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Composition Operators From Harmonic Lipschitz Space Into Weighted Harmonic Zygmund Space

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Abstract. The paper investigate a necessary and sufficient condition for the composition operator from harmonic Lipschitz spaces $Lip_H^\alpha, (0 < \alpha < 1)$ into weighted harmonic Zygmund spaces $\mathcal{Z}_H^\beta, (0 < \beta < \infty)$ to be bounded and compact on the open unit disk. As an application, it estimates the essential norms of such an operator from Lip_H^α into $\mathcal{Z}_H^\beta$ spaces.

1. Introduction

The operator theory has been characterized for spaces of analytic functions with different settings, and a significant number of related papers have appeared in the literature (see, for example, [7], [8], [10], [13], [17], and [22]). However, a similar investigation of the harmonic setting remains limited, see [2] and [14].

In [1], we have examined numerous characterizations of the weighted Bloch spaces and closed separable subspaces of harmonic mappings. We then presented the relationships between the weighted harmonic Bloch space and its Carleson measure. In [3], Aljuaid and Colonna studied the weighted Bloch space as the Banach space for harmonic mappings on an open unit disk. They then showed that the mappings in weighted Bloch space can be characterized in terms of a Lipschitz condition relative to the metric and can also be thought of as the harmonic growth space. Besides, in [5] they studied the harmonic Zygmund spaces and their closed separable subspace called the little harmonic Zygmund space. In [12], Colonna introduced and studied Bloch harmonic mappings on $\mathbb{D}$ as Lipschitz maps from

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the hyperbolic disk into $\mathbb{C}$. In [19], Lusky investigated weighted spaces of harmonic functions on $\mathbb{D}$ and, in [20], isomorphism classes of weighted spaces of holomorphic and harmonic functions with a radial weight on $\mathbb{C}$ and on $\mathbb{D}$. In [21], Yoneda studied harmonic Bloch spaces and harmonic Besov spaces. Characterizations of the isometries between weighted spaces of harmonic functions were provided by Boyd and Rueda in [9]. In [16], Jordá and Zarco studied Banach spaces of harmonic functions and composition operators between weighted Banach spaces of pluriharmonic functions. Isomorphisms on weighted Banach spaces of harmonic and holomorphic functions were treated in [15].

Lately, studies on operator theory acting on spaces of harmonic mappings on the unit disk have been conducted. In [4], the composition operators were studied on the Banach spaces of harmonic mappings on $\mathbb{D}$, including the weighted Bloch spaces, the growth spaces, the Zygmund space, the analytic Besov spaces, and the space BMOA. Chao et al. in [11] studied composition operators in the space of bounded harmonic functions $\mathbb{D}$, and then provided criteria for determining the essential norm of the difference between two composition operators. In [18], Laitila and Tylli characterized the weak compactness of the composition operators on vector-valued harmonic Hardy spaces and on the spaces of vector-valued Cauchy transforms for reflexive Banach spaces.

A *harmonic mapping* with domain $\mathbb{D}$ is a complex-valued function u such that:

$$\Delta u := 4\frac{\partial^2 u}{\partial\zeta\partial\overline{\zeta}} \equiv 0.$$

In this paper, let $H(\mathbb{D})$ denote the space consisting of analytic functions on the unit disk $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, $\mathcal{H}ar(\mathbb{D})$ denote the space consisting all harmonic mappings. The harmonic mapping u is always a representation of the form $h+\overline{f}$, where h and f are analytic functions. Up to the additive constants, this representation is unique. Therefore, $u \in \mathcal{H}ar(\mathbb{D})$ if and only if $u = h + \overline{f}$ where $h, f \in H(\mathbb{D})$ and $f(0) = 0$. For a general reference on the theory of harmonic functions, see [6].

Let $S(\mathbb{D})$ be the set of all analytic or conjugate analytic self-maps of $\mathbb{D}$. The composition operator C_φ induced by $\varphi \in S(\mathbb{D})$ is defined as the operator

$$C_\varphi u = u \circ \varphi,$$

for all $u \in \mathcal{H}ar(\mathbb{D})$. Surely, such an operator preserves harmonicity.

Recall that, for any two normed linear spaces X and Y, the linear operator $T : X \longrightarrow Y$ is said to be bounded if there exists $C > 0$ such that $\|Tu\|_Y \leq C\|u\|_X, \forall u \in X$. Furthermore, a linear operator $T : X \longrightarrow Y$ is said to be compact if it maps every bounded set in X to a relatively compact set in Y (i.e., a set whose closure is compact).

We start with several preliminaries that will be used to get the main results in this work, then we focus on the boundedness and the compactness of the composition operators from the harmonic Lipschitz spaces $Lip_H^\alpha, (0 < \alpha < 1)$ into the harmonic weighted Zygmund spaces $\mathcal{Z}_H^\beta, (0 < \beta < \infty)$. We conclude by approximating the essential norm.

2. Spaces treated in this paper

Most of the research on harmonic mappings in the last two decades has been conducted by analyzing the function theoretic aspects.

Firstly, let $\mathcal{H}_{\mathcal{H}}^{\infty} = \mathcal{H}_{\mathcal{H}}^{\infty}(\mathbb{D})$ denote the space of all bounded harmonic mappings u on $\mathbb{D}$ equipped with the norm

$$\|u\|_{\infty} = \sup_{\zeta\in\mathbb{D}} |u(\zeta)|.$$

The harmonic weighted Bloch space $\mathcal{B}_H^{\alpha}$. For $\alpha \in (0, \infty)$, the harmonic α-Bloch space $\mathcal{B}_H^{\alpha}$ contains all $u \in \mathcal{H}ar(\mathbb{D})$ is defined such that

$$\beta_u^{\alpha} := \sup_{zeta\in\mathbb{D}} (1-|\zeta|^2)^{\alpha}\left(\left|\frac{\partial u(\zeta)}{\partial \zeta}\right| + \left|\frac{\partial u(\zeta)}{\partial \bar{\zeta}}\right|\right) < \infty.$$

If $u \in \mathcal{B}_H^{\alpha}$ is represented as $u = h + \bar{f}$, with $h, f \in H(\mathbb{D})$, the harmonic α-Bloch seminorm β_u^{α} can be characterized as

$$\beta_u^{\alpha} = \sup_{\zeta\in\mathbb{D}} (1-|\zeta|^2)^{\alpha}(|h'(\zeta)| + |f'(\zeta)|) < \infty.$$

The quantity

$$\|u\|_{\mathcal{B}_H^{\alpha}} := |u(0)| + \beta_u^{\alpha},$$

yields a Banach space structure on $\mathcal{B}_H^{\alpha}$; see [3].

The harmonic little α-Bloch space $\mathcal{B}_{H,0}^{\alpha}$ is defined as the subspace of $\mathcal{B}_H^{\alpha}$ consisting of the mappings $u \in \mathcal{H}ar(\mathbb{D})$ such that

$$\beta_{u,0}^{\alpha} = \lim_{|\zeta|\to 1} (1-|\zeta|^2)^{\alpha}\left(\left|\frac{\partial u(\zeta)}{\partial \zeta}\right| + \left|\frac{\partial u(\zeta)}{\partial \bar{\zeta}}\right|\right) = 0.$$

In [12], Bloch harmonic mappings were introduced and the connection between the Lipschitz constant of a bounded harmonic mapping and its supremum norm was studied.

