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SOME IMPLICIT METHODS FOR SOLVING HARMONIC VARIATIONAL INEQUALITIES

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Abstract. In this paper, we use the auxiliary principle technique to suggest an implicit method for solving the harmonic variational inequalities. It is shown that the convergence of the proposed method only needs pseudo monotonicity of the operator, which is a weaker condition than monotonicity.

1. Introduction

Variational inequalities, which were introduced and investigated by Stampacchia [18] in 1964, constitutes a significant extension and generalization of the variational principles. Variational inequality theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics, regional and engineering sciences. The techniques and ideas of variational inequalities are being applied in a variety of diverse areas of pure and applied sciences and prove to productive. The variational inequality is related to simple fact that the minimum of a differentiable convex function on a convex set in a norm space can be characterized by the variational inequality. However, it is remarkable that this theory allows many diversified applications. For the applications, formulation, numerical methods and other aspects of variational inequalities, see [2, 5, 7, 9, 10, 11, 12, 13] and the references therein.

We would like to mention that the harmonic means have applications in electrical circuits. To be more precise, the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, if r_1 and r_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula: $\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 r_2}{r_1 + r_2}$, which is half the harmonic means. The harmonic mean has been used in developing the parallel algorithms for solving various problems. Noor [13] used the harmonic mean to suggest some iterative methods for solving nonlinear equations. Using the weighted harmonic means, one usually defines the harmonic convex set, which can be viewed as another extension of the convex set. This motivated to introduce the concept of the harmonic convex functions, which is a significant generalization of the convex functions, see [8]. Anderson et al [1] have investigated several aspects of the harmonic convex functions. Iscan [6] and Noor et al [17] have derived several Hermite-Hadmard type integral inequalities for the harmonic convex functions and their variant forms. For recent developments, see[15, 16] and the references therein. To the best our knowledge, no one has studied the properties of the differentiable harmonic convex functions. This fact motivated Noor and Noor [14] to investigate the characterizations of the differentiable harmonic convex functions. They have shown that the minimum of the differentiable harmonic convex function can be characterized by a class of variational inequalities, which is called harmonic variational inequality. We here use the auxiliary principle technique[4] to suggest an implicit method for solving the harmonic variational inequalities. Convergence of the proposed implicit method is considered under the pseudomonotonicity of the operator. The ideas and techniques of this paper may be starting point for a wide range of novel and innovative applications of harmonic variational inequalities in various fields. Development of efficient and implementable numerical methods for solving the harmonic variational inequalities is an interesting problem, which needs further efforts.

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2. Preliminaries

Let $\mathcal{C}$ be a nonempty closed and harmonic convex set in the real Hilbert space H. We denote by $\langle .,. \rangle$ and $\|.\|$ be the inner product and norm, respectively.

For a given nonlinear operator T, consider the problem of finding $\eta \in \mathcal{C}$, such that

$$\langle \mathrm{T}\eta, \frac{\eta\xi}{\eta - \xi} \rangle \geq 0, \quad \forall \xi \in \mathcal{C}. \tag{1}$$

The inequality (1) is called the harmonic variational inequality. It has been shown [11,12] that the minimum of a differentiable harmonic convex function can be characterized by harmonic variational inequality of type (1). For the sake of completeness and to convey the main ideas, we include the relevant details.

Definitions 2.1[1,6]. The set $\mathcal{C}$ is said to be a harmonic convex, if

$$\frac{\eta\xi}{\xi + \lambda(\eta - \xi)} \in \mathcal{C}, \quad \forall \eta, \xi \in \mathcal{C}, \quad \lambda \in [0, 1].$$

Definition 2.2[1,6]. The function ϕ on the harmonic convex set $\mathcal{C}$ is said to be harmonic convex, if

$$\phi(\frac{\eta\xi}{\xi + t(\eta - \xi)}) \leq (1 - \lambda)\phi(\eta) + t\phi(\xi), \quad \forall \eta, \xi \in \mathcal{C} \quad \lambda \in [0, 1].$$

The function ϕ is said to be harmonic concave if and only if $-\phi$ is harmonic convex.

We now show that the minimum of a differentiable harmonic convex function on the harmonic convex set $\mathcal{C}$ can be characterized by the harmonic variational inequality (1). This result is mainly due to Noor and Noor [14].

