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ON FIXED POINTS OF GENERALIZED α-ψ CONTRACTIVE TYPE MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. Recently, Samet *et al.* (B. Samet, C. Vetro and P. Vetro, Fixed point theorem for α-ψ contractive type mappings, *Nonlinear Anal.* 75 (2012), 2154–2165) introduced a very interesting new category of contractive type mappings known as α-ψ contractive type mappings. The results obtained by Samet *et al.* generalize the existing fixed point results in the literature, in particular the Banach contraction principle. Further, Karapinar and Samet (E. Karapinar and B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, *Abstract and Applied Analysis* 2012 Article ID 793486, 17 pages doi:10.1155/2012/793486) generalized the α-ψ contractive type mappings and established some fixed point theorems for this generalized class of contractive mappings. In (G. S. Matthews, *Partial metric topology*, Ann. New York Acad. Sci. 728 (1994), 183–197), the author introduced and studied the concept of partial metric spaces, and obtained a Banach type fixed point theorem on complete partial metric spaces. In this paper, we establish the fixed point theorems for generalized α-ψ contractive mappings in the context of partial metric spaces. As consequences of our main results, we obtain fixed point theorems on partial metric spaces endowed with a partial order and that for cyclic contractive mappings. Our results extend and strengthen various known results. Some examples are also given to show that our generalization from metric spaces to partial metric spaces is real.

1. Introduction

The notion of metric space was introduced by Fréchet in 1906. Later, many authors attempted to generalize the notion of metric space such as pseudometric space, quasimetric space, semimetric space etc. In this paper, we consider another generalization of a metric space, so called partial metric space. When compared to metric spaces, the innovation of partial metric spaces is that the self distance of a point is not necessarily zero. Initially, Matthews discussed not only the general topological properties of partial metric spaces but also some properties of convergence of sequences. Matthews also stated and proved the fixed point theorem of contractive mapping on partial metric spaces: Any mapping T of a complete partial metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, has a unique fixed point. Very recently, many authors have focussed on this subject and have generalized some fixed point theorems from the class of metric spaces to the class of partial metric spaces.

The purpose of this work is to establish the fixed point theorems for generalized α-ψ-contractive mappings in the context of partial metric spaces. As consequences of our main results, we obtain fixed point theorems on partial metric spaces endowed with a partial order and that for cyclic contractive mappings. Presented theorems are generalizations of very recent fixed point theorems due to Samet *et al.*[23] and Karapinar and Samet [12]. Some examples are given to show that presented results are real generalizations.

2. Preliminaries

Throughout this work the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Q}$, $\mathbb{N}$ will denote the sets of real numbers, nonnegative real numbers, rational numbers and natural numbers, respectively.

Before presenting our results, we collect relevant definitions and results which will be needed in the proof of our main results.

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Definition 2.1. *(See e.g. [14, 9]). Let X be a nonempty set. The mapping $p : X \times X \to [0, \infty)$ is said to be a partial metric on X if the following conditions hold:*
(P1) $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$,
(P2) $p(x, x) \leq p(x, y)$,
(P3) $p(x, y) = p(y, x)$,
(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$,
for any $x, y, z \in X$. The pair (X, p) is then called a partial metric space (in short PMS).

Let (X, p) be a partial metric space. Then, the functions $d_p, d_m : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

and

$$d_m(x, y) = \max p(x, y) - p(x, x), p(x, y) - p(y, y)$$

are well-known metrics on X. It is easy to check that d_p and d_m are equivalent. Note that each partial metric p on X generates a T_0-topology τ_p with a base of the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$.

Definition 2.2. *(See e.g. [2, 9]). Let (X, p) be a partial metric space.*
(1) A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.
(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(x_n, x_m)$ exists (and finite).
(3) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$.
(4) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subset B_p(f(x_0), \epsilon)$.

