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CATEGORY OF FUZZY TOPOLOGICAL POLYGROUPS

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Abstract. In this paper, the relation between two definitions of a fuzzy topological polygroup is discussed. The collection of all fuzzy continuous functions from a fuzzy topological space Y to a fuzzy topological polygroup Z, denoted by $FC(Y,Z)$ induces a polygroup structure from that of Z. Moreover, we study (fuzzy) topological polygroup of the polygroup $FC(Y,Z)$ when it is equipped with various known topologies and fuzzy topologies. Also, a few properties of fuzzy topological polygroups are established and a category C_{FTP} is formed with objects as FTP and morphisms as the fuzzy topological homomorphisms. The category C_{TP} is seen to be a full subcategory of C_{FTP}.

1. Introduction

The hyperstructure theory was born in 1934 when Marty introduced the notion of hypergroup [20]. In 1979, Foster [13] introduced the concept of fuzzy topological group. Ma and Yu [19] changed the definition of a fuzzy topological group in order to make sure that an ordinary topological group is a special case of a fuzzy topological group. On the other hand, in the last few decades, many connections between hyperstructures and fuzzy sets has been established and investigated. The concept of fuzzy topological polygroup (in short FTP) was introduced and studied in [1]. In [2] we have observed that the collection of all fuzzy continuous functions from a fuzzy topological space Y to a fuzzy topological polygroup Z, denoted by $FC(Y,Z)$ induces a polygroup structure from that of Z. Here we investigate $FC(Y,Z)$ in presence of various known topologies and fuzzy topologies. Then, we show how a fuzzy topological polygroup induced by strong homomorphism. Also, we observe that the collection of all fuzzy topological polygroups and fuzzy topological homomorphisms constitute a category, with we call C_{FTP}. Moreover, the category C_{TP} of topological polygroups and continuous homomorphisms form a full subcategory of C_{FTP}. We recall some basic definitions and results to be used in the sequel.

Let H be a non-empty set. Then a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation*, where $\mathcal{P}^*(H)$ is the family of non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A *hypergroupoid* $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, we have $x \circ H = H = H \circ x$. This condition is called the *reproduction axiom*. The couple $(H, \circ)$ is called a *hypergroup* if it is a *semihypergroup* and a *quasihypergroup* [7, 20].

A special subclass of hypergroups is the class of polygroups. We recall the following definition from [6]. A *polygroup* is a system $P = \langle P, \circ, e, ^{-1} \rangle$, where $\circ : P \times P \to \mathcal{P}^*(P)$, $e \in P$, $^{-1}$ is a unitary operation P and the following axioms hold for all $x, y, z \in P$:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$,
(2) $e \circ x = x = x \circ e$,
(3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

The following elementary facts about polygroups follow easily from the axioms:
$e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$.

A non-empty subset K of a polygroup P is a subpolygroup of P if and only if $a, b \in K$ implies

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$a \circ b \subseteq K$ and $a \in K$ implies $a^{-1} \in K$. The subpolygroup N of P is *normal* in P if and only if $a^{-1} \circ N \circ a \subseteq N$ for all $a \in P$. For a subpolygroup K of P and $x \in P$, denote the *right coset* of K by $K \circ x$ and let P/K be the set of all *right cosets* of K in P. If N is a normal subpolygroup of P, then $(P/N, \odot, N, ^{-1})$ is a polygroup, where $N \circ x \odot N \circ y = \{N \circ z | z \in N \circ x \circ y\}$ and $(N \circ x)^{-1} = N \circ x^{-1}$. For more details about polygroups we refer to [3, 9, 17].

2. Preliminaries

For the sake of convenience and completeness of our study, in this section some basic definition and results of [4, 5, 13, 19, 21, 22], which will be needed in the sequel are recalled here. Throughout this paper, the symbol I will denote the unit interval $[0, 1]$.

Let X be a non-empty set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0, 1]$ which associates with each point $x \in X$ its grade or degree of membership $\mu_A(x) \in [0, 1]$. That is, an element of I^X. We denote by $FS(X)$ the set of all fuzzy sets on X. A family $\mathcal{T} \subseteq FS(X)$ of fuzzy sets is called a fuzzy topology for X if it satisfies the following three axioms:

(1) $\underline{0}, \underline{1} \in \mathcal{T}$.
(2) For all $A, B \in \mathcal{T}$, then $A \wedge B \in \mathcal{T}$.
(3) For all $(A_j)_{j \in J}$, then $\bigvee_{j \in J} A_j \in \mathcal{T}$.

The pair (X,$\mathcal{T}$) is called a fuzzy topological space or FTS, for short. The elements of $\mathcal{T}$ are called fuzzy open sets. A fuzzy set is fuzzy closed if and only if its complement is fuzzy open.

A fuzzy set in X is called a *fuzzy point* if and only if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is λ $(0 < \lambda \leq 1)$, we denote this fuzzy point by x_λ, where the point x is called its *support*. The fuzzy point x_λ is said to be contained in a fuzzy set A, or to belong to A, denoted by $x_\lambda \in A$, if and only if $\lambda \leq \mu_A(x)$. Evidently, every fuzzy set A can be expressed as the union of all the fuzzy points which belong to A.

A fuzzy set A in a fuzzy topological space $(X, \mathcal{T})$ is called a neighborhood of fuzzy point x_λ, if there exists a $B \in \mathcal{T}$ such that $x_\lambda \in B \leq A$. The family consisting of all neighborhood of x_λ is called the system of neighborhood of fuzzy point x_λ. A fuzzy point x_λ is said to be *quasi-coincident* with a fuzzy set A, denoted by $x_\lambda q A$, if $\mu_A(x) + \lambda > 1$. A is said to be *quasi-coincident* with B, denoted by AqB, if there exists $x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$. If this is true, we also say that A and B are quasi-coincident at x. A fuzzy set A in a fuzzy topological space $(X, \mathcal{T})$ is said to be a *Q-neighborhood* of x_λ if there exists a $B \in \mathcal{T}$ such that $x_\lambda q B \leq A$. The family consisting of all Q-neighborhood of x_λ is called the system of Q-neighborhood of fuzzy point x_λ. A fuzzy topological space $(X, \mathcal{T})$ is called a *fully stratified space* if $\mathcal{T}$ contains all constant fuzzy sets.

Given two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, a mapping $f : X \to Y$ is *fuzzy continuous* if, for any fuzzy set $B \in \mathcal{U}$, the inverse image $f^{-1}[B] \in \mathcal{T}$. Conversely, f is fuzzy open if, for any open fuzzy set $A \in \mathcal{T}$, the image $f[A] \in \mathcal{U}$ (see [4]).

Let A be a fuzzy set in the fuzzy topological space $(X, \mathcal{T})$. Then the induced fuzzy topology on A is the family $\mathcal{T}_A$ of fuzzy subsets of A which are the intersection with A of $\mathcal{T}$-open fuzzy sets in X. The pair $(A, \mathcal{T}_A)$ is called a fuzzy subspace of $(X, \mathcal{T})$. For any fuzzy set $A \cap U_j$ of $\mathcal{T}_A$, with $U_j \in \mathcal{T}$, we have $\mu_{U_j \cap A}(x) = \mu_{U_j}(x) \wedge \mu_A(x)$ (see [13]).

Let X and Y be two non-empty subsets, $f : X \to Y$, A be a fuzzy set in X and B a fuzzy set in Y. Then, $f[A]$ is the fuzzy set in Y defined by

$$\mu_{f[A]}(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \varnothing. \\ 0 & \text{otherwise.} \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x | f(x) = y\}$.

$f^{-1}[B]$ is the fuzzy set in X defined by $\mu_{f^{-1}[B]}(x) = \mu_B(f(x))$ for all $x \in X$. Let f be a mapping of FTS $(X, \mathcal{T})$ into FTS $(Y, \mathcal{U})$. If for any fuzzy open Q-neighborhood U of $f(x_\lambda) = [f(x)]_\lambda$ there exists a fuzzy open Q-neighborhood V of x_λ such that $f(V) \leq U$, then we say that f is continuous at x_λ with respect to Q-neighborhood (see [21]).

