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SOME PERTURBED OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE OF BOUNDED VARIATION

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Abstract. The main aim of this paper is to establish some new perturbed Ostrowski type integral inequalities for functions whose first derivatives are of bounded variation. Some perturbed Ostrowski type inequalities for Lipschitzian and monotonic mappings are also obtained.

1. Introduction

In 1938, Ostrowski [20] established a following useful inequality:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty , \tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

The following definitions will be frequently used to prove our results.

Definition 1. *Let $P: a=x_0<x_1<...<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i) = f(x_{i+1})-f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. *Let f be of bounded variation on $[a,b]$, and $\sum \Delta f(P)$ denotes the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number*

$$\bigvee_a^b (f) := \sup\left\{ \sum \Delta f(P) : P\in P([a,b]) \right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [14], Dragomir proved the following Ostrowski type inequalities related functions of bounded variation:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left| \int_a^b f(t)dt - (b-a)f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x-\frac{a+b}{2} \right| \right] \bigvee_a^b (f) \tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

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In the past, many authors have worked on Ostrowski type inequalities for function of bounded variation, see for example ([1]-[4],[6]-[9],[11]-[16],[19]).

For a function of bounded variation $v : [a, b] \to \mathbb{C}$. we define the C*umulative Variation Function* (CVF) $V : [a; b] \to [0, \infty)$ by

$$V(t) := \bigvee_a^t (v),$$

the total variation of v on the interval $[a, t]$ with $t \in [a, b]$.

It is know that the CVF is monotonic nondecreasing on $[a, b]$ and is continuous in a point $c \in [a, b]$ if and only if the generating function v is continuing in that point. If v is *Lipschitzian* with the constant $L > 0$, i.e.

$$|v(t) - v(s)| \leq L\,|t - s|\,, \text{ for any } t, s \in [a, b]\,,$$

then V is also Lipschitzian with the same constant.

A simple proof of the following Lemma was given in [15].

Lemma 1. *Let $f, u : [a, b] \to \mathbb{C}$. If f is continuous on $[a, b]$ and u is of bounded variation on $[a, b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ exist and*

$$\left|\int_a^b f(t)du(t)\right| \leq \int_a^b |f(t)|\, d\left(\bigvee_a^t (u)\right) \leq \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u). \tag{1.3}$$

In [8], authors gave the following Ostrowski type inequality for mapping whoose first derivatives are of bounded variation:

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a, b]$. Then we have the inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f(x) + f(a+b-x)\right]\right.\\
&\left.+\frac{1}{2}\left(x - \frac{3a+b}{4}\right)\left[f'(x) - f'(a+b-x)\right]\right|\\
\leq\;& \frac{1}{16}\left[\frac{5\,(x-a)^2 - 2\,(x-a)\,(b-x) + (b-x)^2}{b-a} + 4\left|x - \frac{3a+b}{4}\right|\right]\bigvee_a^b (f')
\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

For recent related results, see [5],[7] and [9]. Moreover, Dragomir proved some perturbed Ostrowski type inequalities for functions of bounded variation in [17, 18]. The aim of this paper is to obtain new perturbed Ostrowski type inequalities for mappings whose first derivatives are of bounded variation.

2. Some Identities

Before we start our main results, we state and prove following lemma:

Lemma 2. *Let $f : [a, b] \to \mathbb{C}$ be a twice differantiable function on (a, b). Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex number the following identity holds*

$$\begin{aligned}
&\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt\\
&-\frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3}\right]\\
=\;& \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\, d\left[f'(t) - \lambda_1(x)t\right] + \frac{1}{b-a}\int_x^b (t-b)^2\, d\left[f'(t) - \lambda_2(x)t\right]\right],
\end{aligned} \tag{2.1}$$

where the integrals in the right hand side are taken in the Riemann-Stieltjes sense.

