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ON p-VALENTLY MEROMORPHIC-STRONGLY STARLIKE AND CONVEX FUNCTIONS

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ABSTRACT. In this paper, we obtain sufficient conditions for analytic function $f(z)$ in the punctured unit disk to be p-valently meromorphic-strongly starlike and p-valently meromorphic-strongly convex of order β and type α. Some interesting corollaries of the results presented here are also discussed.

1. INTRODUCTION

Let $\mathcal{H}$ be the class of functions that are analytic in the unit disk $\mathbb{U}$ and let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots ,$$

which are analytic in $\mathbb{U}$.

Let $\Sigma(p)$ denote the class of meromorphically p-valent functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \qquad p \in \mathbb{N} := \{1, 2, 3, \ldots\},$$

which are analytic in the punctured unit disk $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. Further, we write that $\Sigma(1) = \Sigma$. For a function $f \in \Sigma(p)$, we say that it is p-valently meromorphic-strongly starlike of order $0 < \beta \le 1$ and type α $(0 \le \alpha < p)$ if

$$\left|\arg\left(-\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi\beta}{2} \qquad z \in \mathbb{U}. \tag{1}$$

The corresponding class is denoted by $\mathcal{ST}_\Sigma(\alpha, \beta)$. We note that $\mathcal{ST}_\Sigma(\alpha, 1)$, is the class of p-valently meromorphic starlike functions of order α (see [6]) and $\mathcal{ST}_\Sigma(0, \beta)$ is the class of p-valently meromorphic-strongly starlike functions of order β. Furthermore, a function $f \in \Sigma(p)$ is said to be in the class $\mathcal{SK}_\Sigma(\alpha, \beta)$ of p-valently meromorphic-strongly convex of order β and type α if and only if

$$\left|\arg\left(-1 - \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi\beta}{2} \qquad z \in \mathbb{U}, \tag{2}$$

for some real $0 < \beta \le 1$ and $0 \le \alpha < p$. In particular, $\mathcal{SK}_\Sigma(\alpha, 1)$, is the class of p-valently meromorphic convex functions of order α (cf. [6]) and $\mathcal{SK}_\Sigma(0, \beta)$ is the class of p-valently meromorphic-strongly convex functions of order β.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{ST}(\beta)$ of strongly starlike function of order β, $0 \le \beta < 1$, if it satisfies the inequality

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi\beta}{2} \qquad z \in \mathbb{U}.$$

A function $f(z)$ belonging to the class $\mathcal{SK}(\beta)$ is said to be strongly convex of order β in $\mathbb{U}$ if and only if $zf'(z) \in \mathcal{ST}(\beta)$ (see [2, 3]).

For proving our results we need the following Lemma.

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Lemma 1. (see [1], [5]). *Let* $b(z) \in \mathcal{H}$ *be continuous on* $\overline{\mathbb{U}}$, $b(0) = 0$, $\sup_{z\in\mathbb{U}} |b(z)| = 1$ *and* $c = \sup_{z\in\mathbb{U}} \int_0^1 |b(tz)|\mathrm{d}t$. *For* $0 < \beta \le 1$ *let*

$$\lambda(\beta) = \frac{\sin(\pi\beta/2)}{\sqrt{1 + 2c\cos(\pi\beta/2) + c^2}}.$$

If $f \in \mathcal{A}$ *and*

$$|f'(z) - 1| \le \lambda(\beta)|b(z)| \qquad z \in \mathbb{U},$$

then f *is strongly starlike of order* β. *Additionally, if*

$$b(t) = \max_{0\le\varphi\le 2\pi} |b(te^{i\varphi})| \qquad 0 \le t \le 1,$$

then the constant $\lambda(\beta)$ *cannot be replaced by any larger number without violating the conclusion.*

The Lemma 1, without the sharpness part, was previously obtained by Ponnusamy and Singh in [4].

In this work, we obtain some sufficient conditions for p-valently meromorphic functions.

2. Main Results

Our first result is contained in the following.

Theorem 2. *Assume that* $f(z) \ne 0$ *for* $z \in \mathbb{U}^*$. *If* $f \in \Sigma(p)$ *satisfies*

$$\left| \left(\frac{f(z)}{z^{-\alpha}}\right)^{\frac{1}{\alpha-p}} \left(\frac{f'(z)}{f(z)} + \frac{\alpha}{z}\right) + p - \alpha \right| < (p-\alpha)\lambda(\beta)|b(z)| \qquad z \in \mathbb{U}^*, \tag{3}$$

then f *is p-valently meromorphic-strongly starlike of order* β *and type* α.

Proof. Assume that $f \in \Sigma(p)$. Let us define the function $g(z)$ by

$$g(z) = \left(\frac{f(z)}{z^{-\alpha}}\right)^{\frac{1}{\alpha-p}} = z + \cdots \qquad z \in \mathbb{U}^*. \tag{4}$$

Then $g(z) \in \mathcal{A}$ and

$$|g'(z) - 1| = \frac{1}{p-\alpha} \left| \left(\frac{f(z)}{z^{-\alpha}}\right)^{\frac{1}{\alpha-p}} \left(\frac{f'(z)}{f(z)} + \frac{\alpha}{z}\right) + p - \alpha \right|.$$

Now, by means of the condition of the theorem and applying Lemma 1 we find that $g(z)$ is strongly starlike function of order β. Note that from (4) we have

$$\frac{zg'(z)}{g(z)} = \frac{1}{p-\alpha}\left(-\frac{zf'(z)}{f(z)} - \alpha\right).$$

Since $g(z)$ is strongly starlike of order β, thus

$$\left|\arg\left\{\frac{1}{p-\alpha}\left(-\frac{zf'(z)}{f(z)} - \alpha\right)\right\}\right| < \frac{\pi\beta}{2}.$$

This shows that the proof is completed. □

Putting $\alpha = 0$ in Theorem 2, we have:

Corollary 3. *Assume that* $f(z) \ne 0$ *for* $z \in \mathbb{U}^*$. *If* $f \in \Sigma(p)$ *satisfies*

$$\left| \frac{1}{\sqrt[p]{f(z)}} \left(\frac{f'(z)}{f(z)}\right) + p \right| < p\lambda(\beta)|b(z)| \qquad z \in \mathbb{U}^*,$$

then f *is p-valently meromorphic-strongly starlike of order* β.