Let f be a function from a fuzzy topological space $(X, \mathcal{T})$ into a fuzzy topological space $(Y, \mathcal{U})$. Then the following are equivalent (see [21]):

(1) f is a fuzzy continuous mapping.
(2) f is continuous with respect to Q-neighborhood at any fuzzy point x_λ.
(3) f is continuous with respect to neighborhood at any fuzzy point x_λ.

3. Topological and fuzzy topological polygroups on $FC(Y,Z)$

Let $P = \langle P, \circ, e, ^{-1} \rangle$ be a polygroup, $A, B \in I^P$ and $C, D \subseteq P$. We define $A \bullet B \in I^P$, $A^{-1} \in I^P$, $C \circ D \subseteq P$ and $C^{-1} \subseteq P$ by the respective formulas: (see [1])

$$(A \bullet B)(x) = \bigvee_{x \in x_1 \circ x_2} (\mu_A(x_1) \wedge \mu_B(x_2))$$

and

$$\mu_{A^{-1}}(x) = \mu_A(x^{-1})$$

for any $x \in P$. Also,

$$C \circ D = \bigcup \{c \circ d : c \in C,\ d \in D\}$$

and

$$C^{-1} = \{c^{-1} : c \in C\}.$$

We denote $A \bullet B$ by AB for short. Then for $A, B \in I^P$, we have

$$(AB)^{-1} = B^{-1}A^{-1} \quad \text{and} \quad (\mathrm{A}^{-1})^{-1} = \mathrm{A}.$$

The following definition of a fuzzy topological polygroup was given in [1].

Definition 3.1. Let $P = \langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $(P, \mathcal{T})$ be a fuzzy topological space. A triad $(P, \circ, \mathcal{T})$ is called a *fuzzy topological polygroup* or FTP for short, if:

(1) For all $x, y \in P$ and any fuzzy (open) Q-neighborhood W of any fuzzy point z_λ of $x \circ y$, there are fuzzy (open) Q-neighborhoods U of x_λ and V of y_λ such that
$$U \bullet V \leq W.$$
(2) For all $x \in P$ and any fuzzy (open) Q-neighborhood V of x_λ^{-1}, there exists a fuzzy (open) Q-neighborhood U of x_λ such that
$$U^{-1} \leq V.$$

Now, we give another definition of a fuzzy topological polygroup.

Definition 3.2. Let $P = \langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $(P, \mathcal{T})$ be a fuzzy topological space. A triad $(P, \circ, \mathcal{T})$ is called a *fuzzy topological polygroup* or FTP for short, if:

(1′) For all $x, y \in P$ and any fuzzy (open) neighborhood W of any fuzzy point z_λ of $x \circ y$, there are fuzzy (open) neighborhoods U of x_λ and V of y_λ such that
$$U \bullet V \leq W.$$
(2′) For all $x \in P$ and any fuzzy (open) neighborhood V of x_λ^{-1}, there exists a fuzzy (open) neighborhood U of x_λ such that
$$U^{-1} \leq V.$$

Proposition 3.3. *For a same polygroup P and a same fuzzy topology $\mathcal{T}$ on P the conditions (2) and (2′) are equivalent.*

Proposition 3.4. *For a same polygroup P and a same fuzzy topology $\mathcal{T}$ on P we have:*

(a) *If $\mathcal{T}$ is a finite set, then (1) implies* (1′).
(b) (1′) *implies (1).*

Proof. (a) Let W be any fuzzy neighborhood of any fuzzy point z_λ of $x \circ y$. If $\mu_W(z) > \lambda$, then W is a fuzzy Q-neighborhood of fuzzy point $z_{1-\lambda}$. By (1), there exist fuzzy Q-neighborhoods U of $x_{1-\lambda}$ and V of $b_{1-\lambda}$ such that $UV \leq W$. Now, it is clear that U and V are fuzzy neighborhoods of x_λ and y_λ respectively and the assertion follows from this. Let $z \in x \circ y$ and $\mu_W(z) = \lambda$, choose a decreasing sequence $\{\varepsilon_i\}_{i=1}^\infty$ of real numbers such that $0 < \varepsilon_i < \lambda$ $(i = 1, 2, \dots)$ and $\lim\limits_{i \to \infty} \varepsilon_i = 0$.

Put $\lambda_i = \lambda - \varepsilon_i$. Then, $\lim\limits_{i \to \infty} \lambda_i = \lambda$. Now, for any λ_i, W is a fuzzy neighborhood of z_λ and

$\mu_W(z) > \lambda_i$. Hence, there exist fuzzy open neighborhoods U_i of x_{λ_i} and V_i of y_{λ_i} such that $U_iV_i \leq W$. We assume that $U = \bigcup_{i=1}^{n} U_i$ and $V = \bigcup_{i=1}^{n} V_i$. It is easy to verify that U and V are fuzzy open neighborhoods of x_λ and y_λ respectively. Since $\mathcal{T}$ is a finite set, we can choose a subsequence $\{U_{i_l}\}_{l=1}^{\infty}$ of the sequence $\{U_i\}_{i=1}^{\infty}$, such that each U_{i_l} is a fuzzy neighborhood of x_λ. Clearly, $\bigcup_{l=1}^{\infty} V_{i_l}$ is a fuzzy open neighborhood of y_λ. It follows from the assumption that there must exist an i_k such that V_{i_k} is a fuzzy neighborhood of y_λ. Now, U_{i_k} and V_{i_k} are fuzzy neighborhoods of x_λ and y_λ respectively and $U_{i_k}V_{i_k} \leq W$.

(b) Let $x, y \in P$ and W be a fuzzy Q-neighborhood of any fuzzy point z_λ of $x \circ y$. Choose a $\lambda_1 \in (0,1)$ such that $1-\lambda < \lambda_1 < \mu_W(z)$. Then W is a fuzzy neighborhood of fuzzy point z_{λ_1}. By $(1')$ there exist fuzzy neighborhoods U of x_{λ_1} and V of y_{λ_1} such that $UV \leq W$. Now, U and V are fuzzy Q-neighborhoods of x_λ and y_λ respectively and $UV \leq W$. □

Let $FC(Y,Z)$ be the set of all fuzzy continuous functions from a fuzzy topological space Y into a fuzzy topological space Z. In this section, we investigate $FC(Y,Z)$ in presence of various known topologies and fuzzy topologies.

Theorem 3.5. [2] *Let $(Y, \mathcal{T}_Y)$ be an FTS, $(Z, \circ, \mathcal{T}_Z)$ an FTP, and $f, g \in FC(Y,Z)$. Then, the maps $f * g$ and f^{-1} from the fuzzy topological space Y into the fuzzy topological space Z with the types,*

$$(f * g)(y) = f(y) \circ g(y)$$

and

$$f^{-1}(y) = (f(y))^{-1}$$

for every $y \in Y$, are fuzzy continuous.