Example 2.3. *Let $X = [0, +\infty)$ and define $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a complete partial metric space. It is clear that p is not a (usual) metric.*

Definition 2.4. *Let (X, p) be a partial metric space and $T : X \to X$ be a given mapping. We say that T is continuous at $x_0 \in X$, if for every $\epsilon > 0$, there exists $\eta > 0$ such that $T(B_p(x_0, \eta)) \subseteq B_p(Tx_0, \epsilon)$.*

The following lemmas have an important role in the proof of our main results.

Lemma 2.1. *(See e.g. [2, 9]). Let (X, p) be a partial metric space.*
(1) A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d_p).
(2) (X, p) is complete if and only if (X, d_p) complete. Moreover,

$$\lim_{n\to\infty} d_p(x_n, x) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_m, x_n).$$

Lemma 2.2. *(See e.g. [2]). Assume that $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Lemma 2.3. *(Sequential characterization of continuity)Let (X, p) be a partial metric space and $T : X \to X$ be a given mapping. T is said to be continuous at $x_0 \in X$ if it is sequentially continuous at x_0, that is, if and only if*

$$\forall \{x_n\} \subset X : \lim_{n\to+\infty} x_n = x_0 \Rightarrow \lim_{n\to+\infty} Tx_n = Tx_0$$

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(i) ψ is nondecreasing.
(ii) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.
These functions are known as (c)-comparison functions in the literature. It can be easily verified that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$.

Recently, Samet *et al.* [23] introduced the following new concepts of α-ψ-contractive type mappings and α-admissible mappings:

Definition 2.5. *Let (X,d) be a metric space and $T : X \to X$ be a given self mapping. T is said to be an α-ψ-contractive mapping if there exists two functions $\alpha : X \times X \to [0,+\infty)$ and $\psi \in \Psi$ such that*
$$\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y))$$
for all $x, y \in X$.

Definition 2.6. *Let $T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. T is said to be α-admissible if*
$$x,y \in X,\ \alpha(x,y) \ge 1 \Rightarrow \alpha(Tx,Ty) \ge 1.$$

The following fixed point theorems are the main results in [23]:

Theorem 2.4. *Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;
(iii) T is continuous.
Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 2.5. *Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then $\alpha(x_n, x) \ge 1$ for all n.
Then, T has a fixed point.

Samet *et al.* [23] added the following condition to the hypotheses of Theorem 2.4 and Theorem 2.5 to assure the uniqueness of the fixed point:
(C): For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \ge 1$ and $\alpha(y,z) \ge 1$.

Recently, Karapinar and Samet [12] introduced the following concept of generalized α-ψ-contractive type mappings:

Definition 2.7. *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalized α-ψ-contractive type mapping if there exists two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have*
$$\alpha(x,y)d(Tx,Ty) \le \psi(M(x,y)),$$
where $M(x,y) = \max\left\{d(x,y), \dfrac{d(x,Tx)+d(y,Ty)}{2}, \dfrac{d(x,Ty)+d(y,Tx)}{2}\right\}.$

Further, Karapinar and Samet [12] established fixed point theorems for this new class of contractive mappings. Also, they obtained fixed point theorems on metric spaces endowed with a partial order and fixed point theorems for cyclic contractive mappings.

3. Main results

Firstly, we present the concept of generalized α-ψ contractive type mappings in the context of partial metric spaces as follows:

Definition 3.1. *Let (X,p) be a partial metric space and $T : X \to X$ be a given mapping. We say that T is a generalized α-ψ-contractive type mapping if there exists two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have*
$$\alpha(x,y)p(Tx,Ty) \le \psi(M(x,y)), \tag{1}$$
where $M(x,y) = \max\left\{p(x,y), \dfrac{p(x,Tx)+p(y,Ty)}{2}, \dfrac{p(x,Ty)+p(y,Tx)}{2}\right\}.$

Now, we present our main results as follows.

