International Journal of Analysis and Applications
ISSN 2291-8639
Volume 12, Number 1 (2016), 49-61
http://www.etamaths.com

DHAGE ITERATION METHOD FOR NONLINEAR FIRST ORDER HYBRID DIFFERENTIAL EQUATIONS WITH A LINEAR PERTURBATION OF SECOND TYPE

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Abstract. In this paper the authors prove algorithms for the existence and approximation of the solutions for an initial and a periodic boundary value problem of nonlinear first order ordinary hybrid differential equations with a linear perturbation of second type via Dhage iteration method. Examples are furnished to illustrate the hypotheses and main abstract results of this paper.

1. Introduction

Given a closed and bounded interval $J = [0, T]$ in the real line $\mathbb{R}$, consider the initial and periodic boundary value problems of first order nonlinear hybrid differential equation (in short HDE),

$$\left.\begin{aligned} \frac{d}{dt}\big[x(t) - f(t, x(t))\big] + \lambda\big[x(t) - f(t, x(t))\big] = g(t, x(t)),\ \ t \in J,\\ x(0) = \alpha_0, \end{aligned}\right\} \tag{1.1}$$

and

$$\left.\begin{aligned} \frac{d}{dt}\big[x(t) - f(t, x(t))\big] + \lambda\big[x(t) - f(t, x(t))\big] = g(t, x(t)),\ \ t \in J,\\ x(0) = x(T), \end{aligned}\right\} \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a *solution* of the HDE (1.1) or (1.2) we mean a function $x \in C(J, \mathbb{R})$ such that
(i) the function $t \mapsto x - f(t, x)$ is differentiable for each $x \in \mathbb{R}$, and
(ii) x satisfies the equations in (1.1) or (1.2),
where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The HDEs (1.1) and (1.2) are linear perturbations of the second type of the nonlinear differential equations

$$\left.\begin{aligned} x'(t) = g(t, x(t)),\ \ t \in J,\\ x(t_0) = \alpha_0, \end{aligned}\right\} \tag{1.3}$$

and

$$\left.\begin{aligned} x'(t) = g(t, x(t)),\ \ t \in J,\\ x(0) = x(T), \end{aligned}\right\} \tag{1.4}$$

and a sharp classification of different types of perturbations of a differential equation appears in Dhage [2] which can be treated with the hybrid fixed point theory (see Dhage [2, 3] and Dhage and Lakshmikantham [14]). The HDE (1.1) with $\lambda = 0$ has been thoroughly discussed in the literature for different basic aspects of the solutions such as existence theorem, differential inequalities, maximal and minimal solutions, comparison principle under some mixed Lipschitz and compactness type conditions.

2010 *Mathematics Subject Classification.* 34A12, 34H34, 47H07, 47H10.
Key words and phrases. hybrid differential equation; Dhage iteration method; hybrid fixed point theorem; approximation theorem.

See for example Dhage and Jadhav [13], Dhage [5, 6] and references therein. However, the HDE (1.2) is new to the literature . In this paper we prove algorithms in terms of successive approximations for proving the existence and approximate solutions of the considered hybrid differential equations. We claim that the results of this paper are new basic and important contribution to the theory of nonlinear ordinary differential equations.

2. Auxiliary Result

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of a partially ordered normed linear space appear in Dhage [3], Heikkilä and Lakshmikantham [16] and the references therein.

Two elements x and y in E are said to be comparable if either the relation $x \preceq$ or $y \preceq x$ holds. A non-empty subset C of E is called a chain or totally ordered if all the elements of C are comparable. It is known that E is regular if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [16] and the references therein.

We need the following definitions in the sequel.

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is called isotone or monotone nondecreasing if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called monotone nonincreasing if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called monotonic or simply monotone if it is either monotone nondecreasing or monotone nonincreasing on E.*

Definition 2.2 (Dhage [4])**.** *A mapping $\mathcal{T} : E \to E$ is called partially continuous at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3 (Dhage [3, 4])**.** *A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. A nondecreasing operator $\mathcal{T}$ on a partially normed linear space E into itself is called partially bounded if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called uniformly partially bounded if all chains C in E, $\mathcal{T}(C)$ are bounded by a unique constant.*

Definition 2.4 (Dhage [3, 4])**.** *A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is compact. A nondecreasing mapping $\mathcal{T} : E \to E$ is called partially compact if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called uniformly partially compact if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called partially totally bounded if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called partially completely continuous on E.*

Definition 2.5 (Dhage [3])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be compatible if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called Janhavi if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a Janhavi metric or Janhavi Banach space.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a Janhavi Banach space.

