \int_{t_1}^{s}|b(v)|\,\Delta v\right]_{t_1}^{t_2}\right. \\
&\left.+\int_{t_1}^{t_2} a(s)\, e_{\ominus a}(t_2, s) \int_{t_1}^{s}|b(v)|\,\Delta v \Delta s\right\} \\
&\leq L\beta \int_{t_1}^{t_2} a(u)\,\Delta u+(N_1+N_2)\, L \int_{t_1}^{t_2}|b(s)|\,\Delta s\left(1+\int_{t_1}^{t_2} a(s)\, e_{\ominus a}(t_2, s)\,\Delta s\right) \\
&\leq L\beta \int_{t_1}^{t_2} a(u)\,\Delta u+2(N_1+N_2)\, L \int_{t_1}^{t_2}|b(s)|\,\Delta s \\
&\leq L\beta \int_{t_1}^{t_2} a(u)\,\Delta u+2L\beta \int_{t_1}^{t_2} a(s)\,\Delta s \\
&\leq 3L\beta k_1\left|t_2-t_1\right| .
\end{aligned} \tag{4.26}$$

Thus, by substituting (4.24)-(4.26) in (4.23), we obtain

$$\begin{aligned}
&\left|(A\varphi)(t_2)-(A\varphi)(t_1)\right| \\
&\leq(\alpha_0 k+\alpha_0 k k_2+L k_3)\left|t_2-t_1\right|+3L\delta k_1\left|t_2-t_1\right|+3L\beta k_1\left|t_2-t_1\right| \\
&\leq K\left|t_2-t_1\right|,
\end{aligned} \tag{4.27}$$

for a constant $K>0$. This shows that $A\varphi$ is Lipschitzian if φ is and that AS_ψ is equicontinuous.

Next, we notice that for arbitrary $\varphi \in S_\psi$, we have

$$\begin{aligned}
|A\varphi(t)| \leq &\left|\frac{c(t)}{1-\tau^{\triangle}(t)} \varphi(t-\tau(t))\right|+\int_0^t|b(s)|\,|G(\varphi(s), \varphi(s-\tau(s)))|\, e_{\ominus a}(t, s)\,\Delta s \\
&+\int_0^t|\mu(s)\, \varphi(s-\tau(s))|\, e_{\ominus a}(t, s)\,\Delta s \\
&\leq L|\alpha(t)|+(N_1+N_2)\, L \int_0^t a(s)\,[|b(s)| / a(s)]\, e_{\ominus a}(t, s)\,\Delta s \\
&+L \int_0^t a(s)\,[|\mu(s)| / a(s)]\, e_{\ominus a}(t, s)\,\Delta s \\
&=q(t),
\end{aligned}$$

because of (4.14)-(4.16). Using a method like the one used for the map A, we see that $q(t) \rightarrow 0$ as $t \rightarrow \infty$. By Theorem 6, we conclude that the set AS_ψ resides in a compact set. □

Theorem 8. *Let $L > 0$. Suppose that the conditions $(H1)-(H3)$, (3.1), (3.2) and (4.14)-(4.16) hold. If ψ is a given initial function which is sufficiently small, then there is a solution $x(t,0,\psi)$ of (1.1) with $|x(t,0,\psi)| \leq L$ and $x(t,0,\psi) \to 0$ as $t \to \infty$.*

Proof. From Lemmas 4 and 7 we have A is bounded by L, Lipschitzian and $(A\phi)(t) \to 0$ as $t \to \infty$. So A maps S_ψ into S_ψ. From Lemmas 5 and 7 for arbitrary, we have $\phi, \varphi \in S_\psi$, $B\varphi + A\phi \in S_\psi$ since both $A\phi$ and $B\varphi$ are Lipschitzian bounded by L and $(B\varphi)(t) \to 0$ as $t \to \infty$. From Lemmas 6 and 7, we have proved that A is continuous and AS_ψ resides in a compact set. Thus, all the conditions of Theorem 5 are satisfied. Therefore, there exists a solution of (1.1) with $|x(t,0,\psi)| \leq L$ and $x(t,0,\psi) \to 0$ as $t \to \infty$. □

5. Stability and Compactness

Referring to Burton [10], except for the fixed point method, we know of no other way proving that solutions of (1.1) converge to zero. Nevertheless, if all we need is stability and not asymptotic stability, then we can avoid conditions (4.14)-(4.16) and still use Krasnoselskii-Burton's theorem on a Banach space endowed with a weighted norm.