Theorem 3.6. [2] *Let $(Y, \mathcal{T}_Y)$ be a fully stratified fuzzy topological space and $(Z, \circ, \mathcal{T}_Z)$ a fuzzy topological polygroup. Then, $(FC(Y,Z), *, e', ^{-1})$ is a polygroup.*

Theorem 3.7. *Let $(FC(Y,Z), *)$ be the polygroup of fuzzy continuous functions from a fully stratified fuzzy topological space $(Y, \mathcal{T}_Y)$ to a fuzzy topological polygroup $(Z, \circ, \mathcal{T}_Z)$.*

(1) *If Z is commutative, then $FC(Y,Z)$ is commutative.*
(2) *If Z contains identity element , then $FC(Y,Z)$ contains identity element.*

Proof. (1) For all $f, g \in FC(Y,Z)$ and $y \in Y$,

$$(f * g)(y) = f(y) \circ g(y) = g(y) \circ f(y) = (g * f)(y).$$

(2) For all $f \in FC(Y,Z)$ and $y \in Y$,

$$(f * e')(y) = f(y) \circ e'(y) = f(y) \circ e = f(y) = e \circ f(y) = e'(y) \circ f(y) = (e' * f)(y).$$

□

Theorem 3.8. [2] *Let $(Y, \mathcal{T}_Y)$ be a fully stratified fuzzy topological space, $(Z, \circ, \mathcal{T}_Z)$ a fuzzy topological polygroup, and $Z_1 \in I^Z$ a fuzzy polygroup. Then, the fuzzy set $\mu \in I^{FC(Y,Z)}$ for which $\mu(f) = \bigwedge_{y \in Y} Z_1(f(y))$, $f \in FC(Y,Z)$ is a fuzzy polygroup.*

Definition 3.9. [14] Let U be a fuzzy open set on an FTS Z and y_λ, $\lambda \in (0,1]$ be a fuzzy point on an FTS Y. By $[y_\lambda, U]$ we denote the subset of $FC(Y,Z)$ where $[y_\lambda, U] = \{f \in FC(Y,Z) : f(y_\lambda) \leq U\}$. The collection of all such $[y_\lambda, U]$ forms a subbase for some topology on $FC(Y,Z)$, called *fuzzy-point fuzzy-open topology (fp-fo)*, denoted by $\mathcal{T}_{(fp-fo)}$.

Definition 3.10. [16] Let $P = \langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $(P, \mathcal{T})$ be a topological space. Then, the system $P = \langle P, \circ, e, ^{-1}, \mathcal{T} \rangle$ is called a *topological polygroup* if the mapping $\circ : P \times P \to \wp^*(P)$ and $^{-1} : P \to P$ are continuous.

Lemma 3.11. [15] *Let P be a polygroup. Then, the hyperoperation $\circ : P \times P \to \wp^*(P)$ is continuous if and only if for every $x, y \in P$ and $W \in \mathcal{T}$ such that $x \circ y \subseteq W$ then there exist $U, V \in \mathcal{T}$ such that $x \in U, y \in V$ and $U \circ V \subseteq W$.*

Theorem 3.12. *Let $(Y, \mathcal{T}_Y)$ be a fully stratified fuzzy topological space, $(Z, \circ, \mathcal{T}_Z)$ a fuzzy topological polygroup. Then $(FC(Y, Z), \mathcal{T}_{(fp-fo)})$ is a topological polygroup.*

Proof. It is clear that $FC(Y, Z)$ is a polygroup and $FC(Y, Z)$ is a topological space with respect to $\mathcal{T}_{(fp-fo)}$. We need to show that the mappings $(f, g) \mapsto f * g$ and $f \mapsto f^{-1}$ are continuous. Suppose that $[y_\lambda, U]$ be a subbasic open set in $FC(Y, Z)$ such that $f*g \subseteq [y_\lambda, U]$. So, for any $h \in f*g \subseteq [y_\lambda, U]$, we have $h(y_\lambda) \leq U$, where $h(y_\lambda) \in (f(y))_\lambda \circ (g(y))_\lambda$. $(Z, \circ, \mathcal{T})$ being an fuzzy topological polygroup , then there exist fuzzy open sets V, W of Z such that $(f(y))_\lambda \in V,\ (g(y))_\lambda \in W$ and $UW \leq V$.

On the other hand, $f(y_\lambda) = (f(y))_\lambda \leq V$ implies that $f \in [y_\lambda, V]$ and similarly $g \in [y_\lambda, W]$. Now, we show that $f * g$ is continuous. We need to show that

$$[y_\lambda, V] * [y_\lambda, W] \subseteq [y_\lambda, U].$$

Let $\xi \in [y_\lambda, V] * [y_\lambda, W]$. Then there exist $\eta \in [y_\lambda, V]$ and $\psi \in [y_\lambda, W]$, such that $\xi = \eta * \psi$.

Since $\eta \in [y_\lambda, V]$ and $\psi \in [y_\lambda, W]$, then we have $\eta(y_\lambda) \leq V$ and $\psi(y_\lambda) \leq W$. So, $(\eta * \psi)(y_\lambda) \leq V * W \leq U$. Therefore, $\xi(y_\lambda) \leq U$ and $\xi \in [y_\lambda, U]$.

Finally we show that f^{-1} is continuous. For any subbasic open set $[y_\lambda, U]$ containing f^{-1}, we get $(f^{-1})(y_\lambda) \leq U$, so $(f^{-1}(y))_\lambda \leq U$ and $(f(y))_\lambda \leq U^{-1}$. Since U is fuzzy open if and only if U^{-1} is fuzzy open and $f(y_\lambda) = (f(y))_\lambda$, we have $f \in [y_\lambda, U^{-1}]$. Now, we show that $[y_\lambda, U]^{-1} \leq [y_\lambda, U^{-1}]$.

Let $\psi \in [y_\lambda, U]^{-1}$. Then, there is some $\eta \in [y_\lambda, U]$ such that $\psi = \eta^{-1}$. Since $\eta \in [y_\lambda, U]$ and $\eta(y_\lambda) \leq U$, so $(\eta^{-1})^{-1}(y_\lambda) \leq U$, then $\psi^{-1}(y_\lambda) \leq U$ and $\psi(y_\lambda) \leq U^{-1}$, $\psi \in [y_\lambda, U^{-1}]$, as desired. $\square$

Definition 3.13. [14, 18] Let Y and Z be two fixed fuzzy topological spaces, $U \in I^Z$ a fuzzy open set of Z, and $y \in Y$. Then by $y_U \in I^{FC(Y,Z)}$ we denote the fuzzy set for which

$$y_U(f) = U(f(y)), \text{ for every } \ f \in FC(Y, Z).$$

The *fuzzy point open topology* $\mathcal{T}_{FP}$ on $FC(Y, Z)$ generated by fuzzy sets of the form y_U, where $y \in Y$ and $U \in I^Z$ is a fuzzy open set of Z.

For each fuzzy compact set K in Y and each fuzzy open set U in Z, a fuzzy set K_U on $FC(Y, Z)$ is given by

$$K_U(g) = \bigwedge_{x \in supp(K)} U(g(x)).$$

The collection of all such K_U forms a subbase for some fuzzy topology on $FC(Y, Z)$, called fuzzy compact open topology and it is denoted by Δ_{co}.

Theorem 3.14. [2] *Let $(Y, \mathcal{T}_Y)$ be a fully stratified fuzzy topological space and $(Z, \circ, \mathcal{T}_Z)$ a fuzzy topological polygroup. Then the triad $(FC(Y, Z), *, \mathcal{T}_{FP})$ is a fuzzy topological polygroup.*

Theorem 3.15. *Let $(Y, \mathcal{T}_Y)$ be a fully stratified fuzzy topological space and $(Z, \circ, \mathcal{T}_Z)$ a fuzzy topological polygroup. Then $FC(Y, Z)$ endowed with fuzzy compact-open topology is a fuzzy topological polygroup.*

Proof. Clearly, by Theorem 3.6 and Definition 3.13, $(FC(Y, Z), *)$ is a polygroup and $(FC(Y, Z), \Delta_{co})$ is a fuzzy topological space. Now, we show that $(FC(Y, Z), *, \Delta_{co})$ satisfies the conditions (1) and (2) in Definition 3.1.

(1) Let K_U be a fuzzy open subbasic Q-neighborhood of any fuzzy point h_λ of $f * g$. We show that there exist fuzzy open subbasic Q-neighborhoods K_V and K_W of f_λ and g_λ, respectively such that

$$K_V \bullet K_W \leq K_U.$$

Since $h_\lambda q K_U$, it follows that $\lambda + K_U(h) > 1$ and $\lambda + \bigwedge_{y \in suppK} U(h(y)) > 1$. So, for all $y \in suppK$, $h(y)_\lambda q U$. That is, the fuzzy set U is a fuzzy open Q-neighborhood of $h(y)_\lambda$.

