International Journal of Analysis and Applications
ISSN 2291-8639
Volume 12, Number 1 (2016), 66-70
http://www.etamaths.com

ON GENERALIZED ABSOLUTE MATRIX SUMMABILITY METHODS

HİKMET SEYHAN ÖZARSLAN*

Abstract. In this paper, we prove a general theorem dealing with absolute matrix summability methods of infinite series. This theorem also includes some new and known results.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1)\,. \tag{1}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (σ_n) of the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [5]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k, k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty. \tag{3}$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{4}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k, k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{5}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |\bar{\Delta} A_n(s)|^k < \infty. \tag{6}$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A, p_n|_k$ summability reduces to $|A, p_n|_k$ summability. If we set $\varphi_n = n$ for all n, $\varphi - |A, p_n|_k$ summability is the same as $|A|_k$ summability (see [7]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$

2010 *Mathematics Subject Classification.* 26D15, 40D15, 40F05, 40G99.

Key words and phrases. Riesz mean; summability factors; absolute matrix summability; infinite series, Hölder inequality; Minkowski inequality.

and $a_{nv}=\frac{p_v}{P_n}$, then we get $|\bar{N},p_n|_k$ summability. If we take $\varphi_n=n$ and $a_{nv}=\frac{p_v}{P_n}$, then we get $|R,p_n|_k$ summability (see [2]). Furthermore, if we take $\varphi_n=n$, $a_{nv}=\frac{p_v}{P_n}$ and $p_n=1$ for all values of n, then $\varphi-|A,p_n|_k$ summability reduces to $|C,1|_k$ summability (see [4]).

Before stating the main theorem we must first introduce some further notations.

Given a normal matrix $A=(a_{nv})$, we associate two lower semimatrices $\bar{A}=(\bar{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

(7) $$\bar{a}_{nv}=\sum_{i=v}^{n}a_{ni}, \quad n,v=0,1,...$$

and

(8) $$\hat{a}_{00}=\bar{a}_{00}=a_{00}, \quad \hat{a}_{nv}=\bar{a}_{nv}-\bar{a}_{n-1,v}, \quad n=1,2,...$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

(9) $$A_n(s) = \sum_{v=0}^{n}a_{nv}s_v=\sum_{v=0}^{n}\bar{a}_{nv}a_v$$

and

(10) $$\bar{\Delta}A_n(s) = \sum_{v=0}^{n}\hat{a}_{nv}a_v.$$

2. Known Result

Bor [3] has proved the following theorem for $\left|\bar{N},p_n\right|_k$ summability method.

Theorem 1. Let (p_n) be a sequence of positive numbers such that

(11) $$P_n=O(np_n) \quad as \quad n\to\infty.$$

If (X_n) is a positive monotonic non-decreasing sequence such that

(12) $$|\lambda_m|X_m=O(1) \quad as \quad m\to\infty,$$

(13) $$\sum_{n=1}^{m}nX_n|\Delta^2\lambda_n|=O(1) \quad as \quad m\to\infty$$

and

(14) $$\sum_{n=1}^{m}\frac{p_n}{P_n}|t_n|^k=O(X_m) \quad as \quad m\to\infty,$$

where

$$t_n=\frac{1}{n+1}\sum_{v=1}^{n}va_v,$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N},p_n|_k$, $k\geq 1$.

3. Main Result

The aim of this paper is to generalize Theorem 1 to $\varphi-|A,p_n|_k$ summability. Now we shall prove the following theorem.

Theorem 2. Let $A=(a_{nv})$ be a positive normal matrix such that

(15) $$\bar{a}_{n0}=1, \quad n=0,1,...,$$

(16) $$a_{n-1,v}\geq a_{nv}, \quad for \quad n\geq v+1,$$

(17) $$a_{nn}=O\left(\frac{p_n}{P_n}\right),$$

(18) $$|\hat{a}_{n,v+1}|=O\left(v\left|\Delta_v\hat{a}_{nv}\right|\right).$$

Let (X_n) be a positive monotonic non-decreasing sequence and $\left(\frac{\varphi_n p_n}{P_n}\right)$ be a non-increasing sequence. If conditions (12)-(13) of Theorem 1 and

$$\sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{19}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$.
It should be noted that if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 2, then we get Theorem 1. In this case, condition (19) reduces to condition (14), condition (18) reduces to condition (11). Also, the condition "$\left(\frac{\varphi_n p_n}{P_n}\right)$ is a non-increasing sequence" and the conditions (15)-(17) are automatically satisfied. We require the following lemma for the proof of Theorem 2.
Lemma 1 ([3]**).** Under the conditions of Theorem 2, we have that