Definition 2.6. *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. An operator $\mathcal{T} : E \to E$ is called partially nonlinear $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi\big(\|x - y\|\big) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lischitz constant k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a partial linear contraction on E with a contraction constant k.

The Dhage iteration principle or method (in short DIP or DIM) developed in Dhage [3, 4, 7] may be described as " *the sequence of successive approximations of a nonlinear equation beginning with a lower or an upper solution as its first or initial approximation converges monotonically to the solution.*" The aforsaid convergence principle forms a basic and powerful tool in the study of nonlinear differential and integral equations. See Dhage [7, 8], Dhage and Dhage [9, 10, 11, 12] and the references therein. The following applicable hybrid fixed point theorem of Dhage [4] containing the DIP is used as a key tool for the work contained in this paper.

Theorem 2.7 (Dhage [4])**.** *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such every compact chain C of E is Janhavi. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is partially continuous and partially compact, and*
(c) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.*

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,..., converges monotonically to x^*.

Remark 2.8. The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it. This simple fact is used to prove the desired characterization of the mild solution of the problem (1.1) on J.

Remark 2.9. We remark that hypothesis (a) of Theorem 2.7 implies that the operator $\mathcal{A}$ is partially continuous and consequently both the operators $\mathcal{A}$ and $\mathcal{B}$ in the theorem are partially continuous on E. The regularity of E in above Theorem 2.7 may be replaced with a stronger continuity condition of the operators $\mathcal{A}$ and $\mathcal{B}$ on E which is a result proved in Dhage [3, 4].

3. Main Results

In this section, we prove an existence and approximation result for the HDE (1.1) on a closed and bounded interval $J = [0, T]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the HDE (1.1) in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.1}$$

and

$$x \leq y \iff x(t) \leq y(t) \quad \text{for all } \ t \in J. \tag{3.2}$$

Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. The following lemma concerning the Janhaviness of subsets of $C(J, \mathbb{R})$ follows immediately form the Arzelá-Ascoli theorem for compactness.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then every partially compact subset of $C(J, \mathbb{R})$ is Janhavi.*

Proof. The proof of the lemma appears in Dhage and Dhage [9] and so we omit the details. □

We consider the following basic hypotheses in what follows.

(A_0) The mapping $x \mapsto x - f(t,x)$ is increasing in $\mathbb{R}$ for each $t \in J$.

(A_1) There exists a $\mathcal{D}$-function ψ such that
$$0 \le f(t,x) - f(t,y) \le \psi(x-y)$$
for all $t \in J$ and $x, y \in \mathbb{R}$ with $x \ge y$. Moreover, $0 < \psi(r) < r$ for $r > 0$.

(A_2) There exists aconstant $M_f > 0$ such that $|f(t,x)| \le M_f$ for all $t \in J$ and $x \in \mathbb{R}$.

(B_1) There exists a $M_g > 0$ such that $|g(t,x)| \le M_g$, for all $t \in J$ and $x \in \mathbb{R}$.

(B_2) $g(t,x)$ is nondecreasing in x for each $t \in J$.

Remark 3.2. *If the hypothesis (A_0) holds, then the function $x \mapsto x - f(t,x)$ is injective in $\mathbb{R}$ for each $t \in J$.*

Remark 3.3. The hypotheses (A_1) and (A_2) hold, in particular if f satisfies the inequality
$$0 \le f(t,x) - f(t,y) \le \frac{L(x-y)}{K + (x-y)}$$
for all $x, y \in \mathbb{R}$ with $x \ge y$, where $L > 0$ and $K > 0$ are constants satisfying $L \le K$.

3.1. **Initial value problem.** The following useful lemma follows from the theory of calculus and linear differential equations.

Lemma 3.4. *Assume that hypothesis (A_0) holds. Then for any continuous function $h : J \to \mathbb{R}$, the function $x \in C(J,\mathbb{R})$ is a solution of the HDE*
$$\left.\begin{aligned} \frac{d}{dt}\,[x(t) - f(t,x(t))] + \lambda[x(t) - f(t,x(t))] = h(t), \ \ t \in J,\\ x(0) = \alpha_0, \end{aligned}\right\} \tag{3.3}$$
if and only if x satisfies the hybrid integral equation (HIE)
$$x(t) = ce^{-\lambda t} + f(t,x(t)) + e^{-\lambda t}\int_0^t e^{\lambda s} h(s)\,ds, \ \ t \in J, \tag{3.4}$$
where $c = \alpha_0 - f(0,\alpha_0)$.

Proof. Let $h \in C(J,\mathbb{R})$. Assume first that x is a solution of the HDE (3.3) defined on J. By definition, the function $t \mapsto x(t) - f(t,x(t))$ is continuous on J, and so, differentiable there, whence $\dfrac{d}{dt}\big[x(t) - f(t,x(t))\big]$ is integrable on J. Applying integration to (3.3) from 0 to t, we obtain the HIE (3.4) on J.