Now, since $(Z, \circ, \mathcal{T}_Z)$ is a fuzzy topological polygroup, it follows that there exist fuzzy open Q-neighborhoods V and W of $f(y)_\lambda$ and $g(y)_\lambda$ such that $VW \leq U$.

We consider the fuzzy sets K_V and K_W. Since V is a fuzzy open Q-neighborhood of $f(y)_\lambda$, then for all $y \in suppK$, $V(f(y)) + \lambda > 1$ and

$$K_V(f) + \lambda = \bigwedge_{x \in suppK} V(f(y)) + \lambda > 1.$$

Hence, $K_V(f)+\lambda>1$ and $f_\lambda q K_V$. Similarly, it can be proved that $g_\lambda q K_W$.

We prove $K_V \bullet K_W \leq K_U$. Let $f \in FC(Y,Z)$. Then, we have

$$\begin{aligned}
(K_V \bullet K_W)(f) &= \bigvee_{f\in f_1 * f_2} [K_V(f_1)\wedge K_W(f_2)] \\
&= \bigvee_{f\in f_1 * f_2} [(\bigwedge_{x\in suppK} V(f_1(x)))\wedge(\bigwedge_{x\in suppK} W(f_2(x)))] \\
&= \bigvee_{f\in f_1 * f_2} [\bigwedge_{x\in suppK} [V(f_1(x))\wedge W(f_2(x))]] \\
&\leq \bigwedge_{x\in suppK} [\bigvee_{f\in f_1 * f_2} [V(f_1(x))\wedge W(f_2(x))]] \\
&\leq \bigwedge_{x\in suppK} [\bigvee_{f(x)\in z_1\circ z_2} [V(z_1)\wedge W(z_2)]] \\
&= \bigwedge_{x\in suppK} [(V\bullet W)(f(x))] \\
&\leq \bigwedge_{x\in suppK} U(f(x)) \\
&= K_U(f).
\end{aligned}$$

Now, the condition (1) in Definition 3.1 follows.

(2) Let $f \in FC(Y,Z)$ and K_U be a fuzzy open Q-neighborhood of f_λ^{-1}. We show that there exists fuzzy open Q-neighborhood K_V of f_λ such that $K_V^{-1} \leq K_U$. Since $f_\lambda^{-1} q K_U$, it follows that $\lambda + U(f^{-1}(y)) > 1$. As $(Z,\circ,\mathcal{T}_Z)$ is a fuzzy topological polygroup then, there exists fuzzy open Q-neighborhood V of f_λ such that $V^{-1} \leq U$. We prove that $K_V^{-1} \leq K_U$. Let $f \in FC(Y,Z)$. Then we have

$$\begin{aligned}
K_V^{-1}(f) &= K_V(f^{-1}) \\
&= \bigwedge_{x\in suppK} V(f^{-1}(x)) \\
&= \bigwedge_{x\in suppK} V^{-1}(f(x)) \\
&\leq \bigwedge_{x\in suppK} U(f(x)) \\
&= K_U(f).
\end{aligned}$$

Hence, $K_V^{-1} \leq K_U$. Now, the condition (2) in Definition 3.1 follows. Therefore, $(FC(Y,Z),*,\Delta_{co})$ is a fuzzy topological polygroup. □

4. Fuzzy topological polygroups induced by strong homomorphisms

In this section, we show how to strong homomorphisms induce fuzzy topological polygroup structure on polygroups.

Definition 4.1. [10] A collection $\mathbb{B}$ of fuzzy neighborhoods of x_λ, for $0<\lambda\leq 1$, is called a fundamental system of fuzzy neighborhoods of x_λ if and only if for any fuzzy neighborhood V of x_λ, there exists $U\in\mathbb{B}$ such that $x_\lambda \leq U \leq V$.

Definition 4.2. [10] A collection Ω of fuzzy sets in an FTS X is called a *prefilterbase* on X if $0_X \notin \Omega$ and for all $A,B\in\Omega$, then there exists $C\in\Omega$ such that $C\leq A\cap B$.

Proposition 4.3. *Let P be a polygroup and $A,B,C \in I^P$. Then the following are hold:*

1. *If $A\leq B$ then $AC\leq BC$ and $CA\leq CB$.*
2. *If $AC=BC$ for any $C\in I^P$, then $A=B$.*
3. *$(AB)C=A(BC)$.*
4. *If $A\leq B$ then $A^{-1}\leq B^{-1}$.*

Proof. It is straightforward. □

Theorem 4.4. *Let $(P,\mathcal{T})$ be a fuzzy topological polygroup. Then the mapping $\phi: P\to P^{-1}$, $x\mapsto x^{-1}$ is homeomorphic mapping.*

Theorem 4.5. *Let $(P,\mathcal{T})$ be a fuzzy topological polygroup. Then, V is fuzzy open if and only if V^{-1} is fuzzy open.*

Proof. For all $x \in P$,
$$\phi^{-1}(V)(x) = V(\phi(x)) = V(x^{-1}) = V^{-1}(x),$$
ϕ is fuzzy continuous and V is fuzzy open, so $V^{-1} = \phi^{-1}(V)$ is fuzzy open.

Converse follows similarly. □

Corollary 4.6. Let $(P, \mathcal{T})$ be a fuzzy topological polygroup. Then, for each λ with $0 < \lambda \leq 1$ and $x \in P$, V is a fuzzy neighborhood of e_λ if and only if V^{-1} is a fuzzy neighborhood of e_λ.

Proof. The proof follows from Theorem 4.4 and the fact that $e_\lambda \leq V$ if and only if $e_\lambda \leq V^{-1}$. □

Definition 4.7. A fuzzy open set U of a fuzzy topological polygroup P is called a *symmetric neighborhood* if $U^{-1} = U$.

Theorem 4.8. *Every fuzzy topological polygroup has a fuzzy open fundamental system of e_λ containing a symmetric fuzzy open fundamental system of e_λ.*

Proof. Suppose that $\mathbb{B}$ is a fuzzy open fundamental system of e_λ. Then, for every $U \in \mathbb{B}$ put $V = U \cap U^{-1}$. So, $V = V^{-1}$ and $V \leq U$. □

Theorem 4.9. *If $\mathbb{B}$ is a fundamental system of fuzzy neighborhoods of e_λ, for $0 < \lambda \leq 1$, then $\mathbb{D} = \{U \cap U^{-1} : U \in \mathbb{B}\}$ is also a fundamental system of fuzzy neighborhoods of e_λ.*

Proof. It is obvious. □

Proposition 4.10. *Let $(P, \mathcal{T})$ be a fuzzy topological polygroup. Then, the family $\mathcal{B} = \{\tilde{A} \in FS(\mathcal{P}^*(P)) \mid A \in \mathcal{T}\}$, where $\mu_{\tilde{A}}(X) = \bigvee_{x \in X} \mu_A(x)$, is a base for a fuzzy topology $\mathcal{T}^*$ on $\mathcal{P}^*(P)$.*

Proof. $\mathcal{B}$ is a base for a fuzzy topology on $\mathcal{P}^*(P)$ because:

(1) For any $\tilde{A}_1, \tilde{A}_2 \in \mathcal{B}$, with $A_1, A_2 \in \mathcal{T}$, it follows that $\tilde{A}_1 \cap \tilde{A}_2 \in \mathcal{B}$, because $\tilde{A}_1 \cap \tilde{A}_2 = \widetilde{A_1 \cap A_2}$ and $A_1 \cap A_2 \in \mathcal{T}$.