The harmonic Lipschitz space. For $\alpha \in (0, 1), Lip_H^{\alpha}$ consists of all complex-valued harmonic mappings u on $\mathbb{D}$ satisfying the condition: there exists a constant $C > 0$ such that

$$|u(w) - u(z)| \leq C|w - z|^{\alpha}, \ \forall \ w, z \in \mathbb{D}.$$

The norm of the harmonic Lipschitz space Lip_H^{α} is defined by the quantity

$$\|u\|_{Lip_H^{\alpha}} = |k(0)| + \sup_{w\neq z} \frac{|u(w) - u(z)|}{|w-z|^{\alpha}}.$$

Let $u \in Lip_H^{\alpha}$ and set $C = \sup\{\frac{|u(w)-u(z)|}{|w-z|^{\alpha}} : w \neq z\}$. Therefore, for $w \in \mathbb{D}$, we have

$$|u(w)| \leq |u(0)| + C|w|^{\alpha} \leq \|u\|_{Lip_H^{\alpha}}.$$

Then, taking the supremum over all $w \in \mathbb{D}$, we get

$$\|u\|_{\infty} \leq \|u\|_{Lip_H^{\alpha}} < \infty. \tag{2.1}$$

The elements of Lip_H^α are characterized by the following harmonic Bloch condition: $u \in Lip_H^\alpha$ if and only if

$$b_\alpha(u) = \sup_{w\in\mathbb{D}}(1-|w|^2)^{1-\alpha}\big(|u_w(w)|+|u_{\overline{w}}(w)|\big) < \infty. \tag{2.2}$$

The weighted harmonic Zygmund space $\mathcal{Z}_H^\beta$. For $\beta \in (0,\infty)$, $\mathcal{Z}_H^\beta$ consists of all complex-valued harmonic mappings $u \in \mathcal{H}ar(\mathbb{D})$ such that

$$\|u\|_{*,\beta} := \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^\beta\left(\left|\frac{\partial^2 u}{\partial\zeta^2}(\zeta)\right|+\left|\frac{\partial^2 u}{\partial\overline{\zeta}^2}(\zeta)\right|\right)< \infty. \tag{2.3}$$

Define

$$\|u\|_{\mathcal{Z}_H^\beta} := |u(0)| + \left|\frac{\partial u}{\partial\zeta}(0)\right|+\left|\frac{\partial u}{\partial\overline{\zeta}}(0)\right|+\|u\|_{*,\beta}.$$

Obviously, $\|.\|_{\mathcal{Z}_H^\beta}$ is a norm on $\mathcal{Z}_H^\beta$ and $\mathcal{Z}_H^\beta$ is a Banach space. For $\beta = 1$, $\mathcal{Z}_H^1$ is with the harmonic Zygmund space $\mathcal{Z}_H$; see [5].

Remark 2.1. *When $u \in H(\mathbb{D})$, the mapping $\frac{\partial u}{\partial\zeta}$ reduces to u' and $\frac{\partial u}{\partial\overline{\zeta}} = \frac{\partial^2 u}{\partial\overline{\zeta}^2} = 0$. Thus, for all $0 < \beta < \infty$, the collection of analytic functions in the space $\mathcal{Z}_H^\beta$ is the classical weighted Zygmund space $\mathcal{Z}^\beta$ and both norms are identical.*

By Theorem 19 provided in [3], we have the following characterization of the harmonic Bloch-type mappings. Given $0 < \alpha < 1$ and let $u \in \mathcal{H}ar(\mathbb{D})$, then

$$\|u\|_{Lip_H^\alpha} \asymp \|u\|_{\mathcal{B}_H^{1-\alpha}} \asymp \|u\|_{\mathcal{Z}^{2-\alpha}}. \tag{2.4}$$

Let $b \in \mathbb{D}$ be fixed, and let $k \in \{1,2,3\}$. Then, for any $\zeta \in \mathbb{D}$, we consider a set of three functions $F_{b,k}^\alpha$ as follows:

$$F_{b,k}^\alpha(\zeta) = \frac{(1-|b|^2)^k}{(1-\overline{b}\zeta)^{k-\alpha}} + \frac{(1-|b|^2)^k}{(1-b\overline{\zeta})^{k-\alpha}}. \tag{2.5}$$

Moreover, it is evident that $\lim_{|b|\to 1} F_{b,k}^\alpha = 0$ uniformly on compact subsets $\overline{\mathbb{D}} \subset \mathbb{D}$. Recall the power series representations of $F_{b,k}^\alpha$ are given as

$$\begin{aligned} F_{b,k}^\alpha(\zeta) &= (1-|b|^2)^k\sum_{j=0}^\infty \frac{\Gamma(j+k-\alpha)}{j!\Gamma(j-\alpha)}\Big\{(\overline{b}\zeta)^j + (b\overline{\zeta})^j\Big\} \\ &\asymp (1-|b|^2)^k\sum_{j=0}^\infty j^{k-\alpha-1}\Big\{(\overline{b}\zeta)^j + (b\overline{\zeta})^j\Big\}. \end{aligned} \tag{2.6}$$

By direct calculation, we know that, for all $n \in \mathbb{N}$ and $k \in \{1,2,3\}$,

$$\begin{aligned} \frac{\partial^n F_{b,k}^\alpha(\zeta)}{\partial\zeta^n} &= \frac{(k+n-\alpha-1)!}{(k-\alpha-1)!}\left[\frac{\overline{b}^n\big(1-|b|^2\big)^k}{(1-\overline{b}\zeta)^{k+n-\alpha}}\right]; \\ \frac{\partial^n F_{b,k}^\alpha(\zeta)}{\partial\overline{\zeta}^n} &= \frac{(k+n-\alpha-1)!}{(k-\alpha-1)!}\left[\frac{b^n\big(1-|b|^2\big)^k}{(1-b\overline{\zeta})^{k+n-\alpha}}\right]. \end{aligned} \tag{2.7}$$

Then, we obtain

$$\begin{aligned} \frac{\partial^n F^{\alpha}_{b,k}(b)}{\partial \zeta^n} &= \frac{(k-\alpha+n-1)!}{(k-\alpha-1)!} \frac{\overline{b}^n}{\left(1-|b|^2\right)^{n-\alpha}}; \\ \frac{\partial^n F^{\alpha}_{b,k}(b)}{\partial \overline{\zeta}^n} &= \frac{(k-\alpha+n-1)!}{(k-\alpha-1)!} \frac{b^n}{\left(1-|b|^2\right)^{n-\alpha}}. \end{aligned}$$

As before, for all $\zeta \in \mathbb{D}$, we have

$$\begin{aligned} \left|\frac{\partial}{\partial \zeta} F^{\alpha}_{b,k}(\zeta)\right| &= (k+\alpha-1)\left|\frac{\overline{b}\left(1-|b|^2\right)^k}{(1-\overline{b}\zeta)^{k+\alpha}}\right| \\ &\leq \frac{2(k-\alpha+1)2^{1-\alpha}}{\left(1-|\zeta|^2\right)^{1-\alpha}}; \\ \left|\frac{\partial}{\partial \overline{\zeta}} F^{\alpha}_{b,k}(\zeta)\right| &= (k+\alpha-1)\left|\frac{b\left(1-|b|^2\right)^k}{(1-\overline{b}\zeta)^{k+\alpha}}\right| \\ &\leq \frac{2(k-\alpha+1)2^{1-\alpha}}{\left(1-|\zeta|^2\right)^{1-\alpha}}. \end{aligned}$$