Conversely, assume that x satisfies the HIE (3.3). Then by direct differentiation we obtain the first equation in (3.4). Again, substituting $t = 0$ in (3.4) yields
$$x(0) - f(0,x(0)) = \alpha_0 - f(0,\alpha_0).$$

Since the mapping $x \mapsto x - f(t,x)$ is increasing in $\mathbb{R}$ for all $t \in J$, the mapping $x \mapsto x - f(0,x)$ is injective in $\mathbb{R}$, whence $x(0) = \alpha_0$. Hence the proof of the lemma is complete. □

We need the following definition in what follows.

Definition 3.5. *A function $u \in C(J,\mathbb{R})$ is called a lower solution of the HDE (1.1) on J if the function $t \mapsto x - f(t,x)$ is differentiable and satisfies the inequalities*
$$\left.\begin{aligned} \frac{d}{dt}\big[x(t) - f(t,x(t))\big] + \lambda\big[x(t) - f(t,x(t))\big] \le g(t,x(t)),\\ x(0) \le \alpha_0, \end{aligned}\right\}$$
for all $t \in J$. Similarly, an upper solution of the HDE (1.1) on J is defined.

(B_3) The HDE (1.1) has a lower solution $u \in C(J,\mathbb{R})$.

Now we are in a position to prove the following existence and approximation theorem for the HDE (1.1) on J.

Theorem 3.6. *Assume that the hypotheses (A_0) through (A_2) and (B_1) through (B_3) hold. Then the HDE (1.1) has a solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$x_0 = u, \quad x_{n+1}(t) = ce^{-\lambda t} + f(t, x_n(t)) + e^{-\lambda t} \int_0^t e^{\lambda s} g(s, x_n(s))\, ds, \;\; t \in J, \tag{3.5}$$

converges monotonically to x^*, where $c = \alpha_0 - f(0, \alpha_0)$.

Proof. Set $E = C(J, \mathbb{R})$. Then, in view of Lemma 3.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ so that every compact chain C is Janhavi in E.

Now, using the hypotheses (A_0) and (A_2), it can be shown by an application of Lemma 3.4 that the HDE (1.1) is equivalent to the nonlinear HIE

$$x(t) = ce^{-\lambda t} + f(t, x(t)) + e^{-\lambda t} \int_0^t e^{\lambda s} g(s, x(s))\, ds \tag{3.6}$$

for $t \in J$.

Define two operators $\mathcal{A}, \mathcal{B} : E \to E$ by

$$\mathcal{A}x(t) = f(t, x(t)), \;\; t \in J, \tag{3.7}$$

and

$$\mathcal{B}x(t) = ce^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s} g(s, x(s))\, ds, \;\; t \in J. \tag{3.8}$$

Then, the HIE (3.6) is transformed into an operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \;\; t \in J. \tag{3.9}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.7. Firstly, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E. Let $x, y \in E$ be such that $x \geq y$. Then, by hypothesis (A_1),

$$\mathcal{A}x(t) = f(t, x(t)) \geq f(t, y(t)) = \mathcal{A}y(t)$$

for all $t \in J$. Similarly, by hypothesis (A_3),

$$\begin{aligned} \mathcal{B}x(t) &= ce^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s} g(s, x(s))\, ds \\ &\geq ce^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s} g(s, y(s))\, ds \\ &= \mathcal{B}y(t) \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on E into E.

From (A_2) it follows that

$$\|\mathcal{A}x\| \leq \sup_{t \in J} |\mathcal{A}x(t)| \leq \sup_{t \in J} |f(t, x)| \leq M_f$$

for all $x \in E$. As a result, $\mathcal{A}$ is bounded and consequently partially bounded on E. Next, we show that $\mathcal{A}$ is a partial nonlinear $\mathcal{D}$-contraction on E with a $\mathcal{D}$ function ψ. Let $x, y \in E$ be such that $x \geq y$. Then, by hypothesis (A_1),

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= |f(t, x(t)) - f(t, y(t))| \\ &= f(t, x(t)) - f(t, y(t)) \\ &\leq \psi(x(t) - y(t)) \\ &= \psi(|x(t) - y(t)|) \\ &\leq \psi(\|x - y\|) \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \psi(\|x - y\|)$$

for all $x, y \in E$ with $x \geq y$. This shows that $\mathcal{A}$ is a partial nonlinear $\mathcal{D}$-contraction on E with the $\mathcal{D}$-function ψ.