Proof. Using the integration by parts for Riemann-Stieltjes, we have

$$\begin{aligned}
&\int_a^x (t-a)^2\, d\left[f'(t)-\lambda_1(x)t\right] \qquad (2.2)\\
&= \int_a^x (t-a)^2\, df'(t) - \lambda_1(x)\int_a^x (t-a)^2\, dt\\
&= \left.(t-a)^2 f'(t)\right|_a^x - 2\int_a^x (t-a)\, f'(t)dt - \frac{\lambda_1(x)}{3}\left.(t-a)^3\right|_a^x\\
&= (x-a)^2 f'(x) - 2\left[\left.(t-a)\, f(t)\right|_a^x - \int_a^x f(t)dt\right] - \frac{\lambda_1(x)}{3}(x-a)^3\\
&= (x-a)^2 f'(x) - 2(x-a)\, f(x) + 2\int_a^x f(t)dt - \frac{\lambda_1(x)}{3}(x-a)^3
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b (t-b)^2\, d\left[f'(t)-\lambda_2(x)t\right] \qquad (2.3)\\
&= \int_x^b (t-b)^2\, df'(t) - \lambda_2(x)\int_x^b (t-b)^2\, dt\\
&= \left.(t-b)^2 f'(t)\right|_x^b - 2\int_x^b (t-b)\, f'(t)dt - \frac{\lambda_1(x)}{3}\left.(t-b)^3\right|_x^b\\
&= -(b-x)^2 f'(x) - 2\left[\left.(t-b)\, f(t)\right|_x^b - \int_x^b f(t)dt\right] - \frac{\lambda_2(x)}{3}(b-x)^3\\
&= (b-x)^2 f'(x) - 2(b-x)\, f(x) + 2\int_x^b f(t)dt - \frac{\lambda_1(x)}{3}(x-a)^3 .
\end{aligned}$$

If we add the equality (2.2) and (2.3) and devide by $2(b-a)$, we obtain required identity. □

Corollary 1. *Under assumption of Lemma 2 with $\lambda_1(x)=\lambda_2(x)=\lambda(x)$, we have*

$$\begin{aligned}
&\left(x-\frac{a+b}{2}\right)f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{\lambda(x)}{6(b-a)}\left[(x-a)^3+(b-x)^3\right] \qquad (2.4)\\
&= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\, d\left[f'(t)-\lambda(x)t\right] + \frac{1}{b-a}\int_x^b (t-b)^2\, d\left[f'(t)-\lambda(x)t\right]\right]
\end{aligned}$$

for all $x\in[a,b]$.

Remark 1. *If we choose $\lambda(x) = 0$ in (2.4), then we have the following identity*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\, df'(t) + \frac{1}{b-a}\int_x^b (t-b)^2\, df'(t)\right] \tag{2.5}$$

for all $x \in [a,b]$.

Corollary 2. *Under assumption of Lemma 2 with $\lambda_1(x) = \lambda_1 \in \mathbb{C}$ and $\lambda_2(x) = \lambda_2 \in \mathbb{C}$, we get*

$$\begin{aligned} &\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt \\ &- \frac{1}{6(b-a)}\left[\lambda_1 (x-a)^3 + \lambda_2 (b-x)^3\right] \\ = &\ \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\, d\left[f'(t) - \lambda_1 t\right] + \frac{1}{b-a}\int_x^b (t-b)^2\, d\left[f'(t) - \lambda_2 t\right]\right]. \end{aligned} \tag{2.6}$$

In particular, taking $\lambda_1 = \lambda_2 = \lambda$ we have

$$\begin{aligned} &\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{\lambda}{6(b-a)}\left[(x-a)^3 + (b-x)^3\right] \\ = &\ \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\, d\left[f'(t) - \lambda t\right] + \frac{1}{b-a}\int_x^b (t-b)^2\, d\left[f'(t) - \lambda t\right]\right]. \end{aligned} \tag{2.7}$$

3. Inequalities for Functions Whose First Derivatives are of Bounded Variation

We denote by $\ell : [a,b] \to [a,b]$ the identity function, namely $\ell(t) = t$ for any $t \in [a,b]$.