Theorem 2.1 [13, 14]**.** Let ϕ be a differentiable harmonic convex function on the harmonic convex set $\mathcal{C}$. Then $\eta \in \mathcal{C}$ is a minimum of ϕ, if and only if, $\eta \in \mathcal{C}$ is the solution of the inequality

$$\langle \phi'(\eta), \frac{\eta\xi}{\eta - \xi} \rangle \geq 0, \quad \forall \xi \in \mathcal{C}, \tag{2}$$

which is called the harmonic variational inequality.

We would like to mention that Theorem 2.1 implies that harmonic convex programming problem can be studied via the harmonic variational inequality (1).

Theorem 2.2.[13, 14] Let ϕ be a differentiable harmonic convex functions on the harmonic convex set $\mathcal{C}$. Then

$$(i). \qquad \phi(\xi) - \phi(\eta) \geq \langle \phi'(\eta), \frac{\eta\xi}{\eta - \xi} \rangle, \quad \forall \eta, \xi \in \mathcal{C}.$$

$$(ii). \qquad \langle \phi'(\eta) - \phi'(\xi), \frac{\eta\xi}{\xi - \eta} \rangle \geq 0, \quad \forall \eta, \xi \in \mathcal{C},$$

where $\phi'(\eta)$ is the differential of ϕ at η in the direction $\frac{\eta\xi}{\xi - \eta}$.

Using Theorem 2.2, we can introduce some new concepts.

Definition 2.3. An operator T is said to be a harmonic monottone operator, if and only if,

$$\langle \mathrm{T}\eta - \mathrm{T}\xi, \frac{\eta\xi}{\xi - \eta} \rangle \geq 0, \quad \forall \eta, \xi \in \mathrm{H}.$$

Definition 2.4. An operator T is said to a harmonic pseudomonotone operator, if

$$\langle T\eta, \frac{\eta\xi}{\eta - \xi} \rangle \geq 0, \quad \text{implies} \quad -\langle T\xi, \frac{\eta\xi}{\eta - \xi} \rangle \geq 0, \quad \forall \eta, \xi \in H.$$

An harmonic monotone operator is a harmonic pseudomonotone operator, but the converse is not true.

3. Main Results

In this Section, we consider use the auxiliary principle technique to suggest an implicit method for solving harmonic variational inequality. This technique is mainly due to Glowinski et al [4].

For a given $\eta \in \mathcal{C}$ satisfying (1), consider the problem of finding $w \in \mathcal{C}$ such that

$$\langle \rho T w, \frac{\xi w}{w-\xi}\rangle + \langle w-\eta, \xi - w\rangle \geq 0, \quad \forall \xi \in \mathcal{C}, \tag{3}$$

which is called the auxiliary harmonic variational inequality. We remark that, if $w = \eta$, then w is a solution of the harmonic variational inequality (1). This observation is used to suggest and analyze an implicit method for solving (1).

Algorithm 3.1. For a given $\eta_0 \in \mathcal{C}$, compute the approximated solution η_{n+1} by the iterative scheme

$$\langle \rho \mathrm{T}\eta_{n+1}, \frac{\xi\xi_{n+1}}{\eta_{n+1}-\xi)}\rangle + \langle \eta_{n+1}-\eta_n, \xi - \eta_{n+1}\rangle \geq 0, \quad \forall \xi \in \mathcal{C}, \tag{4}$$

which is called the proximal-point (implicit) method.

To implement Algorithm 3.1, one usually uses the predictor-corrector technique. Consequently, Algorithm 3.1 can be rewritten in the following equivalent form;

Algorithm 3.2. For a given $\eta_0 \in \mathcal{C}$, compute the approximated solution η_{n+1} by the iterative schemes

$$\begin{aligned} \langle \rho \mathrm{T}\eta_n, \frac{\xi y_n}{y_n-\xi)}\rangle &+ \langle y_n-\eta_n, \xi - y_n\rangle \geq 0, \quad \forall \xi \in \mathcal{C}, \\ \langle \rho \mathrm{T}y_n, \frac{\xi\eta_{n+1}}{\eta_{n+1}-\xi)}\rangle &+ \langle \eta_{n+1}-\eta_n, \xi - \eta_{n+1}\rangle \geq 0, \quad \forall \xi \in \mathcal{C}, \end{aligned}$$

Algorithm 3.2 can be viewed as Koperlevich method for solving the harmonic variational inequalities. We would also like to point out that Algorithm 3.1 and Algorithm 3.2 are equivalent. This equivalence is used to study the convergence analysis of Algorithm 3.2.