Then, we have

$$\left|\frac{\partial}{\partial \zeta} F^{\alpha}_{b,k}(\zeta)\right| + \left|\frac{\partial}{\partial \overline{\zeta}} F^{\alpha}_{b,k}(\zeta)\right| \leq \frac{(k-\alpha+1)2^{3-\alpha}}{\left(1-|\zeta|^2\right)^{1-\alpha}}.$$

Thus, for every $k \in \mathbb{N}$, it can be demonstrated that $F^{\alpha}_{b,k} \in Lip^{\alpha}_{H}$ and

$$\sup_{b \in \mathbb{D}} \|F^{\alpha}_{b,k}\|_{Lip^{\alpha}_{H}} \preceq 1.$$

Throughout this paper, we use the notation $A \preceq B$, which implies that there is a constant $C > 0$ such that $A \leq CB$. Therefore, when $B \preceq A \preceq B$, we use the notation $A \approx B$, meaning that A and B are equivalent.

3. Boundedness

In this section, we characterize the boundedness composition operators from the harmonic Lipschitz space $Lip^{\alpha}_{H}, \alpha \in (0,1)$ into the weighted harmonic Zygmund space $\mathcal{Z}^{\beta}_{H}, 0 < \beta < \infty$.

Let the sequence $p_j(w) = j^{-\alpha}(w^j + \overline{w}^j)$, for $w \in \mathbb{D}$ and $j \geq 0$ is an element of the integers.

Theorem 3.1. *Suppose that $\varphi \in S(\mathbb{D}), 0 < \alpha < 1$ and $0 < \beta < \infty$. Then the following are equivalent:*

(1) *The composition operator $C_{\varphi} : Lip^{\alpha}_{H} \to \mathcal{Z}^{\beta}_{H}$ is bounded.*
(2) $\sup_{j \in \mathbb{N}} \|C_{\varphi} p_j\|_{\mathcal{Z}^{\beta}_{H}} < \infty$.
(3) *The quantities $L_1 = \frac{(1-|\zeta|^2)^{\beta}|\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^{2-\alpha}}$ and $L_2 = \frac{(1-|\zeta|^2)^{\beta}|\varphi''(\zeta)|}{(1-|\varphi(\zeta)|^2)^{1-\alpha}}$ are finite.*

Proof. (1) $\Longrightarrow$ (2). Suppose that $C_{\varphi} : Lip^{\alpha}_{H} \to \mathcal{Z}^{\beta}_{H}$ is bounded. The sequence $\{p_j\}$ is bounded in the harmonic Lip^{α}_{H} space. Then, for each $j \geq 0$ and $0 < \beta < \infty$, we have

$$\|C_{\varphi} p_j\|_{\mathcal{Z}^{\beta}_{H}} \preceq \|C_{\varphi}\|_{*,\beta}.$$

Therefore,

$$\sup_{j\in\mathbb{N}}\|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}<\infty.$$

(2) $\Longrightarrow$ (3). Suppose that $L=\|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}<\infty$.

Since $C_\varphi p_1=\varphi+\overline{\varphi}$, for $\zeta\in\mathbb{D}$, we have

$$\left|\frac{\partial^2[C_\varphi p_1(\zeta)]}{\partial\zeta^2}\right|=\left|\frac{\partial^2[C_\varphi p_1(\zeta)]}{\partial\overline{\zeta}^2}\right|=|\varphi''(\zeta)|.$$

Then,

$$\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^\beta|\varphi''(\zeta)|\le\frac{1}{2}\|C_\varphi p_1\|_{\mathcal{Z}_H^\beta}\le\frac{L}{2}. \tag{3.1}$$

Moreover, we know that $C_\varphi p_2=2^{-\alpha}(\varphi^2+\overline{\varphi}^2)$,

$$\begin{aligned}\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\zeta^2}&=2^{2-\alpha}\Big\{(\varphi'(\zeta))^2+\varphi(\zeta)\varphi''(\zeta)\Big\},\\ \frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\overline{\zeta}^2}&=2^{2-\alpha}\Big\{(\overline{\varphi'(\zeta)})^2+\overline{\varphi(\zeta)\varphi''(\zeta)}\Big\}.\end{aligned}$$

Since $|\varphi(\zeta)|\le 1$ for $\zeta\in\mathbb{D}$, we have

$$|\varphi'(\zeta)|^2\le\frac{1}{2^{3-\alpha}}\left\{\left|\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\overline{\zeta}^2}\right|\right\}+|\varphi''(\zeta)|.$$

Thus,

$$\begin{aligned}\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^\beta|\varphi'(\zeta)|^2&\le\frac{1}{2^{3-\alpha}}\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^\beta\left(\left|\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[C_\varphi p_2(\zeta)]}{\partial\overline{\zeta}^2}\right|\right)\\&\quad+\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^\beta|\varphi''(\zeta)|\\&\le\frac{1}{2^{3-\alpha}}\|C_\varphi p_2\|_{\mathcal{Z}_H^\beta}+\frac{1}{2}\|C_\varphi p_1\|_{\mathcal{Z}_H^\beta}\le\frac{(2^{2-\alpha}+1)L}{2^{3-\alpha}}.\end{aligned} \tag{3.2}$$

On the other hand, by the linearity of the test function (2.6), for $k=1,2,3$ and $\zeta\in\mathbb{D}$, we have

$$\|C_\varphi F^\alpha_{\varphi(\zeta),k}\|_{\mathcal{Z}_H^\beta}\preceq(1-|\varphi(\zeta)|^2)^k\sum_{j=0}^\infty j^k|\varphi(\zeta)|^j\|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}\preceq L. \tag{3.3}$$

From (2.7), for $k=1,2,3$ and $\zeta\in\mathbb{D}$, we obtain

$$\begin{aligned}\frac{\partial^2[C_\varphi F^\alpha_{\varphi(\zeta),k}(\zeta)]}{\partial\zeta^2}&=\frac{(k-\alpha)(k-\alpha+1)[\overline{\varphi(\zeta)}\varphi'(\zeta)]^2}{\left(1-|\varphi(\zeta)|^2\right)^{2-\alpha}}+\frac{(k-\alpha)[\overline{\varphi(\zeta)}\varphi''(\zeta)]}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}},\\ \frac{\partial^2[C_\varphi F^\alpha_{\varphi(\zeta),k}(\zeta)]}{\partial\overline{\zeta}^2}&=\frac{(k-\alpha)(k-\alpha+1)[\varphi(\zeta)\overline{\varphi'(\zeta)}]^2}{\left(1-|\varphi(\zeta)|^2\right)^{2-\alpha}}+\frac{(k-\alpha)[\varphi(\zeta)\overline{\varphi''(\zeta)}]}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}}.\end{aligned}$$

Next, for $k = 1, 2, 3$ we let

$$Q_{\varphi(\zeta),k} = \frac{\partial^2[C_\varphi F^\alpha_{\varphi(\zeta),k}(\zeta)]}{\partial\zeta^2} + \frac{\partial^2[C_\varphi F^\alpha_{\varphi(\zeta),k}(\zeta)]}{\partial\overline{\zeta}^2}. \tag{3.4}$$