Theorem 4. *Let : $f : [a,b] \to \mathbb{C}$ be a twice differantiable function on I° and $[a,b] \subset I^\circ$. If the first derivative f' is of bounded variation on $[a,b]$, then*

$$\begin{aligned} &\Bigg|\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt \\ &- \frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3}\right]\Bigg| \\ \leq &\ \frac{1}{(b-a)}\left[\int_a^x (t-a)\left(\bigvee_t^x (f' - \lambda_1(x)\ell)\right) dt + \int_x^b (b-t)\left(\bigvee_x^t (f' - \lambda_2(x)\ell)\right) dt\right] \\ \leq &\ \frac{1}{2(b-a)}\left[(x-a)^2 \bigvee_a^x (f' - \lambda_1(x)\ell) + (b-x)^2 \bigvee_x^b (f' - \lambda_2(x)\ell)\right] \end{aligned} \tag{3.1}$$

$$\leq \frac{1}{2(b-a)} \times \begin{cases} \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \max\left\{\bigvee_a^x(f'-\lambda_1(x)\ell), \bigvee_x^b(f'-\lambda_2(x)\ell)\right\}(b-a)^2, \\ \max\left\{(x-a)^2,(b-x)^2\right\}\left[\bigvee_a^x(f'-\lambda_1(x)\ell)+\bigvee_x^b(f'-\lambda_2(x)\ell)\right] \end{cases}$$

for any $x\in[a,b]$.

Proof. Taking modulus (2.1) and applying Lemma 1, we get

$$\begin{aligned} &\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt \right. \\ &\left. -\frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3+\lambda_2(x)(b-x)^3}{3}\right]\right| \\ \leq\ & \frac{1}{2}\left[\frac{1}{b-a}\left|\int_a^x (t-a)^2\, d\left[f'(t)-\lambda_1(x)t\right]\right|+\frac{1}{b-a}\left|\int_x^b (t-b)^2\, d\left[f'(t)-\lambda_2(x)t\right]\right|\right] \\ \leq\ & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\, d\left(\bigvee_a^t(f'-\lambda_1(x)\ell)\right)+\int_x^b (t-b)^2\, d\left(\bigvee_a^t(f'-\lambda_2(x)\ell)\right)\right]. \end{aligned} \tag{3.2}$$

Integrating by parts in the Riemann-Stieltjes integral, we get

$$\begin{aligned} &\int_a^x (t-a)^2\, d\left(\bigvee_a^t(f'-\lambda_1(x)\ell)\right) \\ =\ & (t-a)^2\bigvee_a^t(f'-\lambda_1(x)\ell)\Big|_a^x-2\int_a^x (t-a)\left(\bigvee_a^t(f'-\lambda_1(x)\ell)\right)dt \\ =\ & (x-a)^2\bigvee_a^x(f'-\lambda_1(x)\ell)-2\int_a^x (t-a)\left(\bigvee_a^t(f'-\lambda_1(x)\ell)\right)dt \\ =\ & 2\int_a^x (t-a)\left(\bigvee_a^x(f'-\lambda_1(x)\ell)\right)dt-2\int_a^x (t-a)\left(\bigvee_a^t(f'-\lambda_1(x)\ell)\right)dt \\ =\ & 2\int_a^x (t-a)\left(\bigvee_t^x(f'-\lambda_1(x)\ell)\right)dt \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} (3.4)\quad & \int_x^b (t-b)^2\, d\left(\bigvee_a^t (f' - \lambda_2(x)\ell)\right) \\ &= (t-b)^2 \bigvee_a^t (f' - \lambda_2(x)\ell)\Big|_x^b - 2\int_x^b (t-b)\left(\bigvee_a^t (f' - \lambda_2(x)\ell)\right) dt \\ &= -(x-b)^2 \bigvee_a^x (f' - \lambda_2(x)\ell) - 2\int_x^b (t-b)\left(\bigvee_a^t (f' - \lambda_2(x)\ell)\right) dt \\ &= -2\int_x^b (b-t)\left(\bigvee_a^x (f' - \lambda_2(x)\ell)\right) dt + 2\int_x^b (b-t)\left(\bigvee_a^t (f' - \lambda_2(x)\ell)\right) dt \\ &= 2\int_x^b (b-t)\left(\bigvee_x^t (f' - \lambda_2(x)\ell)\right) dt. \end{aligned}$$