Theorem 3.1. *Let (X,d) be a complete partial metric space, $\alpha : X \times X \to [0,\infty)$ be a function, $\psi \in \Psi$ and T be a generalized α-ψ contractive type mapping on X. Suppose that T is α-admissible and continuous. Also, assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then there exists $u \in X$ such that $Tu = u$.*

Proof. Take $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some n, then $x^* = x_n$ is a fixed point of T. Assume that $x_n \neq x_{n+1}$ for all n. Owing to α-admissible property of T, we have

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$$

Continuing this process inductively, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{2}$$

for all $n = 0, 1, 2, \ldots$. Thus for each n, we have

$$\begin{aligned} p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) &\leq \alpha(x_{n-1}, x_n) p(Tx_{n-1}, Tx_n) \\ &\leq \psi(M(x_{n-1}, x_n)) \end{aligned} \tag{3}$$

On the other hand, we have

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{p(x_{n-1}, x_n), \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}, \frac{p(x_{n-1}, x_{n+1}) + p(x_n, x_n)}{2}\right\} \\ &\leq \max\left\{p(x_{n-1}, x_n), \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}, \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}\right\} \\ &\leq \max\left\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\right\}. \end{aligned} \tag{4}$$

From (3), (4) and using the fact that ψ is a nondecreasing function, we get that

$$p(x_{n+1}, x_n) \leq \psi(\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\})$$

for all n. If $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$, then

$$p(x_n, x_{n+1}) \leq \psi(p(x_n, x_{n+1})) < p(x_n, x_{n+1}),$$

which is a contradiction. Thus, $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n)$ for all n. Hence,

$$p(x_n, x_{n+1}) \leq \psi(p(x_{n-1}, x_n))$$

for all n. Continuing this process inductively, we obtain

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \tag{5}$$

for all n. Now, using the definition of partial metric, we have

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leq p(x_n, x_{n+1}) \tag{6}$$

which in view of (5) gives rise to

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leq \psi^n((p(x_0, x_1)) \tag{7}$$

Therefore, owing to (4) and (5), we have

$$\begin{aligned} p^s(x_n, x_{n+1}) &= 2p(x_n, x_{n+1}) - p(x_n, x_n) - p(x_{n+1}, x_{n+1}) \\ &\leq 2p(x_n, x_{n+1}) + p(x_n, x_n) + p(x_{n+1}, x_{n+1}) \\ &\leq 4\psi^n(p(x_0, x_1)). \end{aligned} \tag{8}$$

Now, using inequality (8), we have

$$\begin{aligned} p^s(x_{n+k}, x_n) &\leq p^s(x_{n+k}, x_{n+k-1}) + \ldots + p^s(x_{n+1}, x_n) \\ &\leq 4\psi^{n+k-1}(p(x_0, x_1)) + \ldots + 4\psi^n(p(x_0, x_1)) \\ &\leq 4 \sum_{i=n}^{n+k-1} \psi^i(p(x_0, x_1)) \end{aligned} \tag{9}$$

and as $\sum_{i=0}^{\infty} \psi^i(p(x_0, x_1))$ is convergent, from the last inequality, using Cauchy's criteria for convergent series, we obtain that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s). Now, in view of Lemma 2.1 and the completeness of (X, p), we conclude the completeness of (X, p^s). Therefore, the sequence $\{x_n\}$ is convergent in the space (X, p^s), say $\lim_{n\to\infty} p^s(x_n, u) = 0$. Again from Lemma 2.1, we get

$$p(u, u) = \lim_{n\to\infty} p(x_n, u) = \lim_{m,n\to\infty} p(x_m, x_n) \tag{10}$$

Moreover, since $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s), we have

$$\lim_{m,n\to\infty} p^s(x_m, x_n) = 0 \tag{11}$$

and in view of (7), one gets

$$\lim_{n\to\infty} p(x_n, x_n) = 0 \tag{12}$$

Notice that in view of (11), (12) and definition of p^s, we conclude that

$$\lim_{m,n\to\infty} p(x_m, x_n) = 0 \tag{13}$$

On using (10), we have

$$p(u, u) = \lim_{n\to\infty} p(x_n, u) = \lim_{m,n\to\infty} p(x_m, x_n) = 0 \tag{14}$$