By solving the system of equations (3.4), for $k = 1, 2, 3$, we get

$$\frac{2[\overline{\varphi(\zeta)}\varphi'(\zeta)]^2}{\left(1-|\varphi(\zeta)|^2\right)^{2-\alpha}} + \frac{2[\varphi(\zeta)\overline{\varphi'(\zeta)}]^2}{\left(1-|\varphi(\zeta)|^2\right)^{2-\alpha}} = Q_{\varphi(\zeta),1} - 2Q_{\varphi(\zeta),2} + Q_{\varphi(\zeta),3}. \tag{3.5}$$

Moreover,

$$\begin{aligned} &\frac{\overline{\varphi(\zeta)}\varphi''(\zeta)}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}} + \frac{\varphi(\zeta)\overline{\varphi''(\zeta)}}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}} \\ = &\ -(3-\alpha)Q_{\varphi(\zeta),1} + (5-2\alpha)Q_{\varphi(\zeta),2} - (2-\alpha)Q_{\varphi(\zeta),3}. \end{aligned} \tag{3.6}$$

Thus, from (3.5) we obtain

$$\begin{aligned} &\frac{(1-|\zeta|^2)^\beta|\varphi(\zeta)|^2|\varphi'(\zeta)|^2}{\left(1-|\varphi(\zeta)|^2\right)^{2-\alpha}} \\ \leq &\ \frac{1}{4}\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^\beta\Big(|Q_{\varphi(\zeta),1}| + 2|Q_{\varphi(\zeta),2}| + |Q_{\varphi(\zeta),3}|\Big) \\ \leq &\ \frac{1}{4}\Big(\|C_\varphi F^\alpha_{\varphi(\zeta),1}\|_{\mathcal{Z}^\beta_H} + 2\|C_\varphi F^\alpha_{\varphi(\zeta),2}\|_{\mathcal{Z}^\beta_H} + \|C_\varphi F^\alpha_{\varphi(\zeta),3}\|_{\mathcal{Z}^\beta_H}\Big) \preceq L. \end{aligned} \tag{3.7}$$

Furthermore, from (3.6) we obtain

$$\begin{aligned} &\frac{(1-|\zeta|^2)^\beta|\varphi(\zeta)||\varphi''(\zeta)|}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}} \\ \leq &\ \frac{1}{2}\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^\beta\Big((3-\alpha)|Q_{\varphi(\zeta),1}| + (5-2\alpha)|Q_{\varphi(\zeta),2}| + (2-\alpha)|Q_{\varphi(\zeta),3}|\Big) \\ \leq &\ \frac{1}{2}\Big((3-\alpha)\|C_\varphi F^\alpha_{\varphi(\zeta),1}\|_{\mathcal{Z}^\beta_H} + (5-2\alpha)\|C_\varphi F^\alpha_{\varphi(\zeta),2}\|_{\mathcal{Z}^\beta_H} + (2-\alpha)\|C_\varphi F^\alpha_{\varphi(\zeta),3}\|_{\mathcal{Z}^\beta_H}\Big) \\ \leq &\ (5-2\alpha)L. \end{aligned} \tag{3.8}$$

Now we let $0 < s < 1$, then if $|\varphi(\zeta)| > s$ in (3.7) we have

$$L_1 = \frac{(1-|\zeta|^2)^\beta|\varphi'(\zeta)|^2}{\left(1-|\varphi(\zeta)|^2\right)^{2-\alpha}} \preceq \frac{L}{s^2}. \tag{3.9}$$

On the other hand, if we let $|\varphi(\zeta)| \leq s$ in (3.2) we have

$$L_1 = \frac{(1-|\zeta|^2)^\beta|\varphi'(\zeta)|^2}{\left(1-|\varphi(\zeta)|^2\right)^{2-\alpha}} \leq \frac{(2^{2-\alpha}+1)L}{(1-s^2)^{2-\alpha}2^{3-\alpha}}. \tag{3.10}$$

From (3.9) and (3.10) it follows that the quantity L_1 is finite.

For the second part L_2, we similarly let $0 < s < 1$. Then if $|\varphi(\zeta)| > s$ in (3.8), we have

$$L_2 = \frac{(1-|\zeta|^2)^\beta|\varphi''(\zeta)|}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}} \leq \frac{1}{s}(5-2\alpha)L. \tag{3.11}$$

If we instead let $|\varphi(\zeta)| \leq s$ in (3.1), we have

$$L_2 = \frac{(1-|\zeta|^2)^\beta |\varphi''(\zeta)|}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}} \leq \frac{L}{4(1-s^2)^{1-\alpha}}. \tag{3.12}$$

Therefore, the quantity L_2 is finite.

(3) $\Longrightarrow$ (1). Assume L_1 and L_2 are finite.

Noting that for any $\zeta \in \mathbb{D}$ and $u \in \mathcal{H}ar(\mathbb{D})$, since $Lip_H^\alpha \subset \mathcal{H}_{\mathcal{H}}^\infty$, by (2.1), $\|u\|_\infty \leq \|u\|_{Lip_H^\alpha}$, we have

$$|(C_\varphi u)(0)| = |u(\varphi(0))| \leq \|u\|_{Lip_H^\alpha}.$$

Therefore, because $|\varphi(0)| < 1$, we note that

$$\begin{aligned} \left|\frac{\partial (C_\varphi u)}{\partial \zeta}(0)\right| + \left|\frac{\partial (C_\varphi u)}{\partial \bar{\zeta}}(0)\right| &= \left|\frac{\partial u(\varphi(0))}{\partial \zeta}\varphi'(0)\right| + \left|\frac{\partial u(\varphi(0))}{\partial \bar{\zeta}}\overline{\varphi'(0)}\right| \\ &\preceq \frac{|\varphi'(0)|}{(1-|\varphi(0)|^2)^{1-\alpha}}\|u\|_{Lip_H^\alpha} < \infty. \end{aligned}$$

On the other hand, for any $\zeta \in \mathbb{D}$ and $u \in \mathcal{H}ar(\mathbb{D})$,

$$\begin{aligned} \left|\frac{\partial^2 (C_\varphi u)}{\partial \zeta^2}(\zeta)\right| &= \left|\frac{\partial^2 u(\varphi(\zeta))}{\partial \zeta^2}[\varphi'(\zeta)]^2 + \frac{\partial u(\varphi(\zeta))}{\partial \zeta}\varphi''(\zeta)\right| \\ &\leq |\varphi'(\zeta)|^2\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial \zeta^2}\right| + |\varphi''(\zeta)|\left|\frac{\partial u(\varphi(\zeta))}{\partial \zeta}\right|; \end{aligned} \tag{3.13}$$

$$\begin{aligned} \left|\frac{\partial^2 (C_\varphi u)}{\partial \bar{\zeta}^2}(\zeta)\right| &= \left|\frac{\partial^2 u(\varphi(\zeta))}{\partial \bar{\zeta}^2}[\overline{\varphi'(\zeta)}]^2 + \frac{\partial u(\varphi(\zeta))}{\partial \bar{\zeta}}\overline{\varphi''(\zeta)}\right| \\ &\leq |\varphi'(\zeta)|^2\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial \bar{\zeta}^2}\right| + |\varphi''(\zeta)|\left|\frac{\partial u(\varphi(\zeta))}{\partial \bar{\zeta}}\right|. \end{aligned} \tag{3.14}$$