Next, we show that $\mathcal{B}$ is a partially compact and partially continuous operator on E into E. First we show that $\mathcal{B}$ is a partially continuous on E. Let $\{x_n\}$ be a sequence in a chain C of E converging to a point $x \in C$. Then by the dominated convergence theorem for integration, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[ce^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s} g(s, x_n(s))\, ds \right] \\
&= ce^{-\lambda t} + \lim_{n\to\infty} e^{-\lambda t} \int_0^t e^{\lambda s} g(s, x_n(s))\, ds \\
&= ce^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s} \left[\lim_{n\to\infty} g(s, x_n(s)) \right] ds \\
&= ce^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s} g(s, x(s))\, ds \\
&= \mathcal{B}x(t)
\end{aligned}$$

for all $t \in J$. Moreover, it can be shown as below that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in E. Now, following the arguments similar to that given in Granas and Dugundji [15], it is proved that $\mathcal{B}$ is a a partially continuous operator on E.

Next, we show that $\mathcal{B}$ is a partially compact operator on E. It is enough to show that $\mathcal{B}(C)$ is a uniformly bounded and equi-continuous set in E for every chain C in E. Let $x \in C$ be arbitrary. Then by yhe hypothesis (A_2),

$$\begin{aligned}
|\mathcal{B}x(t)| &\leq \left| ce^{-\lambda t} \right| + \left| e^{-\lambda t} \int_0^t e^{\lambda s} g(s, x(s))\, ds \right| \\
&\leq |\alpha_0 - f(t_0, \alpha_0)| + \int_{t_0}^T e^{\lambda s} M_g\, ds \\
&\leq |\alpha_0 - f(t_0, \alpha_0)| + \frac{(e^{\lambda T} - 1) M_g}{\lambda}
\end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathcal{B}x\| \leq |\alpha_0 - f(t_0, \alpha_0)| + \frac{(e^{\lambda T} - 1) M_g}{\lambda}$$

for all $x \in C$. This shows that $\mathcal{B}$ is uniformly bounded on C.

Again, let $t_1, t_2 \in J$ be arbitrary. Then for any $x \in C$, one has

$$\begin{aligned}
|\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| &= |c| \left|e^{-\lambda t_1} - e^{-\lambda t_2}\right| \\
&\quad + \left| e^{-\lambda t_1} \int_0^{t_1} e^{\lambda s} g(s, x(s))\, ds - e^{-\lambda t_2} \int_0^{t_2} e^{\lambda s} g(s, x(s))\, ds \right| \\
&\le |c| \left|e^{-\lambda t_1} - e^{-\lambda t_2}\right| \\
&\quad + \left|e^{-\lambda t_1} - e^{-\lambda t_2}\right| \left| \int_0^{t_1} e^{\lambda s} g(s, x(s))\, ds \right| \\
&\quad + \left|e^{-\lambda t_2}\right| \left| \int_0^{t_1} e^{\lambda s} g(s, x(s))\, ds - \int_0^{t_2} e^{\lambda s} g(s, x(s))\, ds \right| \\
&\le |c| \left|e^{-\lambda t_1} - e^{-\lambda t_2}\right| + \left[\frac{(e^{\lambda T} - 1) M_g}{\lambda}\right] \left|e^{-\lambda t_1} - e^{-\lambda t_2}\right| \\
&\quad + \left| \int_{t_2}^{t_1} e^{\lambda s} |g(s, x(s))|\, ds \right| \\
&\le \left[|c| + \frac{(e^{\lambda T} - 1) M_g}{\lambda} \right] \left|e^{-\lambda t_1} - e^{-\lambda t_2}\right| + |p(t_1) - p(t_2)|
\end{aligned}$$

where, $p(t) = \displaystyle\int_0^t e^{\lambda s} M_g\, ds$.

Since the functions $t \mapsto e^{-\lambda t}$ and $t \mapsto p(t)$ are continuous on compact J, they are uniformly continuous there. Hence, for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|t_1 - t_2| < \delta \implies |\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| < \epsilon$$

uniformly for all $t_1, t_2 \in J$ and for all $x \in S$. This shows that $\mathcal{B}(C)$ is an equi-continuous set in E. Now the set $\mathcal{B}(C)$ is uniformly bounded and equicontinuous in E, so it is compact by Arzelá-Ascoli theorem. As a result, $\mathcal{B}$ is a partially continuous and partially compact operator on E.

Thus, all the conditions of Theorem 2.7 are satisfied and hence the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence of successive approximations $\{x_n\}$ defined by $x_n = \mathcal{A}x_{n-1} + \mathcal{B}x_{n-1}$ converges monotonically to x^*. As a result, the HDE (1.1) has a solution x^* defined on J and the sequence of successive approximations $\{x_n\}$ defined by (3.5) converges monotonically to x^*. This completes the proof. □

Remark 3.7. We remark that Theorem 3.6 also remains true if we replace the hypothesis (B_3) with the following one:

(B_3') The HDE (1.1) has an upper solution $v \in C(J, \mathbb{R})$.