Let $g : [m_0, \infty) \cap \mathbb{T} \to [1, \infty)$ be any strictly increasing and rd-continuous function with $g(m_0) = 1$, $g(s) \to \infty$ as $s \to \infty$. Let $\left(S, |.|_g\right)$ be the Banach space of rd-continuous function $\varphi : [m_0, \infty) \cap \mathbb{T} \to \mathbb{R}$ for which

$$|\varphi|_g := \sup_{t \geq m_0} \left| \frac{\varphi(t)}{g(t)} \right| < \infty,$$

exists. We continue to use $\|.\|$ as the supremum norm of any $\varphi \in S$ provided φ bounded. Also, we use $\|\psi\|$ as the bound of the initial function. Further, in a similar way as Theorem 7, we can prove that the function $H(\varphi) = \varphi^\sigma - h(\varphi^\sigma)$ is still a large contraction with the norm $|.|_g$.

Theorem 9. *If the conditions of Theorem 8 hold, except for (4.14)-(4.16), then the zero solution of (1.1) is stable.*

Proof. We prove the stability starting at $t_0 = 0$. Let $\epsilon > 0$ be given such that $0 < \epsilon < L$, then for $|x| \leq \epsilon$ find γ^* with $|x^\sigma - h(x^\sigma)| \leq \gamma^*$ and choose a number γ such that

$$\gamma + \gamma^* + \frac{\epsilon}{J} \leq \epsilon. \tag{5.1}$$

In fact, since $x^\sigma - h(x^\sigma)$ is increasing on $(-L, L)$, we may take $\gamma^* = \frac{2\epsilon}{J}$. Thus, inequality (5.1) allows $\gamma > 0$. Now, remove the condition $\varphi(t) \to 0$ as $t \to 0$ from S_ψ defined previously and consider the set

$$\begin{aligned} M_\psi = \{\varphi \in S \mid \varphi \text{ Lipschitzian, } \ |\varphi(t)| \leq \epsilon, \ t \in [m_0, \infty) \cap \mathbb{T} \\ \text{and } \varphi(t) = \psi(t) \ \text{ if } t \in [m_0, 0] \cap \mathbb{T}\}. \end{aligned}$$

Define A, B on M_ψ as before by (4.1), (4.2). We easily check that if $\varphi \in M_\psi$, then $|(A\varphi)(t)| < \epsilon$, and B is a large contraction on M_ψ. Also, by choosing $\|\psi\| < \gamma$ and referring to (5.1), we verify that for $\varphi, \phi \in M_\psi$ $|(B\varphi)(t) + (A\phi)(t)| \leq \epsilon$ and $|(B\varphi)(t)| \leq \epsilon$. AM_ψ is an equicontinuous set. According to [10, Theorem 4.0.1], in the space $\left(S, |.|_g\right)$ the set AM_ψ resides in a compact subset of M_ψ. Moreover, the operator $A : M_\psi \to M_\psi$ is continuous. Indeed, for $\varphi, \phi \in S_\psi$,