Now adding the above expressions (3.13) and (3.13), and multiplying by $(1-|\zeta|^2)^\beta$, then we obtain

$$\begin{aligned} & (1-|\zeta|^2)^\beta\left(\left|\frac{\partial^2 (C_\varphi u)}{\partial \zeta^2}(\zeta)\right| + \left|\frac{\partial^2 (C_\varphi u)}{\partial \bar{\zeta}^2}(\zeta)\right|\right) \\ \leq\ & (1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2\left(\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial \zeta^2}\right| + \left|\frac{\partial^2 u(\varphi(\zeta))}{\partial \bar{\zeta}^2}\right|\right) \\ +\ & (1-|\zeta|^2)^\beta |\varphi''(\zeta)|\left(\left|\frac{\partial u(\varphi(\zeta))}{\partial \zeta}\right| + \left|\frac{\partial u(\varphi(\zeta))}{\partial \bar{\zeta}}\right|\right). \end{aligned}$$

Since $u \in \mathcal{H}ar(\mathbb{D})$, by (2.4), we obtain

$$
\begin{aligned}
& (1-|\zeta|^2)^\beta\left(\left|\frac{\partial^2(C_\varphi u)}{\partial \zeta^2}(\zeta)\right|+\left|\frac{\partial^2(C_\varphi u)}{\partial \overline{\zeta}^2}(\zeta)\right|\right) \\
\leq\ & (1-|\zeta|^2)^\beta|\varphi'(\zeta)|^2\left(\left|\frac{\partial^2 u}{\partial \zeta^2}(\zeta)\right|+\left|\frac{\partial^2 u}{\partial \overline{\zeta}^2}(\zeta)\right|\right) \\
+\ & (1-|\zeta|^2)^\beta|\varphi''(\zeta)|\left(\left|\frac{\partial u}{\partial \zeta}(\zeta)\right|+\left|\frac{\partial u}{\partial \overline{\zeta}}(\zeta)\right|\right) \\
\preceq\ & \frac{(1-|\zeta|^2)^\beta|\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^{2-\alpha}}\|u\|_{\mathcal{Z}^{2-\alpha}}+\frac{(1-|\zeta|^2)^\beta|\varphi''(\zeta)|}{(1-|\varphi(\zeta)|^2)^{1-\alpha}}\|u\|_{\mathcal{B}_H^{1-\alpha}} \\
\preceq\ & (L_1+L_2)\|u\|_{Lip_H^\alpha},
\end{aligned}
$$

Finally, by taking the supremum over all $\zeta \in \mathbb{D}$ the boundedness of $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$ follows from above. The proof of Theorem 3.1 is complete. □

4. Compactness

In this section, we shift our attention to discussing the compactness of $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$. The following criterion lemma for the compactness is similar to the case of Banach spaces of analytic functions (the analytic case), see for example Proposition 3.11 of [13].

Lemma 4.1. *The bounded operator $T : Lip_H^\alpha \to \mathcal{Z}_H^\beta$ is compact if and only if $\|Tu_m\|_{\mathcal{Z}_H^\beta} \to 0$ as $m \to \infty$, for any bounded sequence $\{u_m\}_{m\in\mathbb{N}}$ in Lip_H^α converges to zero uniformly on compact subsets $\mathbb{G} \subset \mathbb{D}$.*

The following result indicates that the compactness of the composition operators can be characterized in terms of the sequence $\|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}$, where $p_j(w) = j^{-\alpha}(w^j + \overline{w}^j)$, for $w \in \mathbb{D}$ and when $j \geq 0$ is an integer.

Theorem 4.1. *Let $\varphi \in S(\mathbb{D}), 0 < \alpha < 1$ and $0 < \beta < \infty$ and assume that the operator $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$ is bounded. Then the following are equivalent:*

(1) *The composition operator $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$ is compact.*
(2) $\lim_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta} = 0$.
(3) $\lim_{|\varphi(\zeta)|\to 1} \frac{(1-|\zeta|^2)^\beta|\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^{2-\alpha}} = 0$ *and* $\lim_{|\varphi(\zeta)|\to 1} \frac{(1-|\zeta|^2)^\beta|\varphi''(\zeta)|}{(1-|\varphi(\zeta)|^2)^{1-\alpha}} = 0$.

Proof. (1) $\Longrightarrow$ (2). As in the proof of Theorem 3.1, since the sequence $\{p_j\}$ is bounded in the harmonic space Lip_H^α and converges to zero uniformly on compact subsets $\mathbb{G} \subset \mathbb{D}$, if $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$ is compact, then by Lemma 4.1,

$$\lim_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta} = 0.$$

(2) $\Longrightarrow$ (3). Assume $\lim_{j\to\infty}\|C_\varphi p_j\|_{\mathcal{Z}^\beta_H}=0$. Next, we let L be an upper bound for $\|C_\varphi p_j\|_{\mathcal{Z}^\beta_H}$. Then, for $\varepsilon>0$, there is $N\in\mathbb{N}$ such that

$$\|C_\varphi p_j\|_{\mathcal{Z}^\beta_H}<\varepsilon,\ \forall\, j\geq N.$$

By using the test function (2.6), for $k=1,2,3$ and $\zeta\in\mathbb{D}$, we have

$$\begin{aligned}\|C_\varphi F^\alpha_{\varphi(\zeta),1}\|_{\mathcal{Z}^\beta_H} &\preceq (1-|\varphi(\zeta)|^2)\sum_{j=0}^{N-1}|\varphi(\zeta)|^j\|C_\varphi p_j\|_{\mathcal{Z}^\beta_H}\Big]\\ &\quad+(1-|\varphi(\zeta)|^2)\sum_{j=N}^{\infty}|\varphi(\zeta)|^j\|C_\varphi p_j\|_{\mathcal{Z}^\beta_H}\\ &< (1-|\varphi(\zeta)|^2)NL+\varepsilon.\end{aligned}$$

Moreover,

$$\begin{aligned}\|C_\varphi F^\alpha_{\varphi(\zeta),2}\|_{\mathcal{Z}^\beta_H} &\preceq (1-|\varphi(\zeta)|^2)^2\left[\left\{\sum_{j=1}^{N}+\sum_{j=N+1}^{\infty}\right\}j|\varphi(\zeta)|^{j-1}\|C_\varphi p_{j-1}\|_{\mathcal{Z}^\beta_H}\right]\\ &< (1-|\varphi(\zeta)|^2)^2\frac{N(N+1)}{2}L+\varepsilon;\end{aligned}$$

$$\begin{aligned}\|C_\varphi F^\alpha_{\varphi(\zeta),3}\|_{\mathcal{Z}^\beta_H} &\preceq (1-|\varphi(\zeta)|^2)^3\left[\left\{\sum_{j=2}^{N+1}+\sum_{j=N+2}^{\infty}\right\}j(j-1)|\varphi(\zeta)|^{j-2}\|C_\varphi p_{j-2}\|_{\mathcal{Z}^\beta_H}\right]\\ &< (1-|\varphi(\zeta)|^2)^3\frac{N(N+1)(N+2)}{6}L+\varepsilon.\end{aligned}$$