Now, we proceed to show that $Tu = u$. Due to the continuity of T, we infer from Lemma 2.3 that

$$p(Tu, Tu) = \lim_{n\to\infty} p(Tx_n, Tu) = \lim_{m,n\to\infty} p(Tx_m, Tx_n) \tag{15}$$

that is,

$$p(Tu, Tu) = \lim_{m,n\to\infty} p(x_{m+1}, x_{n+1}). \tag{16}$$

Notice that in view of (14) and (16),

$$p(u, u) = p(Tu, Tu) = 0 \tag{17}$$

Owing to Lemma 2.2, we have

$$\lim_{n\to\infty} p(x_n, Tu) = p(u, Tu) \tag{18}$$

Therefore, using (15), (17) and (18), we obtain

$$p(Tu, Tu) = p(u, u) = p(u, Tu) = 0$$

implying thereby $Tu = u$. Thus, we conclude that u is a fixed point of T. This completes the proof. □

In the next theorem, we omit the continuity hypothesis of T.

Theorem 3.2. *Let (X, d) be a complete partial metric space, $\alpha : X \times X \to [0, \infty)$ be a function, $\psi \in \Psi$ and T be a generalized α-ψ contractive type mapping on X. Suppose that T is α-admissible and that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k. Then there exists $u \in X$ such that $Tu = u$.*

Proof. Following the proof of Theorem 3.1, we know that the sequence $\{x_n\}$ given by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges to some $u \in X$. From (2) and given hypotheses, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)}, u) \geq 1 \tag{19}$$

for all k. Now, we proceed to show that u is a fixed point of T. Suppose the contrary, then $p(u,Tu)>0$. Therefore, from (1) and (19), we infer that

$$\begin{aligned} p(u,Tu) &\leq p(u,x_{n(k)+1})+p(x_{n(k)+1},Tu)-p(x_{n(k)+1},x_{n(k)+1})\\ &\leq p(u,x_{n(k)+1})+p(x_{n(k)+1},Tu)\\ &= p(u,x_{n(k)+1})+p(Tx_{n(k)},Tu)\\ &\leq p(u,x_{n(k)+1})+\alpha(x_{n(k)},u)p(Tx_{n(k)},Tu)\\ &\leq p(u,x_{n(k)+1})+\psi(M(x_{n(k)},u))) \end{aligned} \tag{20}$$

On the other hand, we have

$$M(x_{n(k)},u)) = \max\left\{p(x_{n(k)},u),\frac{p(x_{n(k)},x_{n(k)+1})+p(u,Tu)}{2},\frac{p(x_{n(k)},Tu)+p(u,x_{n(k)+1})}{2}\right\} \tag{21}$$

Letting $k\to\infty$ in (21) and using the above equality, we get

$$p(u,Tu) \leq \psi\left(\frac{p(u,Tu)}{2}\right)<\frac{p(u,Tu)}{2}, \tag{22}$$

which is a contradiction. Therefore, $p(u,Tu)=0$ and $Tu=u$. □

We demonstrate the use of Theorem 3.1 and Theorem 3.2 with the help of the following examples. These examples also show that our theorems are more general than some other known fixed point results.

Example 3.2. *Let $X=\mathbb{R}^+$, where (X,p) is a complete partial metric space with partial metric p given by $p(x,y)=\max\{x,y\}$. The mapping $T(x)=\dfrac{x^2}{1+x}$ $\forall x\in X$ is continuous. Let us define the function α by*

$$\alpha(x,y)=\begin{cases}1 & x\geq y\\ 0 & otherwise\end{cases}$$

Clearly, T is a generalized α-ψ contractive type mapping with $\psi(t)=\dfrac{t^2}{1+t}$ for all $t\geq 0$. In fact, for all $x,y\in X$, we have

$$\alpha(x,y)p(Tx,Ty)\leq\psi(M(x,y)) \tag{23}$$

Moreover, there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$. In fact, for $x_0=1$, we have

$$\alpha(1,T1)=\alpha(1,1/2)=1 \tag{24}$$

Now we proceed to show that T is α-admissible. For this, we have

$$\alpha(x,y)\geq 1\Rightarrow x\geq y\Rightarrow Tx\geq Ty\Rightarrow\alpha(Tx,Ty)\geq 1 \tag{25}$$