We now study the convergence of the proposed Algorithm 3.1.

Theorem 3.1. Let $\eta \in \mathcal{C}$ be a solution of (1) and let η_{n+1} be the approximated solution obtained from Algorithm 3.1. If the operator T is harmonic pseudomonotone, then

$$\|\eta - \eta_{n+1}\|^2 \leq \|\eta - \eta_n\|^2 - \|\eta_n - \eta_{n+1}\|^2. \tag{5}$$

Proof. Let $\eta \in \mathcal{C}$ be a solution of (1). Then

$$\langle \mathrm{T}\eta, \frac{\eta\xi}{\eta - \xi}\rangle \geq 0, \quad \forall \xi \in \mathcal{C},$$

which implies that

$$\langle \mathrm{T}\xi, \frac{\eta\xi}{\xi - \eta}\rangle \geq 0, \quad \forall \xi \in \mathcal{C}, \tag{6}$$

since T is harmonic pseudomonotone.

Taking $\xi = \eta_{n+1}$ in (6) and $\xi = \eta$ in (4), we have

$$\langle \mathrm{T}\eta_{n+1}, \frac{\eta\eta_{n+1}}{\eta_{n+1} - \eta}\rangle \geq 0. \tag{7}$$

and

$$\langle \rho\mathrm{T}\eta_{n+1}, \frac{\eta\eta_{n+1}}{\eta - \eta_{n+1}}\rangle + \langle \eta_{n+1}-\eta_n, \eta - \eta_{n+1}\rangle \geq 0. \tag{8}$$

From (7) and (8), we have

$$\langle \eta_{n+1}-\eta_n, \eta - \eta_{n+1}\rangle \geq \langle \rho\mathrm{T}\eta_{n+1}, \frac{\eta\eta_{n+1}}{\eta - \eta_{n+1}}\rangle \geq 0. \tag{9}$$

From (9) and using the inequality

$$2\langle \eta, \xi\rangle = \|\eta + \xi\|^2 - \|\eta\|^2 - \|\xi\|^2 \quad \forall \eta, \xi \in \mathrm{H},$$

we obtain

$$\|\eta - \eta_{n+1}\|^2 \leq \|\eta - \eta_n\|^2 - \|\eta_n - \eta_{n+1}\|^2,$$

the required (5). □

Theorem 3.2. Let H be a finite dimensional Hilbert space and let T be harmonic pseudo monotone operator. If η_{n+1} is the approximate solution obtained from Algorithm 3.1 and $\eta \in \mathcal{C}$ is a solution of problem (1), then $\lim_{n\longrightarrow\infty} \eta_n = \eta$.

Proof. Let $\eta \in \mathcal{C}$ be a solution of (1). From (5), we see that the sequence $\{\|\eta - \eta_n\|\}$ is nondecreasing and consequently, the sequence $\{\eta_n\}$ is bounded. Also from (5), we obtain

$$\sum_{n=0}^{\infty} \|\eta_n - \eta_{n+1}\|^2 \leq \|\eta - \eta_0\|^2$$

which implies that

$$\|\eta_n - \eta_{n+1}\| = 0. \tag{10}$$

Let $\hat{\eta}$ be the cluster point of $\{\eta_n\}$ and the subsequent $\{\eta_{n_j}\}$ of the sequence converges to $\hat{\eta} \in \mathcal{C}$. Replacing η_n by η_{n_j} in (4), taking the limit as $n_j \longrightarrow \infty$ and using (10), we obtain

$$\langle \mathrm{T}\hat{\eta}, \frac{\xi\hat{\eta}}{\hat{\eta} - \xi}\rangle \geq 0, \quad \forall \xi \in \mathcal{C},$$

which shows that $\hat{\eta}$ is a solution of the harmonic variational inequality (1) and consequently

$$\|\hat{\eta} - \eta_{n+1}\|^2 \leq \|\hat{\eta} - \eta_n\|^2.$$

Using the above inequality, one can easily sow that the sequence $\{\eta_n\}$ has exactly one cluster point and $\lim_{n\longrightarrow\infty} \eta_n = \hat{\eta}$, the required result. □

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