Indeed, for any $X \in \mathcal{P}^*(P)$, we have
$$\begin{aligned}
\mu_{\widetilde{A_1 \cap A_2}}(X) &= \bigvee_{x \in X} \mu_{(A_1 \cap A_2)}(x) = \bigvee_{x \in X} (\mu_{A_1}(x) \wedge \mu_{A_2}(x)) \\
&= (\bigvee_{x \in X} \mu_{A_1}(x)) \wedge (\bigvee_{x \in X} \mu_{A_2}(x)) \\
&= \mu_{\tilde{A}_1}(X) \wedge \mu_{\tilde{A}_2}(X) \\
&= \mu_{(\tilde{A}_1 \cap \tilde{A}_2)}(X).
\end{aligned}$$

(2) Since $\underline{1} \in \mathcal{T}$, it follows that $\mu_{\tilde{\underline{1}}}(X) = 1$, for any $X \in \mathcal{P}^*(P)$ and thus
$$\bigcup_{\tilde{A} \in \mathcal{B}} = 1.$$
□

Lemma 4.11. *Let U be a fuzzy open subset of a fuzzy topological polygroup P. Then, $a_\lambda U$ and $U a_\lambda$ are fuzzy open subsets of P for every $a \in P$.*

Proof. Suppose that U be a fuzzy open subset of P. Then,
$$\begin{aligned}
({}_{a^{-1}}\phi^{-1}(\tilde{U}))(z) &= \tilde{U}({}_{a^{-1}}\phi(z)) = \tilde{U}(a^{-1} \circ z) \\
&= \bigvee_{t \in a^{-1} \circ z} U(t) = \bigvee_{z \in a \circ t} U(t) \\
&= a_\lambda U(z).
\end{aligned}$$

Since the mapping ${}_{a^{-1}}\phi^{-1} : P \to \mathcal{P}^*(P), x \mapsto a^{-1} \circ x$, is fuzzy continuous, thus $a_\lambda U$ is fuzzy open. Similarly, we can prove that $U a_\lambda$ is fuzzy open. □

Lemma 4.12. *Let $(P, \mathcal{T})$ be a fuzzy topological polygroup and $\mathbb{B}$ be a fuzzy open fundamental system of fuzzy neighborhood of e_λ. Then, the families $\{x_\lambda U\}$ and $\{U x_\lambda\}$, are fuzzy open fundamental system of fuzzy neighborhood of x_λ.*

Proof. Suppose that W is a fuzzy open subset of P and $x_\lambda \leq W$. Since

$$\begin{aligned}(x_\lambda U)(x) &= \bigvee_{x\in x_1\circ x_2}[x_\lambda(x_1)\wedge U(x_2)]\\ &= \bigvee_{x\in x\circ x_2}[\lambda\wedge U(x_2)]\\ &\geq \lambda\wedge U(e)\\ &= \lambda,\end{aligned}$$

we conclude that $x_\lambda \leq x_\lambda U$. Since $e_\lambda \leq x_\lambda^{-1}W$, it follows that there exists $U \in \mathbb{B}$ such that $e_\lambda \leq U \leq x_\lambda^{-1}W$. So, $x_\lambda U \leq W$. Thus, W is a union of fuzzy open subsets $x_\lambda U$. Therefore, $\{x_\lambda U\}$ is a fuzzy open fundamental system for P. Similarly, the family $\{Ux_\lambda\}$ is a fuzzy open fundamental system for P. $\square$

In the next theorem we characterize a fuzzy topological polygroup via the fundamental system of fuzzy neighborhoods of e_λ.

Theorem 4.13. *If P is a fuzzy topological polygroup, then there exists a fundamental system of fuzzy neighborhoods $\mathbb{B}$ of e_λ $(0 < \lambda \leq 1)$, such that the following conditions hold:*

1. *Each member of $\mathbb{B}$ is symmetric.*
2. *For all $U \in \mathbb{B}$, there exists $V \in \mathbb{B}$ such that $V \bullet V \leq U$.*
3. *For all $U \in \mathbb{B}$, there exists $V \in \mathbb{B}$ such that $V^{-1} \leq U$.*
4. *For all $U \in \mathbb{B}$ and $x_\lambda \leq U$ there exists $V \in \mathbb{B}$ such that $x_\lambda V \leq U$.*

Conversely, given a polygroup P and a prefilterbase $\mathbb{B}$ of e_λ satisfying the conditions (1)-(4), there exists a unique fuzzy topology $\mathcal{T}$ on P such that $(P, \mathcal{T})$ forms a fuzzy topological polygroup such that $\mathbb{B}$ forms a fundamental system of fuzzy neighborhoods of e_λ.

Proof. (1) Let P be a fuzzy topological polygroup. Consider any fundamental system $\mathbb{D}$ of fuzzy neighborhoods of e_λ (viz. consider the fuzzy open sets containing e_λ). Then, we get a fundamental system of fuzzy neighborhoods $\mathbb{B}$ of e_λ such that each member of $\mathbb{B}$ is symmetric.

(2) For any $U \in \mathbb{B}$, since P is an FTP, there exist $V_1, V_2 \in \mathbb{B}$ such that $V_1V_2 \leq U$. Let $V = V_1 \cap V_2$. So, $VV \leq V_1V_2 \leq U$.

(3) For any $U \in \mathbb{B}$, $x_\lambda \leq P$, since P is an FTP, there exists $V \in \mathbb{B}$ such that $V^{-1} \leq U$.

(4) Let $U \in \mathbb{B}$ and $x_\lambda \leq U$. As $x_\lambda = (xe)_\lambda = x_\lambda e_\lambda$, by fuzzy topological polygroup of P, there exist fuzzy neighborhoods W_1, V_1 of x_λ and e_λ respectively such that $W_1V_1 \leq U$. Since

$$(x_\lambda V_1)(z) = \bigvee_{z\in z_1\circ z_2}[x_\lambda(z_1)\wedge V_1(z_2)],$$

it follows that

$$(x_\lambda V_1)(z) = \begin{cases}\bigvee[\lambda\wedge V_1(z_2)] & \text{if } z\in x\circ z_2,\\ 0 & \text{if } x\neq z_1.\end{cases}$$

As $W_1(x) \geq \lambda$, $(W_1V_1)(z) \geq (x_\lambda V_1)(z)$. Hence, $x_\lambda V_1 \leq W_1V_1 \leq U$.

Conversely, suppose $\mathbb{B}$ is a prefilterbase at e_λ, such that it satisfies (1)-(4). For each $x \in P$, it is easy to see that $\mathbb{B}_x = \{x_\lambda U : U \in \mathbb{B}\}$ forms a prefilterbase at x_λ. Then $\cup\mathbb{B}_x$ generates a unique fuzzy topology on P with $\mathbb{B}$ as a fundamental system of fuzzy neighborhoods of e_λ. In order to show that P is a fuzzy topological polygroup, we have to show P satisfies the conditions (1) and (2) in Definition 3.1.

Suppose $z_\lambda U$ is a fuzzy open neighborhood of z_λ, where $U \in \mathbb{B}$ and $z_\lambda \in x \circ y$. Then by (2), there are $V, W \in \mathbb{B}$ such that $VW \leq U$. It is clear that $(x_\lambda V)(y_\lambda W) = (xy)_\lambda VW$ and consequently $(x_\lambda V)(y_\lambda W) \leq z_\lambda U$.