If we put the identities (3.3) and (3.4) in (3.2), then we obtain the first inequality in (3.1). Moreover, we have,

$$(3.5)\qquad \int_a^x (t-a)\left(\bigvee_t^x (f' - \lambda_1(x)\ell)\right) dt \le \frac{1}{2}(x-a)^2 \bigvee_a^x (f' - \lambda_1(x)\ell)$$

and

$$(3.6)\qquad \int_x^b (b-t)\left(\bigvee_x^t (f' - \lambda_2(x)\ell)\right) dt \le \frac{1}{2}(b-x)^2 \bigvee_x^b (f' - \lambda_2(x)\ell).$$

With the inequalities (3.5) and (3.6), the proof of Theorem 4 is completed. □

Corollary 3. *If we chosose $\lambda_1(x) = \lambda_2(x) = 0$, then we have the following inequality*

$$\begin{aligned} & \left| \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\le \frac{1}{(b-a)}\left[\int_a^x (t-a)\left(\bigvee_t^x (f')\right) dt + \int_x^b (b-t)\left(\bigvee_x^t (f'\ell)\right) dt\right] \\ &\le \frac{1}{2(b-a)}\left[(x-a)^2 \bigvee_a^x (f') + (b-x)^2 \bigvee_x^b (f')\right] \\ &\le \frac{b-a}{2}\begin{cases} \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\left[\frac{1}{2}\bigvee_a^b (f') + \frac{1}{2}\left|\bigvee_a^x (f') - \bigvee_x^b (f')\right|\right], \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2 \bigvee_a^b (f') \end{cases} \end{aligned}$$

for all $x \in [a,b]$.

Corollary 4. *Under assumption of Theorem 4 with* $\lambda_1(x) = \lambda_2(x) = \lambda(x)$, *we have*

(3.7)
$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt-\frac{\lambda(x)}{6(b-a)}\left[(x-a)^3+(b-x)^3\right]\right|$$
$$\leq \frac{1}{(b-a)}\left[\int_a^x (t-a)\left(\bigvee_t^x(f'-\lambda(x)\ell)\right)dt+\int_x^b (b-t)\left(\bigvee_x^t(f'-\lambda(x)\ell)\right)dt\right]$$
$$\leq \frac{1}{2(b-a)}\left[(x-a)^2\bigvee_a^x(f'-\lambda(x)\ell)+(b-x)^2\bigvee_x^b(f'-\lambda(x)\ell)\right]$$
$$\leq \frac{b-a}{2}\begin{cases}\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \\ \times\left[\frac{1}{2}\bigvee_a^b(f'-\lambda(x)\ell)+\frac{1}{2}\left|\bigvee_a^x(f'-\lambda(x)\ell)-\bigvee_x^b(f'-\lambda(x)\ell)\right|\right], \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2\bigvee_a^b(f'-\lambda(x)\ell)\end{cases}$$

for all $x \in [a,b]$.

Corollary 5. *If we choose* $\lambda(x) = \lambda$ *and* $x = \frac{a+b}{2}$ *in (3.7), then we have the following identity*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-f(x)-\frac{\lambda(b-a)^2}{24}\right|$$
$$\leq \frac{1}{(b-a)}\left[\int_a^{\frac{a+b}{2}} (t-a)\left(\bigvee_t^{\frac{a+b}{2}}(f'-\lambda\ell)\right)dt+\int_{\frac{a+b}{2}}^b (b-t)\left(\bigvee_{\frac{a+b}{2}}^t(f'-\lambda\ell)\right)dt\right]$$
$$\leq \frac{(b-a)}{8}\bigvee_a^b(f'-\lambda(x)\ell).$$

4. Inequalities for Functions Whose First Derivatives are Lipschitzian

Theorem 5. *Let* $f : [a,b] \to \mathbb{C}$ *be a twice differantiable function on* I° *and* $[a,b] \subset I^\circ$. *If there exist the positive numbers* $K_1(x)$ *and* $K_2(x)$ *such that* $f' - \lambda_1(x)\ell$ *is Lipschitzian with the constant* $K_1(x)$ *on the interval* $[a,x]$ *and* $f' - \lambda_2(x)\ell$ *is Lipschitzian with the constant* $K_2(x)$ *on the interval* $[x,b]$, *then we have for any* $x \in [a,b]$