Remark 3.8. We note that if the HDE (1.1) has a lower solution u as well as an upper solution v such that $u \le v$, then under the given conditions of Theorem 3.6 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \le x^*$. Hence they are the minimal and maximal solutions of the HDE (1.1) in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set of elements in $C(J, \mathbb{R})$ defined by

$$[u, v] = \{x \in C(J, \mathbb{R}) \mid u \le x \le v\}.$$

This is because the order relation $\le$ defined by (3.2) is equivalent to the order relation defined by the order cone $\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x \ge \theta\}$ which is a closed set in $C(J, \mathbb{R})$.

Example 3.9. *Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the HDE*

$$\left.\begin{aligned}
\frac{d}{dt}\left[x(t) - \tan^{-1} x(t)\right] + \left[x(t) - \tan^{-1} x(t)\right] &= \tanh x(t), \quad t \in J, \\
x(0) &= 1.
\end{aligned}\right\} \tag{3.10}$$

Here, $\lambda = 1$ and the functions f and g are given by

$$f(t, x) = \tan^{-1} x \quad \text{and} \quad g(t, x) = \tanh x$$

for all $t \in J$ and $x \in \mathbb{R}$. We show that the functions f and g satisfy all the hypotheses of Theorem 3.6. First we show that f satisfies the hypotheses (A_0)-(A_2). Now,

$$\frac{\partial}{\partial x}\Big[x - f(t,x)\Big] = \frac{d}{dx}\Big[x - \tan^{-1} x\Big] = 1 - \frac{1}{1+x^2} > 0,$$

for all $x \in \mathbb{R}$ and $t \in J$, so that the function $x \mapsto x - f(t,x)$ is increasing in $\mathbb{R}$ for each $t \in J$. Therefore, hypothesis (A_0) holds. Next, let $x, y \in \mathbb{R}$ be such that $x \geq y$. Then,

$$0 \leq f(t,x) - f(t,y) \leq \tan^{-1} x - \tan^{-1} y = \frac{1}{1+\xi^2}(x-y)$$

for all $x > \xi > y$, showing that f satisfies the hypothesis (A_1) with $\mathcal{D}$-function ψ given by

$$\psi(r) = \frac{r}{1+\xi^2} < r, \ \ r > 0,$$

where $\xi \neq 0$. Again,

$$|f(t,x)| = |\tan^{-1} x| \leq \frac{\pi}{2},$$

for all $t \in J$ and $x \in \mathbb{R}$. This shows that f satisfies hypothesis (A_2) with $M_f = \dfrac{\pi}{2}$.

Furthermore,

$$|g(t,x)| = |\tanh x| \leq 1,$$

for all $t \in J$ and $x \in \mathbb{R}$, so that the hypothesis (B_1) holds with $M_g = 1$. Again, since the function $x \mapsto \tanh x$ is nondecreasing in $\mathbb{R}$ and so the hypothesis (B_2) is satisfied. Finally, the function $u(t) = -(t+3)$ is a lower solution of the HDE (3.10) defined on $J = [0,1]$.

Thus the functions f and g satisfy all the conditions of Theorem 3.6. Hence we apply and conclude that the HDE (3.10) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0 = u, \quad x_{n+1}(t) = 1 - \frac{\pi}{2} + \tan^{-1} x_n(t) + e^{-t}\int_0^t e^s \tanh x_n(s)\, ds,$$

for each $t \in J$, converges monotonically to x^*. A similar conclusion also remains true if we replace the lower solution u with the upper solution $v(t) = t + 3$, $t \in J$.

3.2. **Periodic boundary value problem.** The following useful lemma is obvious and may be found in Dhage [1] and Nieto [17].

Lemma 3.10. *For any function $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= \sigma(t), \ \ t \in J,\\ x(0) &= x(T), \end{aligned}\right\} \tag{3.11}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_\lambda(t,s)\,\sigma(s)\,ds \tag{3.12}$$

where,

$$G_\lambda(t,s) = \begin{cases} \dfrac{e^{\lambda s - \lambda t + \lambda T}}{e^{\lambda T} - 1}, & \text{if} \quad 0 \leq s \leq t \leq T,\\[2ex] \dfrac{e^{\lambda s - \lambda t}}{e^{\lambda T} - 1}, & \text{if} \quad 0 \leq t < s \leq T. \end{cases} \tag{3.13}$$

Notice that the Green's function G_λ is continuous and nonnegative on $J \times J$ and therefore, the number

$$K_\lambda := \max\{\, |G_\lambda(t,s)| \,:\, t,s \in [0,T]\,\}$$

exists for all $\lambda \in \mathbb{R}^+$. For the sake of convenience, we write $G_\lambda(t,s) = G(t,s)$ and $K_\lambda = K$.