Thus, T is α-admissible. Now, all the hypotheses of Theorem 3.1 are satisfied. Consequently, T has a fixed point. In this case, 0 is a fixed point.
The same conclusion cannot be obtained by Theorem 2.3 from [12]. *Indeed, using $p^s(a,b)=2p(a,b)-p(a,a)-p(b,b)$, and then taking p^s instead p, $x=3$, $y=2$ in (1), we obtain*

$$\alpha(3,2)p^s(T3,T2)=\frac{11}{12}\nleq\frac{1}{2}=\psi(1). \tag{26}$$

Therefore, this example shows that our generalization from metric spaces to partial metric spaces is real.

Example 3.3. *Let $X=\{0,1,2,3\}$ and the function $p:X\times X\to[0,+\infty)$ defined by $p(1,2)=p(2,3)=1,p(1,3)=\frac{3}{2},p(1,1)=p(3,3)=\frac{1}{2},p(2,2)=0$ and $p(x,y)=p(y,x)$. Obviously, p is a partial metric on X but not a metric (since $p(x,x)\neq 0$ for $x=1$ and $x=3$). Let us define the self-mapping T on X by $T0=1$, $T1=2$, $T2=2$, $T3=1$ for each $x\in X$. Clearly, T is a generalized α-ψ contractive type mapping with $\psi(t)=\frac{2}{3}t$ for $t\geq 0$. In fact, for all $x,y\in X$, we have*

$$\alpha(x,y)p(Tx,Ty)\leq\psi(M(x,y)), \tag{27}$$

where

$$\alpha(x,y) = \begin{cases} 1 & (x,y) \neq (0,0) \\ 0 & otherwise \end{cases}$$

Moreover, there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$. *In fact, for* $x_0 = 1$, *we have*

$$\alpha(1, T1) = \alpha(1, 2) = 1 \tag{28}$$

Let $\{x_n\}$ *be a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x$ *as* $n \to +\infty$ *for some* $x \in X$. *From the definition of* α, *for all* n, *we have* $x_n \neq 0$ *for all* n. *Thus,* $x \neq 0$ *and we have* $\alpha(x_n, x) \geq 1$ *for all* n. *Now we proceed to show that* T *is* α*-admissible. For this, we have*

$$\alpha(x,y) \geq 1 \Rightarrow x \neq 0, y \neq 0 \Rightarrow Tx \neq 0, Ty \neq 0 \Rightarrow \alpha(Tx, Ty) \geq 1 \tag{29}$$

Thus, T *is* α*-admissible. Now, all the hypotheses of Theorem 3.2 are satisfied. Consequently,* T *has a fixed point. In this case, 2 is a fixed point.*
The same conclusion cannot be obtained by Theorem 2.4 from [12]. *Indeed, using* $p^s(a,b) = 2p(a,b) - p(a,a) - p(b,b)$, *and then taking* p^s *instead* p, $x = 1$, $y = 3$ *in (1), we obtain*

$$\alpha(1,3)p^s(T1, T3) = \frac{3}{2} \nleq \frac{4}{3} = \psi(2). \tag{30}$$

Therefore, this example shows that our generalization from metric spaces to partial metric spaces is real.