Suppose $x_\lambda U$ is a fuzzy open neighborhood of $x_\lambda U$, where $U \in \mathbb{B}$. By the condition (3), there is $V \in \mathbb{B}$ such that $V^{-1} \leq U$. It follows from the symmetric of the members of $\mathbb{B}$. $\square$

Definition 4.14. [9] Let $\langle P_1, \cdot, e_1, ^{-1}\rangle$ and $\langle P_2, *, e_2, ^{-I}\rangle$ be two polygroups. Let f be a mapping from P_1 to P_2 such that $f(e_1) = e_2$. Then, f is called a *strong homomorphism* or a *good homomorphism* if

$$f(x \cdot y) = f(x) * f(y), \text{ for all } x, y \in P_1.$$

Since P_1 is a polygroup, $e_1 \in a \circ_1 a^{-1}$ for all $a \in P_1$, it follows that $f(e_1) \in f(a) \circ_2 f(a^{-1})$ or $e_2 \in f(a) \circ_2 f(a^{-1})$ which implies $f(a^{-1}) \in f(a)^{-1} \circ_2 e_2$. Therefore, $f(a^{-1}) = f(a)^{-1}$ for all $a \in P_1$. Moreover, if f is a fuzzy topological homomorphism from P_1 into P_2, then the *kernel* of f is the set $kerf = \{x \in P_1 | f(x) = e_2\}$. It is trivial that $kerf$ is a subpolygroup of P_1 but in general is not normal in P_1. As in polygroup, if f is a fuzzy topological homomorphism from P_1 into P_2, then f is injective if and only if $kerf = \{e_1\}$.

Definition 4.15. A fuzzy topology that makes a polygroup FTP is called a fuzzy topology compatible with the polygroup structure.

Theorem 4.16. *Let $(P, \mathcal{T})$ be a fuzzy topological polygroup. If $f : Q \to P$ is a strong homomorphism from any polygroup Q to P then, f induces a unique compatible fuzzy topology on Q that makes f fuzzy continuous.*

Proof. Let $\mathbb{B}$ be a fundamental system of fuzzy neighborhoods of e_λ in P. Then it is enough to show that $f^{-1}(\mathbb{B})$ determines a unique fuzzy topology on Q such that $f^{-1}(\mathbb{B})$ forms a fundamental system of fuzzy neighborhoods of e_λ in Q. It is clear that $f^{-1}(\mathbb{B})$ is a prefilterbase at e_λ in Q. In view of Theorem 4.13, it is now to verify that $f^{-1}(\mathbb{B})$ satisfies the conditions $(1) - (4)$ of Theorem 4.13.

(1) Any element of $f^{-1}(\mathbb{B})$ is of the form $f^{-1}(V)$, for some $V \in \mathbb{B}$. Now, for all $x \in P$

$$\begin{aligned}(f^{-1}(V))^{-1}(x) \quad &= f^{-1}(V)(x^{-1}) = V(f(x^{-1})) = V(f^{-1}(x)) \\ &= V^{-1}(f(x)) = V(f(x)) = f^{-1}(V)(x).\end{aligned}$$

Hence, $(f^{-1}(V))^{-1} = f^{-1}(V)$, showing that each member of $f^{-1}(\mathbb{B})$ is symmetric.

(2) Let $f^{-1}(U) \in f^{-1}(\mathbb{B})$, for some $U \in \mathbb{B}$. Then as $U \in \mathbb{B}$, there exists $V \in \mathbb{B}$, such that $VV \leq U$. For any $z \in Q$,

$$\begin{aligned}(f^{-1}(V)f^{-1}(V))(z) \quad &= \bigvee_{z \in z_1 \circ z_2} [(f^{-1}(V))(z_1) \wedge (f^{-1}(V))(z_2)] \\ &= \bigvee_{z \in z_1 \circ z_2} [V(f(z_1)) \wedge V(f(z_2))] \\ &= \bigvee_{f(z) \in f(z_1) \circ f(z_2)} [V(f(z_1)) \wedge V(f(z_2))] \\ &= (VV)(f(z)) \leq U(f(z)) = f^{-1}(U)(z).\end{aligned}$$

(3) Let $f^{-1}(U) \in f^{-1}(\mathbb{B})$, for some $U \in \mathbb{B}$. Then as $U \in \mathbb{B}$, there exists $V \in \mathbb{B}$, such that $V^{-1} \leq U$. For any $z \in Q$,

$$\begin{aligned}f^{-1}(V)(z) \quad &= V(f^{-1}(z)) = V^{-1}(f(z)) \\ &\leq U(f(z)) = U^{-1}(f(z)) \\ &= f^{-1}(U^{-1})(z).\end{aligned}$$

(4) Let $a \in Q$ and $f^{-1}(U) \in f^{-1}(\mathbb{B})$. Then $f(a) \in P$ and $U \in \mathbb{B}$ so that there exists $V \in \mathbb{B}$ such that $f(a)_\lambda V \leq U$,

$$(a_\lambda f^{-1}(V))(z) = \bigvee_{z \in z_1 \circ z_2} [a_\lambda(z_1) \wedge f^{-1}(V)(z_2)].$$

This implies that

$$\begin{aligned}(a_\lambda f^{-1}(V))(z) \quad &= \begin{cases} \bigvee[\lambda \wedge f^{-1}(V)(z_2)] & \text{if } z \in a \circ z_2, \\ 0 & \text{if } x \neq z_1. \end{cases} \\ &\leq \bigvee_{z \in a \circ z_2} [\lambda \wedge V(f(z_2))] \\ &= \bigvee_{f(z) \in f(a) \circ f(z_2)} [\lambda \wedge V(f(z_2))] \\ &= \bigvee_{f(z) \in f(a) \circ x} [\lambda \wedge V(f(z_2))] \\ &= (f(a)_\lambda V)(f(z)) \\ &\leq U(f(z)) = f^{-1}(U)(z).\end{aligned}$$

□

Corollary 4.17. Any subpolygroup of a fuzzy topological polygroup is a fuzzy topological polygroup.

Proof. Let $(P, \mathcal{T})$ be a fuzzy topological polygroup and K be a subpolygroup of P. If $\mathbb{B}$ is a fundamental system of fuzzy neighborhoods of e_λ in P, and $i : K \to P$ given by $i(x) = x$, is the inclusion homomorphism, then by the Theorem 4.16 $f^{-1}(\mathbb{B})$ determines a unique compatible fuzzy topology on K such that $f^{-1}(\mathbb{B})$ forms a fundamental system of fuzzy neighborhoods of e_λ in K. $\square$

Corollary 4.18. Let P be an FTP and $f : P \to Q$ is a strong homomorphism from P onto Q with *kernel* K such that K is a normal subpolygroup of P. Then, $\bar{f} : P/N \to Q$ induces a compatible fuzzy topology on P/N that makes $\bar{f}$ fuzzy continuous.

Proof. It is straightforward. $\square$

Theorem 4.19. *Let $(P, \mathcal{T})$ be an FTP and $f : P \to Q$ is a strong homomorphism from P onto any polygroup Q. Then, P induces a fuzzy topology compatible with the polygroup structure on Q that makes f fuzzy continuous.*

Proof. Let $\mathbb{B}$ be a fundamental system of fuzzy neighborhoods of e_λ in Q. In view of Theorem 4.13, it is enough to show that $f(\mathbb{B})$ is a fundamental system of fuzzy neighborhoods of e_λ in Q.

(1) For any $U \in \mathbb{B}$, $U = U^{-1}$ and so, $f(U) = f(U^{-1})$. Now, for all $z \in Q$,

$$\begin{aligned} f(U^{-1})(z) &= \bigvee_{f(t)=z} U^{-1}(t) = \bigvee_{f(t)=z} U(t^{-1}) \\ &= \bigvee_{f(t^{-1})=z^{-1}} U(t^{-1}) = f(U)(z^{-1}) \\ &= f^{-1}(U)(z). \end{aligned}$$

Consequently, $f(U) = f^{-1}(U)$.