Lemma 3.11. *If there exists a differentiable function $u \in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t)+\lambda u(t) \le \sigma(t), \ \ t\in J,\\ u(0)\le u(T), \end{aligned}\right\} \tag{3.14}$$

then

$$u(t) \le \int_0^T G_\lambda(t,s)\,\sigma(s)\,ds, \tag{3.15}$$

for all $t\in J$, where $G(t,s)$ is a Green's function given by (3.5).

Proof. Suppose that the function $u \in C(J,\mathbb{R})$ satisfies the inequalities given in (3.14). Multiplying the first inequality in (3.14) by $e^{\lambda t}$,

$$\Big(e^{\lambda t}u(t)\Big)' \le e^{\lambda t}\sigma(t).$$

A direct integration of above inequality from 0 to t yields

$$e^{\lambda t}u(t) \le u(0) + \int_0^t e^{\lambda s}\sigma(s)\,ds, \tag{3.16}$$

for all $t\in J$. Therefore, in particular,

$$e^{\lambda T}u(T) \le u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds. \tag{3.17}$$

Now $u(0)\le u(T)$, so one has

$$u(0)e^{\lambda T} \le u(T)e^{\lambda T}. \tag{3.18}$$

From (3.17) and 3.18) it follows that

$$e^{\lambda T}u(0) \le u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds \tag{3.19}$$

which further yields

$$u(0) \le \int_0^T \frac{e^{\lambda s}}{(e^{\lambda T}-1)}\sigma(s)\,ds. \tag{3.20}$$

Substituting (3.20) in (3.16) we obtain

$$u(t) \le \int_0^T G(t,s)\sigma(s)\,ds,$$

for all $t\in J$. This completes the proof. □

We need the following definition in what follows.

Definition 3.12. *A function $u \in C(J,\mathbb{R})$ is called a lower solution of the HDE (1.1) if the function $t\mapsto x-f(t,x)$ is differentiable and satisfies the inequalities*

$$\left.\begin{aligned} \frac{d}{dt}\big[u(t)-f(t,u(t))\big]+\lambda\big[u(t)-f(t,u(t))\big] \le g(t,u(t)),\\ u(0)\le u(T), \end{aligned}\right\}$$

for all $t\in J$. Similarly, an upper solution $v\in C(J,\mathbb{R})$ of the HDE (1.2) is defined.

We need the following hypotheses in what follows.

- (B_4) The function $f(t,x)$ is periodic in t with period T for all $x\in\mathbb{R}$, i.e., $f(0,x)=f(T,x)$ for all $x\in\mathbb{R}$.
- (B_5) The HDE (1.2) has a lower solution $u\in C(J,\mathbb{R})$.

Lemma 3.13. *Assume that hypothesis* (A_0) *holds. Then for any* $\lambda \in \mathbb{R}_+$, *the function* $x \in C(J, \mathbb{R})$ *is a solution of the HDE*

$$\left.\begin{aligned} \frac{d}{dt}\,[x(t) - f(t, x(t))] + \lambda[x(t) - f(t, x(t))] = g(t, x(t)),\ \ t \in J,\\ x(0) = x(T), \end{aligned}\right\} \tag{3.21}$$

if and only if x *satisfies the hybrid integral equation (HIE)*

$$x(t) = f(t, x(t)) + \int_0^T G(t, s) g(s, x(s))\, ds,\ \ t \in J. \tag{3.22}$$

Proof. Assume first that x is a solution of the HDE (3.22) defined on J. By definition, the function $t \mapsto x(t) - f(t, x(t))$ is continuous on J, and so, differentiable there, whence $\frac{d}{dt}\big[x(t) - f(t, x(t))\big]$ is integrable on J. Again by hypothesis (B_2), $x(0) - f(0, x(0)) = x(T) - f(T, x(T))$. So by a direct application of Lemma 3.10, we obtain the HIE (3.22) on J.

Conversely, assume that x satisfies the HIE (3.21) on J. Then by a direct differentiation of (3.22), we obtain the first equation in (3.21). Again, substituting $t = 0$ in (3.2) yields

$$x(0) - f(0, x(0)) = x(T) - f(0, x(T)).$$

Since the mapping $x \mapsto x - f(t, x)$ is increasing in $\mathbb{R}$ for all $t \in J$, the mapping $x \mapsto x - f(0, x)$ is injective in $\mathbb{R}$, and so $x(0) = x(T)$. Hence the proof of the lemma is complete. □

Now we are in a position to prove the following existence and approximation theorem for the HDE (1.2) on J.