4. Fixed Point Theorems on Partial Metric Spaces Endowed with a Partial Order

Fixed point theory has developed rapidly in partially ordered metric spaces. The first result in this direction was given by Turinici [24], where he extended the Banach contraction principle in partially ordered sets. Some applications of Turinici's theorem to matrix equations were presented by Ran and Reurings [20]. Subsequently, Nieto and Rodríguez-López [17] extended this result and applied it to obtain a unique solution for periodic boundary value problems. Further results were obtained by several authors (see, for example, [4, 8, 13, 7, 16] and the references cited therein). Altun and Erduran [5] used the idea of partial order and established fixed point theorems to the frame of ordered partial metric spaces. Aydi [6], Samet et al. [22], Abbas and Nazir [1] also studied fixed point results on partially ordered partial metric spaces. Before presenting our result, we collect relevant concepts which will be needed in the proof of our results.

Definition 4.1. *Let* $(X, \preceq)$ *be a partially ordered set and* $T : X \to X$ *be a given mapping. We say that* T *is nondecreasing with respect to* $\preceq$ *if*

$$x, y \in X, x \preceq y \Rightarrow Tx \preceq Ty.$$

Definition 4.2. *Let* $(X, \preceq)$ *be a partially ordered set. A sequence* $\{x_n\} \subset X$ *is said to be nondecreasing with respect to* $\preceq$ *if* $x_n \preceq x_{n+1}$ *for all* n.

Definition 4.3. [12] *Let* $(X, \preceq)$ *be a partially ordered set and* p *be a partial metric on* X. *We say that* $(X, \preceq, p)$ *is regular if for every nondecreasing sequence* $\{x_n\} \subset X$ *such that* $x_n \to x \in X$ *as* $n \to \infty$, *there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $x_{n(k)} \preceq x$ *for all* k.

Now, we have the following result.

Corollary 4.1. *Let* $(X, \preceq)$ *be a partially ordered set and* p *be a partial metric on* X *such that* (X, p) *is complete. Let* $T : X \to X$ *be a nondecreasing mapping with respect to* $\preceq$. *Suppose that there exists a function* $\psi \in \Psi$ *such that*

$$p(Tx, Ty) \leq \psi(M(x,y)), \tag{31}$$

for all $x, y \in X$ *with* $x \succeq y$. *Suppose also that the following conditions hold:*
(i) there exists $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*;*
(ii) T *is continuous or* $(X, \preceq, d)$ *is regular.*
Then, T *has a fixed point.*

Proof. Let us define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y \text{ or } x \succeq y \\ 0 & \textit{otherwise} \end{cases} \tag{32}$$

for all $x, y \in X$. In view of condition (i), we obtain $\alpha(x_0, Tx_0) \geq 1$. Moreover, for all $x, y \in X$, from the monotone property of T, we get

$$\alpha(x,y) \geq 1 \Rightarrow x \succeq y \textit{ or } x \preceq y \Rightarrow Tx \succeq Ty \textit{ or } Tx \preceq Ty \Rightarrow \alpha(Tx, Ty) \geq 1. \tag{33}$$

Thus, T is α-admissible. Now, if T is continuous, the existence of a fixed point follows from Theorem 3.1. Suppose now that $(X, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. So, from the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. Notice that in view of definition of α, we obtain that $\alpha(x_{n(k)}, x) \geq 1$ for all k. Thus, we get the existence of a fixed point from Theorem 3.2. □

The following corollaries can be straightway derived from Corollary 4.1

Corollary 4.2. *Let $(X, \preceq)$ be a partially ordered set and p be a partial metric on X such that (X, p) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$p(Tx, Ty) \leq \psi(p(x, y)), \tag{34}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:
(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(ii) T is continuous or $(X, \preceq, p)$ is regular.
Then T is a fixed point.

Corollary 4.3. *Let $(X, \preceq)$ be a partially ordered set and p be a partial metric on X such that (X, p) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1)$ such that*

$$p(Tx, Ty) \leq \max\left\{p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}, \frac{p(x, Ty) + p(y, Tx)}{2}\right\}, \tag{35}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:
(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(ii) T is continuous or $(X, \preceq, p)$ is regular.
Then T is a fixed point.