(2) If $f(U) \in f(\mathbb{B})$, then $U \in \mathbb{B}$ and so, there exists $V \in \mathbb{B}$ such that $VV \le U$. For any $z \in Q$,

$$\begin{aligned} (f(V)f(V))(z) &= \bigvee_{z \in z_1 \circ z_2} [(f(V))(z_1) \wedge (f(V))(z_2)] \\ &= \bigvee_{z \in z_1 \circ z_2} [(\bigvee_{f(y_1)=z_1} V(y_1)) \wedge (\bigvee_{f(y_2)=z_2} V(y_2))] \\ &= \bigvee_{z \in z_1 \circ z_2} [\bigvee_{\substack{f(y_1)=z_1 \\ f(y_2)=z_2}} [V(y_1) \wedge V(y_2)]] \\ &\le \bigvee_{f(t)=z} (VV)(t) = f(VV)(z) \le f(U)(z). \end{aligned}$$

(3) If $f(U) \in f(\mathbb{B})$, then $U \in \mathbb{B}$ and so, there exists $V \in \mathbb{B}$, such that $V^{-1} \le U$. For any $z \in Q$,

$$\begin{aligned} (f(V))^{-1}(z) &= (\bigvee_{f(t)=z} V(t))^{-1} = \bigvee_{f(t)=z} V^{-1}(t) \\ &\le \bigvee_{f(t)=z} U(t) = f(U)(z). \end{aligned}$$

(4) Let $b \in Q$ and $f(U) \in f(\mathbb{B})$. Then there exists $a \in P$ with $f(a) = b$. So, there exists $V \in \mathbb{B}$ such that $a_\lambda V \le U$. It is to show that $b_\lambda f(V) \le f(U)$. For any $z \in Q$, it is easy to see that

$$b_\lambda f(V)(z) = f(a_\lambda)f(V)(z) \le f(a_\lambda V)(z) \le f(U)(z).$$

Hence, all the condition (1)-(4) are satisfied proving $f(\mathbb{B})$ to be a fundamental system of fuzzy neighborhoods of e_λ in Q. Let U be any fuzzy neighborhood of e_λ in Q, by definition of fundamental system of fuzzy neighborhoods, there exists some $B \in \mathbb{B}$ such that $e_\lambda \le f(B) \le U$. As $B \le f^{-1}(f(B)) \le f^{-1}(U)$ and $e_\lambda \in B$ it follows that $f^{-1}(U)$ is a fuzzy neighborhood of e_λ in Q. Therefore, f is fuzzy continuous. $\square$

Corollary 4.20. Let P be a polygroup and N be a normal subpolygroup of P. The polygroup epimorphism $\pi : P \to P/N$ given by $\pi(x) = xN$ induces a fuzzy topology compatible with the polygroup P/N that makes π fuzzy continuous.

Proof. The proof follows from Corollary 4.18 and Theorem 4.19. $\square$

Theorem 4.21. *Let P be an FTP. If Q is an FTP induced from a strong homomorphism $f : P \to Q$ and $N = ker f$ such that N is a normal subpolygroup of P then, the fuzzy topology compatible with P/N induced from $\bar{f} : P/N \to Q$ and the fuzzy topology compatible with P/N induced from $k : P \to P/N$ are same.*

Proof. Let $\mathbb{B}$ be a fundamental system of fuzzy neighborhoods of e_λ in P. It follows from the Theorem 4.19 that $\{\pi(V) : V \in \mathbb{B}\}$ is a fundamental system of fuzzy neighborhoods of e_λ for the compatible fuzzy topology on P/N induced by π and $\{\bar{f}^{-1}(f(V)) : V \in \mathbb{B}\}$ is a fundamental system of fuzzy neighborhoods of e_λ for the compatible fuzzy topology on P/N induced by $\bar{f}$. Since, $\bar{f}\pi = f$, it follows that

$$\begin{aligned}\bar{f}^{-1}(f(V))(xN) &= f(V)(\bar{f}(xN)) = f(V)(\bar{f}\pi)(x) = f(V)(f(x)) \\ &= \bigvee_{f(y)=f(x)} V(y) = \bigvee_{yx^{-1}\in N} V(y) = \bigvee_{yN=xN} V(y) \\ &= \bigvee_{\pi(y)=xN} V(y) = \pi(V)(xN).\end{aligned}$$

Hence, both the fundamental systems are identical leading to the same compatible topology on P/N. □

5. Fuzzy topological polygroups and the category C_{FTP}

In this section we introduce the category C_{FTP}, which the objects in this category are fuzzy topological polygroups, morphisms are fuzzy topological homomorphisms and compositions is the usual composition of functions. Also, we show that the category C_{TP} of topological polygroups and continuous topological homomorphisms form a full subcategory of C_{FTP}.

Theorem 5.1. *In a fuzzy topological polygroup P, V is a fuzzy Q-neighborhood of e_λ if and only if V^{-1} is a fuzzy Q-neighborhood of e_λ.*

Proof. Let V be a fuzzy Q-neighborhood of e_λ. Then there exists fuzzy open set A such that $e_\lambda qA \leq V$, that is, $A(e) + \lambda > 1$ and $A \leq V$. For all $x \in P$, $A(x^{-1}) \leq V(x^{-1})$, so $A^{-1}(x) \leq V^{-1}(x)$ and $A^{-1} \leq V^{-1}$. Now, $A^{-1}(e) + e_\lambda(e) = A^{-1}(e) + \lambda > 1$. Hence, $e_\lambda qA^{-1}$ and $A^{-1} \leq V^{-1}$. Therefore, V^{-1} is a fuzzy Q-neighborhood of e_λ.

Conversely, let V^{-1} be a fuzzy Q-neighborhood of e_λ. Then there exist fuzzy open set A such that $e_\lambda qA \leq V^{-1}$. As above, $A^{-1} \leq V$ and $e_\lambda qA^{-1}$. That is, V is a fuzzy Q-neighborhood of e_λ. □

Proposition 5.2. [1] *Let $(P_1, \mathcal{T}_1)$ and $(P_2, \mathcal{T}_2)$ be two fuzzy topological polygroups and $f : P_1 \to P_2$ be a homomorphism. Then, f is fuzzy continuous if and only if f is continuous at e_λ (here e is the unit of P_1) for any $\lambda \in (0, 1]$.*

Definition 5.3. [1] Let $\langle P_1, \circ_1, e_1, ^{-1}, \mathcal{T}_1\rangle$ and $\langle P_2, \circ_2, e_2, ^{-I}, \mathcal{T}_2\rangle$ be fuzzy topological polygroups. A mapping f from P_1 into P_2 is said to be a *fuzzy topological homomorphism* if for all $a, b \in P_1$:

(1) $f(e_1) = e_2$.
(2) $f(a \circ_1 b) = f(a) \circ_2 f(b)$.
(3) f is fuzzy continuous mapping of FTS $(P_1, \mathcal{T}_1)$ into FTS $(P_2, \mathcal{T}_2)$.

Theorem 5.4. *The collection of all fuzzy topological polygroups and fuzzy topological homomorphisms form a category.*

Proof. Consider the collection of all FTP as objects, morphisms are fuzzy topological homomorphisms and compositions is the usual composition of functions. In checking that C_{FTP} is a category, one must note that for each object P, $i : P \to P$ given by $i(x) = x$ is the identity morphism. Consequently, it forms a category. □

Remark 1. It is well known that corresponding to any topological space $(X, \mathcal{T})$, one can obtain the characteristic fuzzy topological space $(X, \mathcal{T}_f)$.

Theorem 5.5. *If $(P, \mathcal{T})$ is a topological polygroup, then $(P, \mathcal{T}_f)$ is a fuzzy topological polygroup.*

Proof. Clearly $(P, \mathcal{T}_f)$ is a fuzzy topological space. Now, we show that $(P, \mathcal{T}_f)$ satisfies the conditions (1) and (2) in Definition 3.1.

(1) Let $x, y \in P$ and W be a fuzzy open Q-neighborhood of any fuzzy point z_λ of $x \circ y$. We show that there exist fuzzy open Q-neighborhood U and V of x_λ and y_λ respectively, such that $UV \leq W$.