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\right. \tag{4.1}$$
$$\left.-\frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3+\lambda_2(x)(b-x)^3}{3}\right]\right|$$
$$\leq \frac{(b-a)^2}{6}\left[K_1(x)\left(\frac{x-a}{b-a}\right)^3+K_2(x)\left(\frac{b-x}{b-a}\right)^3\right]$$

$$\leq \frac{(b-a)^2}{6}\begin{cases} \left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right]\max\left\{K_1(x),K_2(x)\right\}, \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{3p}+\left(\frac{b-x}{b-a}\right)^{3p}\right]^{\frac{1}{p}}\left[(K_1(x))^q+(K_1(x))^q\right]^{\frac{1}{q}} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^3\left[K_1(x)+K_2(x)\right]. \end{cases}$$

Proof. It is known that, if $g:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$, then the Riemann-Stieltje integral $\int\limits_c^d g(t)du(t)$ exist and

$$\left|\int\limits_c^d g(t)du(t)\right|\leq K\int\limits_c^d |g(t)|\,dt.$$

Taking the madulus (2.1), we get

$$\begin{aligned} &\Bigg|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int\limits_a^b f(t)dt \\ &-\frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3+\lambda_2(x)(b-x)^3}{3}\right]\Bigg| \\ \leq\ & \frac{1}{2(b-a)}\left[\left|\int\limits_a^x (t-a)^2\,d\left[f'(t)-\lambda_1(x)t\right]\right|+\left|\int\limits_x^b (t-b)^2\,d\left[f'(t)-\lambda_2(x)t\right]\right|\right] \\ \leq\ & \frac{1}{2(b-a)}\left[K_1(x)\int\limits_a^x\left|(t-a)^2\right|dt+K_2(x)\int\limits_x^b\left|(t-b)^2\right|dt\right] \\ =\ & \frac{(b-a)^2}{6}\left[K_1(x)\,(x-a)^3+K_2(x)\,(b-x)^3\right] \\ =\ & \frac{(b-a)^2}{6}\left[K_1(x)\left(\frac{x-a}{b-a}\right)^3+K_2(x)\left(\frac{b-x}{b-a}\right)^3\right]. \end{aligned}$$

This completes the proof of first inequality in (4.1).

Using the Hölder's inequality, we have

$$K_1(x)\left(\frac{x-a}{b-a}\right)^3+K_2(x)\left(\frac{b-x}{b-a}\right)^3$$

$$\leq \begin{cases} \left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right]\max\left\{K_1(x),K_2(x)\right\}, \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{3p}+\left(\frac{b-x}{b-a}\right)^{3p}\right]^{\frac{1}{p}}\left[(K_1(x))^q+(K_1(x))^q\right]^{\frac{1}{q}} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^3\left[L_1(x)+L_2(x)\right] \end{cases}$$

which completes the proof. □

Corollary 6. *Under assumption of Theorem 5 with $K_1(x) = K_2(x) = K$ and $\lambda_1(x) = \lambda_2(x) = \lambda(x)$, we have*

$$\left|\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{\lambda(x)}{6(b-a)}\left[(x-a)^3 + (b-x)^3\right]\right| \tag{4.2}$$
$$\leq \frac{1}{6}\left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]^3 K(b-a)^2.$$

Corollary 7. *If we choose $x = \frac{a+b}{2}$ and $\lambda(x) = \lambda \in \mathbb{C}$ in (4.2), we get the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{\lambda(b-a)^2}{48}\right| \leq \frac{1}{48}K(b-a)^2.$$