Corollary 4.4. *Let $(X, \preceq)$ be a partially ordered set and p be a partial metric on X such that (X, p) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists constants $A, B, C \geq 0$ with $(A + 2B + 2C) \in (0, 1)$ such that*

$$p(Tx, Ty) \leq Ap(x, y) + B[p(x, Tx) + p(y, Ty)] + C[p(x, Ty) + p(y, Tx)], \tag{36}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:
(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(ii) T is continuous or $(X, \preceq, p)$ is regular.
Then T is a fixed point.

Corollary 4.5. *Let $(X, \preceq)$ be a partially ordered set and p be a partial metric on X such that (X, p) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1/2)$ such that*

$$p(Tx, Ty) \leq \lambda[p(x, Tx) + p(y, Ty)], \tag{37}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:
(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(ii) T is continuous or $(X, \preceq, p)$ is regular.
Then T is a fixed point.

Corollary 4.6. *Let $(X, \preceq)$ be a partially ordered set and p be a partial metric on X such that (X, p) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1/2)$ such that*

$$p(Tx, Ty) \leq \lambda[p(x, Ty) + p(y, Tx)], \tag{38}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:
(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(ii) T is continuous or $(X, \preceq, p)$ is regular.
Then T is a fixed point.

5. Fixed Point Theorems for Cyclic Contractive Mappings

As a generalization of the Banach contraction mapping principle, Kirk-Srinivasan-Veeramani developed the cyclic contraction. A contraction $T : A \cup B \to A \cup B$ on non-empty sets A, B is called cyclic if $T(A) \subset B$ and $T(B) \subset A$ hold for closed subsets A, B of a complete metric space X. In the last decade, several authors have used the cyclic representations and cyclic contractions to obtain various fixed point results. See e.g., ([3, 10, 11, 18, 19, 21]). In this section, we will show that, from our Theorem 3.1 and 3.2, we can deduce some fixed point theorems for cyclic contractive mappings.
Now, we have the following result.

Corollary 5.1. *Let $\{A_i\}$ be nonempty closed subsets of a complete partial metric space (X, p) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*
(i) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;
(ii) there exists a function $\psi \in \Psi$ such that

$$p(Tx, Ty) \leq \psi(M(x, y)), \ \forall (x, y) \in A_1 \times A_2. \tag{39}$$

Then T has a fixed point that belongs to $A_1 \cap A_2$.

Proof. Due to the fact that A_1 and A_2 are closed subsets of the complete metric space (X, d), we get completeness of the space (Y, d). Let us define the mapping $\alpha : Y \times Y \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 & \textit{otherwise} \end{cases} \tag{40}$$

Notice that in view of definition α and condition (ii), we infer that

$$\alpha(x, y)p(Tx, Ty) \leq \psi(M(x, y)), \tag{41}$$

for all $x, y \in Y$. Thus T is a generalized α-ψ contractive mapping. Now, we proceed to show that T is α-admissible. For thus, let $(x, y) \in Y \times Y$ such that $\alpha(x, y) \geq 1$. If $(x, y) \in A_1 \times A_2$, then from (i), we have $(Tx, Ty) \in A_2 \times A_1$, thereby implying $\alpha(Tx, Ty) \geq 1$. Again from (i), we obtain that $(x, y) \in A_2 \times A_1$ implies that $(Tx, Ty) \in A_1 \times A_2$, which further implies that $\alpha(Tx, Ty) \geq 1$. Thus, we have $\alpha(Tx, Ty) \geq 1$ in all the cases. Therefore, we obtain that T is α-admissible.
Also, in view of (i), for any $u \in A_1$, we have $(u, Tu) \in A_1 \times A_2$, which suggest that $\alpha(u, Tu) \geq 1$.
Now, we consider that $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. This suggest from the definition of α that

$$(x_n, x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \tag{42}$$

for all n. Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the Euclidean metric, we obtain that

$$(x, x) \in (A_1 \times A_2) \cup (A_2 \times A_1), \tag{43}$$

which refer that $x \in A_1 \cap A_2$. Consequently, we get from the definition of α that $\alpha(x_n, x) \geq 1$ for all n. Now, all the hypotheses of Theorem 3.2 are satisfied, and we conclude that T has a fixed point that belongs to $A_1 \cap A_2$(from (i)). □

The following results are immediate consequences of Corollary 5.1.