$$
\begin{aligned}
&\frac{|(A\varphi)(t) - (A\phi)(t)|}{g(t)} \\
&\leq \frac{1}{g(t)} \{\alpha_0 |\varphi(t-\tau(t)) - \phi(t-\tau(t))| \\
&+ \left| \int_0^t b(s) [G(\varphi(s), \varphi(s-\tau(s))) - G(\phi(s-\tau(s)))] e_{\ominus a}(t,s) \Delta s \right| \\
&+ \left| \int_0^t \mu(s) [\varphi^\sigma(s-\tau(s)) - \phi^\sigma(s-\tau(s))] e_{\ominus a}(t,s) \Delta s \right| \Bigg\} \\
&\leq \alpha_0 |\varphi - \phi|_g \\
&+ \int_0^t |b(s)| \left(\frac{N_1 |\varphi(s) - \phi(s)|}{g(t)} + \frac{N_2 |\varphi(s-\tau(s)) - \phi(s-\tau(s))|}{g(t)} \right) e_{\ominus a}(t,s) \Delta s \\
&+ \int_0^t |\mu(s)| \frac{|\varphi^\sigma(s-\tau(s)) - \phi^\sigma(s-\tau(s))|}{g(t)} e_{\ominus a}(t,s) \Delta s \\
&\leq \alpha_0 |\varphi - \phi|_g + \int_0^t |b(s)| \left[\frac{N_1 |\varphi(s) - \phi(s)|}{g(s)} \frac{g(s)}{g(t)} \right] e_{\ominus a}(t,s) \Delta s \\
&+ \int_0^t |b(s)| \left[\frac{N_2 |\varphi(s-\tau(s)) - \phi(s-\tau(s))|}{g(s-\tau(s))} \frac{g(s-\tau(s))}{g(t)} \right] e_{\ominus a}(t,s) \Delta s \\
&+ \int_0^t |\mu(s)| \left[\frac{|\varphi^\sigma(s-\tau(s)) - \phi^\sigma(s-\tau(s))|}{g^\sigma(s-\tau(s))} \frac{g^\sigma(s-\tau(s))}{g(t)} \right] e_{\ominus a}(t,s) \Delta s \\
&\leq \alpha_0 |\varphi - \phi|_g + |\varphi - \phi|_g \int_0^t |b(s)| \left[\frac{N_1 g(s) + N_2 g(s-\tau(s))}{g(t)} \right] e_{\ominus a}(t,s) \Delta s \\
&+ \delta |\varphi - \phi|_g \int_0^t a(s) \frac{g^\sigma(s-\tau(s))}{g(t)} e_{\ominus a}(t,s) \Delta s \\
&\leq \alpha_0 |\varphi - \phi|_g + \beta |\varphi - \phi|_g \int_0^t a(s) e_{\ominus a}(t,s) \Delta s + \delta |\varphi - \phi|_g \int_0^t a(s) e_{\ominus a}(t,s) \Delta s \\
&\leq \frac{1}{J} |\varphi - \phi|_g .
\end{aligned}
$$

The conditions of Theorem 5 are satisfied on M_ψ, and so there exists a fixed point lying in M_ψ and solving (1.1). □

References

[1] M. Adıvar, Y. N. Raffoul, Existence of periodic solutions in totally nonlinear delay dynamic equations. Electron. J. Qual. Theory Differ. Equ., Spec. Ed. 1 (2009), 1–20.

[2] A. Ardjouni, I. Derrardjia and A. Djoudi, Stability in totally nonlinear neutral differential equations with variable delay, Acta Math. Univ. Comenianae, 83 (2014), 119-134.

[3] A. Ardjouni, A Djoudi, Existence of periodic solutions for nonlinear neutral dynamic equations with functional delay on a time scale, Acta Univ. Palacki. Olomnc., Fac. rer. nat., Mathematica 52, 1 (2013) 5-19.

[4] A. Ardjouni, A Djoudi, Stability in neutral nonlinear dynamic equations on time scale with unbounded delay, Stud. Univ. Babeç-Bolyai Math. 57(2012), No. 4, 481-496.

[5] A. Ardjouni, A Djoudi, Fixed points and stability in linear neutral differential equations with variable delays, Nonlinear Analysis 74 (2011), 2062-2070.

[6] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhauser, Boston, 2001.

[7] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

[8] T. A. Burton, Liapunov functionals, fixed points and stability by Krasnoselskii's theorem, Nonlinear Stud. 9 (2001), 181–190.

[9] T. A. Burton, Stability by fixed point theory or Liapunov theory: A Comparaison, Fixed Point Theory, 4(2003), 15-32.

[10] T. A. Burton, Stability by Fixed Point Theory for Functional Differential Equations, Dover Publications, New York, 2006.

[11] I. Derrardjia, A. Ardjouni and A. Djoudi, Stability by Krasnoselskii's theorem in totally nonlinear neutral differential equations, Opuscula Math. 33(2) (2013), 255-272.

[12] S. Hilger, Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph. D. thesis, Universität Würzburg, Würzburg, 1988.

[13] E. R. Kaufmann, Y. N. Raffoul, Stability in neutral nonlinear dynamic equations on a time scale with functional delay, Dynamic Systems and Applications 16 (2007) 561-570.

[14] D. R. Smart, Fixed point theorems, Cambridge Tracts in Mathematics, no. 66, Cambridge University Press, London–New York, 1974.

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