5. Inequalities for Mappings Whose First Derivatives are Monotonic Function

Theorem 6. *Let $f : [a,b] \to \mathbb{C}$ be a twice differantiable function on I° and $[a,b] \subset I^\circ$. If $\lambda_1(x)$ and $\lambda_2(x)$ are real numbers such that $f' - \lambda_1(x)\ell$ is monotonic nondecreasing on the interval $[a,x]$ and $f' - \lambda_2(x)\ell$ is monotonic nondecreasing on the interval $[x,b]$, then for any $x \in [a,b]$ the following inequalities hold:*

$$\left|\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt \right. \tag{5.1}$$
$$\left. -\frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3}\right]\right|$$
$$\leq \frac{1}{2(b-a)}\left[(x-a)^2\left[f'(x) - f'(a) - \lambda_1(x)(x-a)\right]\right.$$
$$\left. + (b-x)^2\left[f'(b) - f'(x) - \lambda_2(x)(b-x)\right]\right]$$
$$\leq \frac{1}{2(b-a)}\begin{cases} \left[\frac{1}{2}\left[f'(b) - f'(a) - \lambda_1(x)(x-a) - \lambda_2(x)(b-x)\right]\right. \\ \left. + \left|f'(x) - \frac{f'(a)+f'(b)}{2} - \frac{1}{2}\lambda_1(x)(x-a) + \frac{1}{2}\lambda_2(x)(b-x)\right|\right] \\ \times\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)^2, \\ \\ \max\left\{(x-a)^2, (b-x)^2\right\} \\ \times\left[f'(b) - f'(a) - \lambda_1(x)(x-a) - \lambda_2(x)(b-x)\right]. \end{cases}$$

Proof. Taking the madulus (2.1), we have

$$\left|\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt \right. \tag{5.2}$$
$$\left. -\frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3}\right]\right|$$
$$\leq \frac{1}{2(b-a)}\left[\left|\int_a^x (t-a)^2 d\left[f'(t) - \lambda_1(x)t\right]\right| + \left|\int_x^b (t-b)^2 d\left[f'(t) - \lambda_2(x)t\right]\right|\right]$$

Since $f' - \lambda_1(x)\ell$ is monotonic nondecreasing on the interval $[a,x]$, we have

$$\begin{aligned}\int_a^x (t-a)^2\, d\left[f'(t)-\lambda_1(x)t\right] &\le (x-a)^2\left[f'(x)-\lambda_1(x)x-f'(a)+\lambda_1(x)a\right] \\ &= (x-a)^2\left[f'(x)-f'(a)-\lambda_1(x)(x-a)\right]\end{aligned} \tag{5.3}$$

and similarly, since $f' - \lambda_2(x)\ell$ is monotonic nondecreasing on the interval $[x,b]$, we have

$$\begin{aligned}\int_x^b (t-b)^2\, d\left[f'(t)-\lambda_2(x)t\right] &\le (b-x)^2\left[f'(b)-\lambda_2(x)b-f'(x)+\lambda_2(x)x\right] \\ &= (b-x)^2\left[f'(b)-f'(x)-\lambda_2(x)(b-x)\right].\end{aligned} \tag{5.4}$$

If we put (5.3) and (5.4) in (5.2), we obtain the first inequality in (5.1).

The proofs of last inequalities are obvious, they are omitted. □

Corollary 8. *Under assumption of Theorem 6 with $\lambda_1(x)=\lambda_2(x)=\lambda(x)$, we have*

$$\begin{aligned}&\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt-\frac{\lambda(x)}{6(b-a)}\left[(x-a)^3+(b-x)^3\right]\right| \\ &\le \frac{1}{2(b-a)}\Big[(x-a)^2\left[f'(x)-f'(a)-\lambda(x)(x-a)\right] \\ &\qquad +(b-x)^2\left[f'(b)-f'(x)-\lambda(x)(b-x)\right]\Big] \\ &\le \frac{b-a}{2} \\ &\times\begin{cases}\left[\frac{f'(b)-f'(a)}{2}-\frac{1}{2}\lambda(x)(b-a)\left|f'(x)-\frac{f'(a)+f'(b)}{2}-\lambda(x)\left(x-\frac{a+b}{2}\right)\right|\right. \\ \times\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right], \\ \\ \left[f'(b)-f'(a)-\lambda(x)(b-a)\right]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2.\end{cases}\end{aligned} \tag{5.5}$$

Corollary 9. *If we choose $x=\frac{a+b}{2}$ and $\lambda(x)=\lambda$ in (5.5), we get the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)-\frac{\lambda(b-a)^2}{48}\right| \le \frac{(b-a)}{8}\left[f'(b)-f'(a)-\lambda(b-a)\right].$$

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