Corollary 5.2. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete partial metric space (X,p) and $T:Y\to Y$ be a given mapping, where $Y=A_1\cup A_2$. Suppose that the following conditions hold:*
(i) $T(A_1)\subseteq A_2$ and $T(A_2)\subseteq A_1$;
(ii) there exists a function $\psi\in\Psi$ such that

$$p(Tx,Ty)\le\psi(p(x,y)),\ \forall(x,y)\in A_1\times A_2. \tag{44}$$

Then T has a fixed point that belongs to $A_1\cap A_2$.

Corollary 5.3. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete partial metric space (X,p) and $T:Y\to Y$ be a given mapping, where $Y=A_1\cup A_2$. Suppose that the following conditions hold:*
(i) $T(A_1)\subseteq A_2$ and $T(A_2)\subseteq A_1$;
(ii) there exists a constant $\lambda\in(0,1)$ such that

$$p(Tx,Ty)\le\lambda\max\left\{p(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(y,Tx)}{2}\right\},\quad\forall(x,y)\in A_1\times A_2. \tag{45}$$

Then T has a fixed point that belongs to $A_1\cap A_2$.

Corollary 5.4. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete partial metric space (X,p) and $T:Y\to Y$ be a given mapping, where $Y=A_1\cup A_2$. Suppose that the following conditions hold:*
(i) $T(A_1)\subseteq A_2$ and $T(A_2)\subseteq A_1$;
(ii) there exists constants $A,B,C\ge 0$ with $(A+2B+2C)\in(0,1)$ such that

$$p(Tx,Ty)\le Ap(x,y)+B[d(x,Tx)+d(y,Ty)]+C[d(x,Ty)+d(y,Tx)],\quad\forall(x,y)\in A_1\times A_2. \tag{46}$$

Then T has a fixed point that belongs to $A_1\cap A_2$.

Corollary 5.5. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete partial metric space (X,p) and $T:Y\to Y$ be a given mapping, where $Y=A_1\cup A_2$. Suppose that the following conditions hold:*
(i) $T(A_1)\subseteq A_2$ and $T(A_2)\subseteq A_1$;
(ii) there exists a constant $\lambda\in(0,1)$ such that

$$p(Tx,Ty)\le\lambda p(x,y),\ \forall(x,y)\in A_1\times A_2. \tag{47}$$

Then T has a fixed point that belongs to $A_1\cap A_2$.

Corollary 5.6. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete partial metric space (X,p) and $T:Y\to Y$ be a given mapping, where $Y=A_1\cup A_2$. Suppose that the following conditions hold:*
(i) $T(A_1)\subseteq A_2$ and $T(A_2)\subseteq A_1$;
(ii) there exists a constant $\lambda\in(0,1/2)$ such that

$$p(Tx,Ty)\le\lambda[p(x,Tx)+p(y,Ty)],\ \forall(x,y)\in A_1\times A_2. \tag{48}$$

Then T has a fixed point that belongs to $A_1\cap A_2$.

Corollary 5.7. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete partial metric space (X,p) and $T:Y\to Y$ be a given mapping, where $Y=A_1\cup A_2$. Suppose that the following conditions hold:*
(i) $T(A_1)\subseteq A_2$ and $T(A_2)\subseteq A_1$;
(ii) there exists a constant $\lambda\in(0,1/2)$ such that

$$p(Tx,Ty)\le\lambda[p(x,Ty)+p(y,Tx)],\ \forall(x,y)\in A_1\times A_2. \tag{49}$$

Then T has a fixed point that belongs to $A_1\cap A_2$.

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