$$nX_n |\Delta \lambda_n| = O(1) \quad as \quad n \to \infty, \tag{20}$$

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty. \tag{21}$$

4. Proof of Theorem 2

Let (I_n) denotes A-transform of the series $\sum a_n \lambda_n$. Then, by (9) and (10), we have

$$\bar{\Delta} I_n = \sum_{v=0}^{n} \hat{a}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \frac{\hat{a}_{nv} \lambda_v}{v} v a_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned} \bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} \lambda_n}{n} \sum_{r=1}^{n} r a_r \\ &= \frac{n+1}{n} a_{nn} \lambda_n t_n + \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v \left(\hat{a}_{nv}\right) \lambda_v t_v \\ &+ \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta \lambda_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \frac{t_v}{v} \\ &= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}. \end{aligned}$$

To complete the proof of Theorem 2, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \mid I_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{22}$$

First, by using Abel's transformation, we have that

$$\begin{aligned} \sum_{n=1}^{m} \varphi_n^{k-1} |I_{n,1}|^k &= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} a_{nn}^k |\lambda_n|^k |t_n|^k \\ &= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |\lambda_n|^{k-1} |\lambda_n| |t_n|^k \\ &= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |\lambda_n| |t_n|^k \\ &= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \varphi_v^{k-1} \left(\frac{p_v}{P_v}\right)^k |t_v|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |t_n|^k \\ &= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m \\ &= O(1) \quad as \quad m \to \infty, \end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 1. Now, applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, as in $I_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|\,|t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|^k\,|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|^k\,|t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k a_{vv}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k|\lambda_v|\,|t_v|^k\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.
Now, using Hölder's inequality we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k\,|t_v|^k|\Delta_v(\hat{a}_{nv})|\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k\,|t_v|^k|\Delta_v(\hat{a}_{nv})|\right)\\
&= O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^k\,|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^{k-1}(v|\Delta\lambda_v|)\,|t_v|^k\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k v|\Delta\lambda_v||t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\varphi_r^{k-1}\left(\frac{p_r}{P_r}\right)^k|t_r|^k+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v|+O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.
Finally by using (18), as in $I_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} |I_{n,4}|^k &\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}| \frac{|t_v|}{v} \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}| |t_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}|^k |t_v|^k \right) \times \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_{v+1}|^k |t_v|^k \right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| |t_v|^k \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} \sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| |t_v|^k a_{vv} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^k |\lambda_{v+1}| \, |t_v|^k \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by virtue of hypotheses of Theorem 2 and Lemma 1.
This completes the proof of Theorem 2.

5. Conclusions

It should be noted that, if we take $\varphi_n = \frac{P_n}{p_n}$, then we get a theorem dealing with $|A, p_n|_k$ summability. Also, if we take $a_{nv} = \frac{p_v}{P_n}$, then we have a result dealing with $\varphi - |\bar{N}, p_n|_k$ summability. Furthermore, if we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get another result dealing with $\varphi - |C, 1|_k$ summability. When we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C, 1|_k$ summability. Finally, if we take $k = 1$ and $a_{nv} = \frac{p_v}{P_n}$, then we get a result for $\left|\bar{N}, p_n\right|$ summability and in this case the condition "$\left(\frac{\varphi_n p_n}{P_n}\right)$ is a non-increasing sequence" is not needed.

References

[1] H. Bor, On two summability methods, Math. Proc. Cambridge Philos. Soc. 97 (1985), 147-149.
[2] H. Bor, On the relative strength of two absolute summability methods, Proc. Amer. Math. Soc. 113 (1991), 1009-1012.
[3] H. Bor, On absolute summability factors, Proc. Amer. Math. Soc. 118 (1993), 71-75.
[4] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. London Math. Soc. 7 (1957), 113-141.
[5] G. H. Hardy, Divergent Series, Oxford University Press, Oxford, 1949.
[6] W. T. Sulaiman, Inclusion theorems for absolute matrix summability methods of an infinite series. IV, Indian J. Pure Appl. Math. 34 (11) (2003), 1547-1557.
[7] N. Tanovič-Miller, On strong summability, Glas. Mat. Ser. III 14 (34) (1979), 87-97.

Department of Mathematics, Erciyes University, 38039 Kayseri, Turkey

*Corresponding author: seyhan@erciyes.edu.tr