Theorem 3.14. *Assume that the hypotheses* (A_0) - (A_2), (B_1) - (B_2) *and* (B_4) - (B_5) *hold. Then the HDE (1.2) has a solution* x^* *defined on* J *and the sequence* $\{x_n\}_{n=0}^{\infty}$ *of successive approximations defined by*

$$x_0 = u, \quad x_{n+1}(t) = f(t, x_n(t)) + \int_0^T G(t, s) g(s, x_n(s))\, ds,\ \ t \in J, \tag{3.23}$$

converges monotonically to x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then, in view of Lemma 3.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ so that every compact chain C is Janhavi set in E.

Now, using the hypotheses (A_0), it can be shown by an application of Lemma 3.4 that the HDE (1.1) is equivalent to the nonlinear HIE

$$x(t) = f(t, x(t)) + \int_0^T G(t, s) g(s, x(s))\, ds \tag{3.24}$$

for $t \in J$.

Define two operators $\mathcal{A}, \mathcal{B} : E \to E$ by

$$\mathcal{A}x(t) = f(t, x(t)),\ \ t \in J, \tag{3.25}$$

and

$$\mathcal{B}x(t) = \int_0^T G(t, s) g(s, x(s))\, ds,\ \ t \in J. \tag{3.26}$$

Then, the HIE (3.24) is transformed into an operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t),\ \ t \in J. \tag{3.27}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 3.6. Now, it can be shown as in the proof of Theorem 3.6 that $\mathcal{A}$ is a nondecreasing partially bounded and partially nonlinear $\mathcal{D}$-contraction on E with a $\mathcal{D}$-function ψ. Next, we show that the operator $\mathcal{B}$ is a nondecreasing, partially continuous and partially compact operator on E. Let $x, y \in E$ be such that $x \geq y$. Then, by hypothesis (B$_3$),

$$\mathcal{B}x(t) = \int_0^T G(t,s)g(s,x(s))\,ds \geq \int_0^T G(t,s)g(s,y(s))\,ds = \mathcal{B}y(t)$$

for all $t \in J$. This shows that $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on E into E.

Next, let $\{x_n\}$ be a sequence in a chain C of E converging to a point $x \in C$. Then by dominated convergence theorem for integration, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[\int_0^T G(t,s)g(s,x_n(s))\,ds\right] \\
&= \int_0^T G(t,s)\left[\lim_{n\to\infty} g(s,x_n(s))\right]\,ds \\
&= \int_0^T G(t,s)g(s,x(s))\,ds \\
&= \mathcal{B}x(t)
\end{aligned}$$

for all $t \in J$. Moreover, it can be shown as below that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in E. Now, following the arguments similar to that given in Granas and Dugundji [15], it is proved that $\mathcal{B}$ is a a partially continuous operator on E.

Next, we show that $\mathcal{B}$ is a partially compact operator on E. It is enough to show that $\mathcal{B}(C)$ is a uniformly bounded and equi-continuous set in E for every chain C in E. Let $x \in C$ be arbitrary. Then by hypothesis (A$_2$),

$$|\mathcal{B}x(t)| \leq \left|\int_0^T G(t,s)g(s,x(s))\,ds\right| \leq \int_0^T G(t,s)|g(s,x(s))|\,ds \leq KTM_g$$

for all $t \in J$. Taking the supremum over t, $\|\mathcal{B}x\| \leq KTM_g$ for all $x \in C$. This shows that $\mathcal{B}$ is uniformly bounded on C.

Again, let $t_1, t_2 \in J$ be arbitrary. Then for any $x \in C$, one has

$$\begin{aligned}
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &= \left|\int_0^T G(t_1,s)\tilde{f}(s,x_n(s))\,ds - \int_0^T G(t_2,s)g(s,x_n(s))\,ds\right| \\
&\leq \int_0^T |G(t_1,s) - G(t_2,s)||g(s,x_n(s))|\,ds \\
&\leq M_g \int_0^T |G(t_1,s) - G(t_2,s)|\,ds \\
&\to 0 \quad as \quad t_2 - t_1 \to 0.
\end{aligned}$$

Since the functions $t \mapsto G(t,s)$ is continuous on compact J, it is uniformly continuous there. Hence, for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|t_1 - t_2| < \delta \quad \Longrightarrow \quad |\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| < \epsilon$$

uniformly for all $t_1, t_2 \in J$ and for all $x \in S$. This shows that $\mathcal{B}(C)$ is an equi-continuous set in E. Now the set $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E, so it is compact by Arzelá-Ascoli theorem. As a result, $\mathcal{B}$ is a partially continuous and partially compact operator on E.