On the other hand, for any $\zeta\in\mathbb{D}$ let $0<s<1$ sufficiently close to 1 such that $|\varphi(\zeta)|>s$, thus

$$\|C_\varphi F^\alpha_{\varphi(\zeta),k}\|_{\mathcal{Z}^\beta_H}<2\varepsilon,\ \text{ for } k=1,2,3.$$

Since ε is arbitrary, for $k=1,2,3$, it follows that

$$\lim_{|\varphi(\zeta)|\to1}\|C_\varphi F^\alpha_{\varphi(\zeta),k}\|_{\mathcal{Z}^\beta_H}=0. \tag{4.1}$$

Going back to the proof of Theorem 3.1, from (3.7) we know

$$\frac{(1-|\zeta|^2)^\beta|\varphi'(\zeta)|^2}{\big(1-|\varphi(\zeta)|^2\big)^{2-\alpha}}\leq\frac{1}{|\varphi(\zeta)|^2}\max_{1\leq k\leq3}\|C_\varphi h_{\varphi(\zeta),k}\|_{\mathcal{Z}^\beta_H}. \tag{4.2}$$

Moreover, from (3.8) we know

$$\frac{(1-|\zeta|^2)^\beta|\varphi''(\zeta)|}{\big(1-|\varphi(\zeta)|^2\big)^{1-\alpha}}\leq\frac{(5-2\alpha)}{|\varphi(\zeta)|}\max_{1\leq k\leq3}\|C_\varphi h_{\varphi(\zeta),k}\|_{\mathcal{Z}^\beta_H}. \tag{4.3}$$

Using (4.2), we have

$$\lim_{|\varphi(\zeta)|\to1}\frac{(1-|\zeta|^2)^\beta|\varphi'(\zeta)|^2}{\big(1-|\varphi(\zeta)|^2\big)^{2-\alpha}}=0.$$

Moreover, by (4.3), we have

$$\lim_{|\varphi(\zeta)|\to 1} \frac{(1-|\zeta|^2)^\beta |\varphi''(\zeta)|}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}} = 0.$$

(3) $\Longrightarrow$ (1). Suppose that (3) holds. By Theorem 3.1 and the boundedness of $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$, we see that

$$L_1 = \frac{(1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^{2-\alpha}} < \infty, \quad L_2 = \frac{(1-|\zeta|^2)^\beta |\varphi''(\zeta)|}{(1-|\varphi(\zeta)|^2)^{1-\alpha}} < \infty.$$

For any $\varepsilon > 0$, from (3), there is $0 < s < 1$, where $1 > |\varphi(\zeta)| > s$, such that

$$\frac{(1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2}{\left(1-|\varphi(\zeta)|^2\right)^{2-\alpha}} < \varepsilon, \quad \frac{(1-|\zeta|^2)^\beta |\varphi''(\zeta)|}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}} < \varepsilon. \tag{4.4}$$

Now we let a sequence $\{h_m\}$ in the harmonic space Lip_H^α with

$$M = \sup_{m\in\mathbb{N}} \|h_m\|_{Lip_H^\alpha} < \infty,$$

and $h_m \to 0$ uniformly on compact subsets $\mathbb{G} \subset \mathbb{D}$, as $m \to \infty$. To prove the compactness of $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$, it suffices to show that

$$\lim_{m\to\infty} \|C_\varphi h_m\|_{\mathcal{Z}_H^\beta} = 0.$$

Thus, using (4.4), for $|\varphi(w)| > s$ we have

$$\begin{aligned}
&(1-|\zeta|^2)^\beta \left(\left| \frac{\partial^2 [C_\varphi h_m(\zeta)]}{\partial \zeta^2} \right| + \left| \frac{\partial^2 [C_\varphi h_m(\zeta)]}{\partial \bar{\zeta}^2} \right| \right) \\
\le\ & \sup_{1>|\varphi(w)|>s} (1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2 \left(\left| \frac{\partial^2 h_m(\varphi(\zeta))}{\partial \zeta^2} \right| + \left| \frac{\partial^2 h_m(\varphi(\zeta))}{\partial \bar{\zeta}^2} \right| \right) \\
+\ & \sup_{1>|\varphi(w)|>s} (1-|\zeta|^2)^\beta |\varphi''(\zeta)| \left(\left| \frac{\partial h_m(\varphi(\zeta))}{\partial \zeta} \right| + \left| \frac{\partial h_m(\varphi(\zeta))}{\partial \bar{\zeta}} \right| \right). \\
\preceq\ & \|h_m\|_{Lip_H^\alpha} \Big(\frac{(1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^{2-\alpha}} + \frac{(1-|\zeta|^2)|\varphi''(\zeta)|}{(1-|\varphi(\zeta)|^2)^{1-\alpha}} \Big) \preceq \varepsilon M.
\end{aligned} \tag{4.5}$$

Once again, going back to the proof of Theorem 3.1, from (3.1) and (3.2), we know

$$\sup_{\zeta\in\mathbb{D}} (1-|\zeta|^2)^\beta |\varphi''(\zeta)| \le \frac{L}{2} \text{ and } \sup_{\zeta\in\mathbb{D}} (1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2 \le \frac{(2^{2-\alpha}+1)L}{2^{3-\alpha}}. \tag{4.6}$$

We know by Cauchy's estimates that, all the sequences $\left\{\frac{\partial h_m}{\partial \zeta}\right\}$, $\left\{\frac{\partial h_m}{\partial \bar{\zeta}}\right\}$, $\left\{\frac{\partial^2 h_m}{\partial \zeta^2}\right\}$ and $\left\{\frac{\partial^2 h_m}{\partial \bar{\zeta}^2}\right\}$ are convergent to zero on compact subsets $\mathbb{G} \subset \mathbb{D}$. Thus using (4.6), for any $0 < s < 1$ if $|\varphi(\zeta)| \le s$, we obtain

$$\begin{aligned}
&(1-|\zeta|^2)^\beta\left(\left|\frac{\partial^2[C_\varphi h_m(\zeta)]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[C_\varphi h_m(\zeta)]}{\partial\bar{\zeta}^2}\right|\right)\\
\leq\quad &\frac{(2^{2-\alpha}+1)L}{2^{3-\alpha}}\left(\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial\zeta^2}\right|+\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial\bar{\zeta}^2}\right|\right)\\
&+\frac{L}{2}\left(\left|\frac{\partial h_m(\varphi(\zeta))}{\partial\zeta}\right|+\left|\frac{\partial h_m(\varphi(\zeta))}{\partial\bar{\zeta}}\right|\right), \qquad (4.7)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\lim_{m\to\infty}(1-|\zeta|^2)^\beta\left(\left|\frac{\partial^2[C_\varphi h_m(\zeta)]}{\partial\zeta^2}\right|+\left|\frac{\partial^2[C_\varphi h_m(\zeta)]}{\partial\bar{\zeta}^2}\right|\right)\\
\leq\quad &\lim_{m\to\infty}\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial\zeta^2}\right|+\lim_{m\to\infty}\left|\frac{\partial^2 h_m(\varphi(\zeta))}{\partial\bar{\zeta}^2}\right|\\
&+\lim_{m\to\infty}\left|\frac{\partial h_m(\varphi(\zeta))}{\partial\zeta}\right|+\lim_{m\to\infty}\left|\frac{\partial h_m(\varphi(\zeta))}{\partial\bar{\zeta}}\right|=0. \qquad (4.8)
\end{aligned}$$

Therefore, $\lim_{m\to\infty}|C_\varphi h_m(0)|=0$ and $\lim_{m\to\infty}\left|\frac{\partial[C_\varphi h_m](0)}{\partial\bar{\zeta}}\right|=0$. Thus, we obtain

$$\lim_{m\to\infty}\|C_\varphi h_m\|_{\mathcal{Z}_H^\beta}=0. \qquad (4.9)$$

By Lemma 4.1, we verify that $C_\varphi : Lip_H^\alpha\to\mathcal{Z}_H^\beta$ is compact. □

The proof of the main theorem of this section is complete. Our next goal of this paper is to provide an approximation of the essential norm.