Let W be a fuzzy open Q-neighborhood on $(P, \mathcal{T}_f)$ with $z_\lambda qW$. Then $W = \chi_A$ for some $A \in \mathcal{T}$. Hence,

$$z_\lambda q\ \chi_A \Rightarrow \chi_A(z) + \lambda > 1 \Rightarrow z \in A.$$

Since $(P, \mathcal{T})$ is a topological polygroup, there exist open sets $B, C \in \mathcal{T}$ such that $x \in B$, $y \in C$ and $BC \subseteq A$. Then $x_\lambda q\ \chi_B$ and $y_\lambda q\ \chi_C$, where $\chi_B, \chi_C \in \mathcal{T}_f$. In order to complete the proof, we show $\chi_B\ \chi_C \leq\ \chi_A = W$. For all $t \in P$,

$$\begin{aligned}
(\chi_B\ \chi_C)(t) &= \bigvee_{t \in t_1 \circ t_2} (\chi_B(t_1) \wedge \chi_C(t_2)) \\
&= \begin{cases} 1 & \text{if } t_1 \in B, t_2 \in C, \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} 1 & \text{if } t \in BC, \\ 0 & \text{otherwise.} \end{cases} \\
&= \chi_{BC}(t) \\
&\leq \chi_A(t).
\end{aligned}$$

So, the condition (1) in Definition 3.1 follows.

(2) Let $x \in P$ and U be a fuzzy open Q-neighborhood of x_λ^{-1}. We show that there exists fuzzy open Q-neighborhood V of x_λ such that $V^{-1} \leq U$.

Since $x^{-1} qU$, it follows that $U = \chi_A$ for some $A \in \mathcal{T}$. Hence,

$$x^{-1} q\ \chi_A \Rightarrow \chi_A(x^{-1}) + \lambda > 1 \Rightarrow x^{-1} \in A.$$

Since $(P, \mathcal{T})$ is a topological polygroup, there exists open set $B \in \mathcal{T}$ such that $x \in B$ and $B^{-1} \subseteq A$. Then $\chi_B(x) + \lambda > 1$, where $\chi_B \in \mathcal{T}_f$. In order to complete the proof, we show $\chi_B^{-1} \leq \chi_A = U$. For all $t \in P$,

$$\begin{aligned}
\chi_B^{-1}(t) &= \begin{cases} 1 & \text{if } t \in B^{-1}, \\ 0 & \text{if } t \notin B^{-1}. \end{cases} \\
&\leq \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A. \end{cases} \\
&= \chi_A(t).
\end{aligned}$$

Now, the condition (2) in Definition 3.1 follows. Therefore, $(P, \mathcal{T}_f)$ is a fuzzy topological polygroup. □

Theorem 5.6. *If f is a continuous topological homomorphism from a topological polygroup $(P_1, \mathcal{T}_1)$ to a topological polygroup $(P_2, \mathcal{T}_2)$ then $f : (P_1, \mathcal{T}_{1_f}) \to (P_2, \mathcal{T}_{2_f})$ is a fuzzy topological homomorphism between the corresponding fuzzy topological polygroup.*

Proof. The proof is straightforward. □

Theorem 5.7. *If C_{TP} is the category of topological polygroups and continuous topological homomorphisms, then C_{TP} is a full subcategory of C_{FTP}.*

Proof. We know that any object of C_{TP} can be viewed as an object of C_{FTP} and any morphism between two objects of C_{TP} is a morphism between the corresponding objects of C_{FTP}. Hence, C_{TP} is a subcategory of C_{FTP}. Let the inclusion functor $i : C_{TP} \to C_{FTP}$ that sends $(P, \mathcal{T})$ to its characteristic fuzzy topological space $(P, \mathcal{T}_f)$ and $f : (P, \mathcal{T}) \to (Q, \sigma)$ to $f^* : (P, \mathcal{T}_f) \to (Q, \sigma_f)$. To show that the functor i is full. Let $(P, \mathcal{T})$ and (Q, σ) be two objects in C_{TP} and $f^* : (P, \mathcal{T}_f) \to (Q, \sigma_f)$ a morphism in C_{FTP}. If $U \in \sigma$ then $\chi_U \in \sigma_f$ and so, ${f^*}^{-1}(\chi_U) = \chi_{{f^*}^{-1}(U)} \in \mathcal{T}_f$, which in turn gives ${f^*}^{-1}(U) \in \mathcal{T}$. Hence, there exist $f^* : (P, \mathcal{T}) \to (Q, \sigma)$a morphism in C_{TP} such that $i(f^*) = f^*$, that is, i is full. Consequently, $C_{TP}$is a full subcategory of C_{FTP}. □

Theorem 5.8. *Let $(P, \mathcal{T})$ be a fuzzy topological polygroup. Then, for all $0 \leq \lambda < 1$, $(P, i_\lambda(\mathcal{T}))$ is a topological polygroup.*

Proof. It is clear that $(P, i_\lambda(\mathcal{T}))$ is a topological space. We need to show that the mappings $(x, y) \mapsto x \circ y$ and $x \mapsto x^{-1}$ are continuous.

Let $x, y \in P$ and W be any open set in $(P, i_\lambda(\mathcal{T}))$ such that $x \circ y \subseteq W$. There exists a fuzzy open set γ in $(P, \mathcal{T})$ such that $\gamma^\lambda = W$. So, for any $z_\lambda \in x \circ y$, we have $z_\lambda < \gamma$. Since $(P, \mathcal{T})$ is a fuzzy topological polygroup, there exist fuzzy open sets U and V such that $x_\lambda < U$, $y_\lambda < V$ and $UV \leq \gamma$. Then $x \in U^\lambda$ and $y \in V^\lambda$. We shall show that $U^\lambda V^\lambda \subseteq W$.

If $r \in U^\lambda V^\lambda$, then $r \in st$ where $s \in U^\lambda$ and $t \in V^\lambda$, that is, $U(s) > \lambda$ and $V(t) > \lambda$. Now,,

$$(UV)(r) = \bigvee_{r \in r_1 r_2} [U(r_1) \wedge V(r_2)] \geq U(s) \wedge V(t) > \lambda.$$

So, $\gamma(r) > \lambda$, $r \in \gamma^\lambda = W$. Hence, $U^\lambda V^\lambda \subseteq W$. This show that the mapping $(x, y) \mapsto x \circ y$ is continuous. Now, we prove that $x \mapsto x^{-1}$ is continuous. Let $x \in P$ and V be an open set of $(P, i_\lambda(\mathcal{T}))$ containing x^{-1}. There is a fuzzy open set γ on $(P, \mathcal{T})$ such that $\gamma^\lambda = V$. So, $x^{-1} \in \gamma^\lambda$ and we have $x_\lambda^{-1} < \gamma$. Since $(P, \mathcal{T})$ is a fuzzy topological polygroup, there exists fuzzy open set U containing x_λ such that $x_\lambda < U$ and $U^{-1} \leq \gamma$. We shall show $(U^{-1})^\lambda \subseteq V$.

Let $t \in (U^{-1})^\lambda$, then $\gamma(t) \geq U^{-1}(t) > \lambda$. So, $t \in \gamma^\lambda$. Hence, $(U^{-1})^\lambda \subseteq V$. □

Theorem 5.9. *A function $f : (X, \mathcal{T}) \to (Y, \sigma)$ is fuzzy continuous if and only if $f : (X, i_\lambda(\mathcal{T})) \to (Y, i_\lambda(\sigma))$ is continuous for each $0 \leq \lambda < 1$, where $(X, \mathcal{T}), (Y, \sigma)$ are fuzzy topological spaces.*

Proof. The proof is straightforward. □

Theorem 5.10. *A function $f : (P_1, \mathcal{T}_1) \to (P_2, \mathcal{T}_2)$ is fuzzy topological homomorphism if and only if $f : (P_1, i_\lambda(\mathcal{T}_1)) \to (P_2, i_\lambda(\mathcal{T}_2))$ is continuous topological homomorphism for each $0 \leq \lambda < 1$, where $(P_1, \mathcal{T}_1)$ and $(P_2, \mathcal{T}_2)$ are fuzzy topological polygroups.*

Proof. The proof follows from Theorem 5.9. □

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