Finally, we show that u is a lower solution of the operator equation $\mathcal{A}x + \mathcal{B}x = x$. Since u is a lower solution of the HDE (1.2) on J, we have

$$\left.\begin{aligned}
\frac{d}{dt}\big[u(t) - f(t,u(t))\big] + \lambda\big[u(t) - f(t,u(t))\big] \leq g(t,u(t)), \\
u(0) \leq u(T),
\end{aligned}\right\}$$

for all $t \in J$. Again, since the hypotheses (A_0) and (B_4) hold, one has

$$\left.\begin{aligned} \frac{d}{dt}\big[u(t) - f(t,u(t))\big] + \lambda\big[u(t) - f(t,u(t))\big] \le g(t,u(t)), \\ \big[u(0) - f(0,u(0))\big] \le \big[u(T) - f(T,u(T))\big], \end{aligned}\right\}$$

for all $t \in J$. Now an application Lemma 3.11 yields that

$$u(t) \le f(t,u(t)) + \int_0^T G(t,s) g(s,u(s))\,ds \tag{3.28}$$

for $t \in J$. This further in view of definitions of the operators $\mathcal{A}$ and $\mathcal{B}$ implies that $u \le \mathcal{A}u + \mathcal{B}u$ and that u is a lower solution of the operator equation $\mathcal{A}x + \mathcal{B}x = x$.

Thus, all the conditions of Theorem 3.6 are satisfied by the operators $\mathcal{A}$ and $\mathcal{B}$ and hence the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence of successive approximations $\{x_n\}$ defined by $x_n = \mathcal{A}x_{n-1} + \mathcal{B}x_{n-1}$ converges monotonically to x^*. Consequently, the HDE (1.1) has a solution x^* defined on J and the sequence of successive approximations $\{x_n\}$ defined by (3.3) converges monotonically to x^*. This completes the proof. □

Remark 3.15. We remark that Theorem 3.14 also remains true if we replace the hypothesis (B_5) with the following one:

(B_5') The HDE (1.2) has an upper solution $v \in C(J,\mathbb{R})$.

Remark 3.16. We note that if the HDE (1.2) has a lower solution u as well as an upper solution v such that $u \le v$, then under the given conditions of Theorem 3.14 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \le x^*$. Hence they are the minimal and maximal solutions of the HDE (1.2) in the vector segment $[u,v]$ of the Banach space $E = C(J,\mathbb{R})$, where the vector segment $[u,v]$ is a set of elements in $C(J,\mathbb{R})$ defined by

$$[u,v] = \{x \in C(J,\mathbb{R}) \mid u \le x \le v\}.$$

This is because the order relation $\le$ defined by (3.2) is equivalent to the order relation defined by the order cone $\mathcal{K} = \{x \in C(J,\mathbb{R}) \mid x \ge \theta\}$ which is a closed set in $C(J,\mathbb{R})$.

Example 3.17. *Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the HDE*

$$\left.\begin{aligned} \frac{d}{dt}\big[x(t) - \tan^{-1} x(t)\big] + \big[x(t) - \tan^{-1} x(t)\big] = \tanh x(t), \;\; t \in J, \\ x(0) = x(1). \end{aligned}\right\} \tag{3.29}$$

Here, $\lambda = 1$ and the functions f and g are given by

$$f(t,x) = \tan^{-1} x \quad \text{and} \quad g(t,x) = \tanh x$$

for all $t \in J$ and $x \in \mathbb{R}$. Now, it can be shown that the functions f and g satisfy all the hypotheses of Theorem 3.14 with $u(t) = -4e^t,\ t \in [0,1]$. Hence we conclude that the HDE (3.29) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0 = u, \quad x_{n+1}(t) = \tan^{-1} x_n(t) + \int_0^1 G(t,s) \tanh x_n(s)\,ds,$$

for each $t \in J$, converges monotonically to x^*, where $G(t,s)$ is a Green's function associated with the homogeneous PBVP

$$\left.\begin{aligned} x'(t) + x(t) = 0, \;\; t \in J, \\ x(0) = x(1), \end{aligned}\right\} \tag{3.30}$$

given by

$$G(t,s) = \begin{cases} \dfrac{e^{s-t+1}}{e-1}, & \text{if} \quad 0 \le s \le t \le 1, \\[2ex] \dfrac{e^{s-t}}{e-1}, & \text{if} \quad 0 \le t < s \le 1. \end{cases}$$

Again, a similar conclusion holds if we replace the lower solution u with the upper solution $v(t) = 4e^t$, $t \in [0, 1]$ in view of Remark 3.14.

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