5. Essential norm

In this section, we characterize the essential norms of the composition operators from Lip_H^α to $\mathcal{Z}_H$. We know that the essential norm $\|T\|_e$ of an operator T is its distance from the compact operators in the operator norm. Precisely, consider X and Y to be Banach spaces and let $T : X\to Y$ be a bounded linear operator, then the essential norm of T between X and Y is then given by

$$\|T\|_{e,X\to Y}=\inf\{\|T-\mathcal{T}\|_{X\to Y}|\mathcal{T} : X\to Y \text{ is compact}\}.$$

First, we define

$$\begin{aligned}
B_1 &= \limsup_{|\varphi(\zeta)|\to1}\frac{(1-|\zeta|^2)^\beta|\varphi'(\zeta)|^2}{\left(1-|\varphi(\zeta)|^2\right)^{2-\alpha}},\\
B_2 &= \limsup_{|\varphi(\zeta)|\to1}\frac{(1-|\zeta|^2)^\beta|\varphi''(\zeta)|}{\left(1-|\varphi(\zeta)|^2\right)^{1-\alpha}}.
\end{aligned}$$

Theorem 5.1. *Let $\varphi \in S(\mathbb{D})$ and consider the bounded operator $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$. Then*

$$\begin{aligned} \|C_\varphi\|_{e,Lip_H^\alpha \to \mathcal{Z}_H^\beta} &\approx \max_{1\le k\le 3}\Big\{\limsup_{|b|\to 1}\|C_\varphi F_{b,k}^\alpha(\zeta)\|_{\mathcal{Z}_H^\beta}\Big\} \\ &\approx \max\{B_1, B_2\}. \end{aligned}$$

Proof. By using the test function (2.5), for $k = 1,2,3$ and $\zeta \in \mathbb{D}$, we prove that

$$\max_{1\le k\le 3}\Big\{\limsup_{|b|\to 1}\|C_\varphi F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta}\Big\} \preceq \|C_\varphi\|_{e,Lip_H^\alpha \to \mathcal{Z}_H^\beta}.$$

Fix $b \in \mathbb{D}$, since for all $1 \le k \le 3$, $F_{b,k}^\alpha \in Lip_H^\alpha$ and $F_{b,k}^\alpha$ converges uniformly to 0 on compact subsets $\mathbb{G} \subset \mathbb{D}$. Then, for a compact operator $\mathcal{T} : Lip_H^\alpha \to \mathcal{Z}_H^\beta$, we have

$$\lim_{|b|\to 1}\|\mathcal{T} F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta} = 0, \ \forall\, k = 1,2,3.$$

Thus,

$$\begin{aligned} \|C_\varphi - \mathcal{T}\|_{Lip_H^\alpha \to \mathcal{Z}_H^\beta} &\succeq \limsup_{|b|\to 1}\|(C_\varphi - \mathcal{T})F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta} \\ &\ge \limsup_{|b|\to 1}\|C_\varphi F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta} - \limsup_{|b|\to 1}\|\mathcal{T} F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta}. \end{aligned}$$

Hence, we obtain

$$\|C_\varphi\|_{e,Lip_H^\alpha \to \mathcal{Z}_H^\beta} = \inf_{\mathcal{T}}\|C_\varphi - \mathcal{T}\| \succeq \max_{1\le k\le 3}\Big\{\limsup_{|b|\to 1}\|C_\varphi F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta}\Big\}.$$

Next, to prove that $\|C_\varphi\|_{e,Lip_H^\alpha \to \mathcal{Z}_H^\beta} \succeq \max\{B_1, B_2\}$, we define the sequence $\{w_i\}$ such that $\lim_{i\to\infty}|\varphi(w_i)| = 1$, for $w \in \mathbb{D}$. Moreover, we define

$$\begin{aligned} G_i(\zeta) &= F_{\varphi(w_i),1}^\alpha(\zeta) - \frac{5-2\alpha}{3-\alpha}F_{\varphi(w_i),2}^\alpha(\zeta) + \frac{2-\alpha}{3-\alpha}F_{\varphi(w_i),3}^\alpha(\zeta), \\ K_i(\zeta) &= F_{\varphi(w_i),1}^\alpha(\zeta) - 2F_{\varphi(w_i),2}^\alpha(\zeta) + F_{\varphi(w_i),3}^\alpha(\zeta). \end{aligned}$$

For all $\zeta \in \mathbb{D}$, it can be proven that $G_i, K_i \in Lip_H^\alpha$ and

$$\lim_{|\varphi(w_i)|\to 1} G_i = \lim_{|\varphi(w_i)|\to 1} K_i = 0,$$

uniformly on compact subsets $\overline{G} \subset \mathbb{D}$. Moreover, by direct calculation we see that

$$\begin{aligned} G_i(\varphi(w_i)) &= K_i(\varphi(w_i)) = 0, \\ \left|\frac{\partial G_i(\varphi(w_i))}{\partial \zeta}\right| &= \left|\frac{\partial G_i(\varphi(w_i))}{\partial \overline{\zeta}}\right| = \frac{1}{3-\alpha}\frac{|\varphi(w_i)|}{(1-|\varphi(w_i)|^2)^{1-\alpha}}, \\ \frac{\partial^2 G_i(\varphi(w_i))}{\partial \zeta^2} &= \frac{\partial^2 G_i(\varphi(w_i))}{\partial \overline{\zeta}^2} = 0, \\ \frac{\partial K_i(\varphi(w_i))}{\partial \zeta} &= \frac{\partial K_i(\varphi(w_i))}{\partial \overline{\zeta}} = 0, \\ \left|\frac{\partial^2 K_i(\varphi(w_i))}{\partial \zeta^2}\right| &= \left|\frac{\partial^2 K_i(\varphi(w_i))}{\partial \overline{\zeta}^2}\right| = \frac{2|\varphi(w_i)|^2}{(1-|\varphi(w_i)|^2)^{2-\alpha}}. \end{aligned}$$