Setting $b(z) = z$ and $p = \beta = 1$ in Theorem 2, we obtain the following result:

Corollary 4. *Assume that $f(z) \neq 0$ for $z \in \mathbb{U}^*$. If $f \in \Sigma$ satisfies*

$$\left| \left(\frac{f(z)}{z^{-\alpha}} \right)^{\frac{1}{\alpha-1}} \left(\frac{f'(z)}{f(z)} + \frac{\alpha}{z} \right) + 1 - \alpha \right| < \frac{2}{\sqrt{5}}(1-\alpha) \qquad z \in \mathbb{U}^*,$$

then f is meromorphic starlike function of order α.

If we take $\alpha = 0$ in Corollary 4, we obtain the following result:

Corollary 5. *Assume that $f(z) \neq 0$ for $z \in \mathbb{U}^*$. If $f \in \Sigma$ satisfies*

$$\left| \left(\frac{1}{f(z)} \right)^2 f'(z) + 1 \right| < \frac{2}{\sqrt{5}} \approx 0.894427... \qquad z \in \mathbb{U}^*,$$

then f is meromorphic starlike functions.

Next we derive the following.

Theorem 6. *Assume that $f'(z) \neq 0$ for $z \in \mathbb{U}^*$. If $f \in \Sigma(p)$ satisfies*

$$\left| \left(\frac{f'(z)}{-pz^{-p-1}} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{zf''(z)}{f'(z)} + p \right) \right| < (p-\alpha)\lambda(\beta)|b(z)| \qquad z \in \mathbb{U}^*, \tag{5}$$

then f is p-valently meromorphic-strongly convex of order β and type α.

Proof. Let $f \in \Sigma(p)$ and define the function $p(z)$ by

$$p(z) = \int_0^z \left(\frac{f'(t)}{-pt^{-p-1}} \right)^{\frac{1}{\alpha-p}} \mathrm{d}t = z + \cdots \qquad z \in \mathbb{U}^*. \tag{6}$$

Further, let

$$h(z) = zp'(z) = z \left(\frac{f'(z)}{-pz^{-p-1}} \right)^{\frac{1}{\alpha-p}} = z + \cdots \qquad z \in \mathbb{U}^*. \tag{7}$$

We see that $p(z)$ and $h(z)$ belongs to $\mathcal{A}$. Differentiating from (7), we have

$$h'(z) = \frac{1}{\alpha - p} \left(\frac{f'(z)}{-pz^{-p-1}} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{zf''(z)}{f'(z)} + \alpha \right).$$

Further we have

$$|h'(z) - 1| = \frac{1}{p-\alpha} \left| \left(\frac{f'(z)}{-pz^{-p-1}} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{zf''(z)}{f'(z)} + p \right) - (\alpha - p) \right| < \lambda(\beta)|b(z)|.$$

Therefore, applying of the Lemma 1 gives us that

$$h(z) = zp'(z) \in \mathcal{ST}(\beta) \Rightarrow p(z) \in \mathcal{SK}(\beta).$$

Since

$$\frac{zp''(z)}{p'(z)} = \frac{1}{\alpha - p} \left(\frac{zf''(z)}{f'(z)} + 1 + p \right),$$

therefore

$$\left| \arg \left(1 + \frac{zp''(z)}{p'(z)} \right) \right| = \left| \arg \frac{1}{p-\alpha} \left(-1 - \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi\beta}{2},$$

which imply that $f(z)$ is p-valently meromorphic-strongly convex of order β and type α. This completes the proof. □

Putting $\alpha = 0$ in Theorem 6, we have:

Corollary 7. *Assume that $f'(z) \neq 0$ for $z \in \mathbb{U}^*$. If $f \in \Sigma(p)$ satisfies*

$$\left| \sqrt[p]{\frac{-pz^{-p-1}}{f'(z)}} \left(1 + \frac{zf''(z)}{f'(z)} + p \right) \right| < p\lambda(\beta)|b(z)| \qquad z \in \mathbb{U}^*,$$

then f is p-valently meromorphic-strongly convex of order β.

Setting $b(z) = z$ and $p = \beta = 1$ in Theorem 6, we obtain the following result:

Corollary 8. *Assume that $f'(z) \neq 0$ for $z \in \mathbb{U}^*$. If $f \in \Sigma$ satisfies*

$$\left| \left(-z^2 f'(z)\right)^{\frac{1}{\alpha - 1}} \left(2 + \frac{zf''(z)}{f'(z)}\right) \right| < \frac{2}{\sqrt{5}}(1-\alpha) \qquad z \in \mathbb{U}^*,$$

then f is meromorphic convex function of order α.

If we take $\alpha = 0$ in Corollary 8, we obtain the following result:

Corollary 9. *Assume that $f'(z) \neq 0$ for $z \in \mathbb{U}^*$. If $f \in \Sigma$ satisfies*

$$\left| \frac{-1}{z^2 f'(z)} \left(2 + \frac{zf''(z)}{f'(z)}\right) \right| < \frac{2}{\sqrt{5}} \approx 0.894427\ldots \qquad z \in \mathbb{U}^*,$$

then f is convex meromorphic function.

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