Since $\mathcal{T} : Lip_H^\alpha \to \mathcal{Z}_H^\beta$ is a compact operator, by Lemma 4.1 we have

$$\begin{aligned}
\|C_\varphi - \mathcal{T}\|_{Lip_H^\alpha \to \mathcal{Z}_H^\beta} &\succeq \limsup_{i\to\infty} \|C_\varphi G_i\|_{\mathcal{Z}_H^\beta} - \limsup_{i\to\infty} \|\mathcal{T} G_i\|_{\mathcal{Z}_H^\beta} \\
&\geq \limsup_{i\to\infty} (1-|w_i|^2)^\beta \left\{ \left| \frac{\partial^2 (G_i(\varphi(w_i)))}{\partial \zeta^2} \right| + \left| \frac{\partial^2 (G_i(\varphi(w_i)))}{\partial \bar{\zeta}^2} \right| \right\} \\
&= \limsup_{i\to\infty} (1-|w_i|^2)^\beta |\varphi''(w_i)| \left\{ \left| \frac{\partial (G_i)}{\partial \zeta}(\varphi(w_i)) \right| + \left| \frac{\partial (G_i)}{\partial \bar{\zeta}}(\varphi(w_i)) \right| \right\} \\
&\succeq \limsup_{i\to\infty} (1-|w_i|^2)^\beta \frac{|\varphi(w_i)||\varphi''(w_i)|}{(1-|\varphi(w_i)|^2)^{1-\alpha}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|C_\varphi - \mathcal{T}\|_{Lip_H^\alpha \to \mathcal{Z}_H^\beta} &\succeq \limsup_{i\to\infty} \|C_\varphi K_i\|_{\mathcal{Z}_H^\beta} - \limsup_{i\to\infty} \|\mathcal{T} K_i\|_{\mathcal{Z}_H^\beta} \\
&\geq \limsup_{i\to\infty} (1-|w_i|^2)^\beta \left\{ \left| \frac{\partial^2 (K_i(\varphi(w_i)))}{\partial \zeta^2} \right| + \left| \frac{\partial^2 (K_i(\varphi(w_i)))}{\partial \bar{\zeta}^2} \right| \right\} \\
&= \limsup_{i\to\infty} (1-|w_i|^2)^\beta |\varphi'(w_i)| \left\{ \left| \frac{\partial^2 (K_i)}{\partial \zeta^2}(\varphi(w_i)) \right| + \left| \frac{\partial^2 (K_i)}{\partial \bar{\zeta}^2}(\varphi(w_i)) \right| \right\} \\
&\succeq \limsup_{i\to\infty} (1-|w_i|^2)^\beta \frac{|\varphi(w_i)|^2|\varphi'(w_i)|^2}{(1-|\varphi(w_i)|^2)^{2-\alpha}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|C_\varphi\|_{e, Lip_H^\alpha \to \mathcal{Z}_H^\beta} &= \inf_{\mathcal{T}} \|C_\varphi - \mathcal{T}\| \\
&\succeq \limsup_{i\to\infty} (1-|w_i|^2)^\beta \frac{|\varphi(w_i)||\varphi''(w_i)|}{(1-|\varphi(w_i)|^2)^{1-\alpha}} \\
&= \limsup_{|\varphi(w)|\to 1} (1-|w|^2)^\beta \frac{|\varphi(w)||\varphi''(w)|}{(1-|\varphi(w)|^2)^{1-\alpha}} = B_2,
\end{aligned}$$

and

$$\begin{aligned}
\|C_\varphi\|_{e, Lip_H^\alpha \to \mathcal{Z}_H^\beta} &= \inf_{\mathcal{T}} \|C_\varphi - \mathcal{T}\| \\
&\succeq \limsup_{i\to\infty} (1-|w_i|^2)^\beta \frac{|\varphi(w_i)|^2|\varphi'(w_i)|^2}{(1-|\varphi(w_i)|^2)^{2-\alpha}} \\
&= \limsup_{|\varphi(w)|\to 1} (1-|w|^2)^\beta \frac{|\varphi(w)|^2|\varphi'(w)|^2}{(1-|\varphi(w)|^2)^{2-\alpha}} = B_1.
\end{aligned}$$

Hence, we obtain

$$\|C_\varphi\|_{e, Lip_H^\alpha \to \mathcal{Z}_H^\beta} = \inf_{\mathcal{T}} \|C_\varphi - \mathcal{T}\| \succeq \max\{B_1, B_2\}.$$

Secondly, we prove that

$$\|C_\varphi\|_{e, Lip_H^\alpha \to \mathcal{Z}_H^\beta} \preceq \max_{1\leq k\leq 3} \left\{ \limsup_{|b|\to 1} \|C_\varphi F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta} \right\}.$$

For any $0 \leq \delta < 1$, let the operator $\mathcal{T}_\delta : \mathcal{H}ar(\mathbb{D}) \to \mathcal{H}ar(\mathbb{D})$ such that

$$(\mathcal{T}_\delta u)(w) = u_\delta(w) = u(\delta w), \quad u \in \mathcal{H}ar(\mathbb{D}).$$

Without a doubt, $u_\delta \to u$ uniformly on compact subsets of the unit disk as $\delta \to 1$. Moreover, $\mathcal{T}_\delta$ is a compact operator on Lip_H^α and $\|\mathcal{T}_\delta\|_{Lip_H^\alpha \to Lip_H^\alpha} \leq 1$.

For $\{\delta_i\} \subset (0,1)$ a sequence such that $\delta_i \to 1$ as $i \to \infty$. Thus, for all positive integers i, we obtain

$$C_\varphi \mathcal{T}_{\delta_i} : Lip_H^\alpha \to \mathcal{Z}_H^\beta$$

is a compact operator. However, the definition of the essential norm, indicates that

$$\|C_\varphi\|_{e, Lip_H^\alpha \to \mathcal{Z}_H^\beta} \leq \lim_{i\to\infty} \sup \|C_\varphi - C_\varphi \mathcal{T}_{\delta_i}\|_{Lip_H^\alpha \to \mathcal{Z}_H^\beta}. \tag{5.1}$$

Thus, we only need to demonstrate that

$$\limsup_{i\to\infty} \|(C_\varphi - C_\varphi \mathcal{T}_{\delta_i}\|_{Lip_H^\alpha \to \mathcal{Z}_H^\beta} \preceq \max_{1\leq k\leq 3} \Big\{ \limsup_{|b|\to 1} \|C_\varphi F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta} \Big\}.$$

Let $u \in Lip_H^\alpha$ such that $\|u\|_{Lip_H^\alpha} \leq 1$, then

$$\begin{aligned} & \|(C_\varphi - C_\varphi \mathcal{T}_{\delta_i})u\|_{\mathcal{Z}_H^\beta} \\ = \; & |u(\varphi(0)) - u(\delta_i \varphi(0))| \\ & + |\varphi'(0)| \left\{ \left| \frac{\partial (u - u_{\delta_i})}{\partial \zeta}(\varphi(0)) \right| + \left| \frac{\partial (u - u_{\delta_i})}{\partial \bar{\zeta}}(\varphi(0)) \right| \right\} \\ & + \sup_{\zeta\in\mathbb{D}} (1 - |\zeta|^2)^\beta \left\{ \left| \frac{\partial^2[(u - u_{\delta_i}) \circ \varphi(\zeta)]}{\partial \zeta^2} \right| + \left| \frac{\partial^2[(u - u_{\delta_i}) \circ \varphi(\zeta)]}{\partial \bar{\zeta}^2} \right| \right\}. \end{aligned} \tag{5.2}$$

It is obvious that

$$\begin{aligned} & \lim_{i\to\infty} |u(\varphi(0)) - u(\delta_i \varphi(0))| \\ = \; & \lim_{i\to\infty} \left| \frac{\partial (u - u_{\delta_i})}{\partial \zeta}(\varphi(0)) \right| |\varphi'(0)| \\ = \; & \lim_{i\to\infty} \left| \frac{\partial (u - u_{\delta_i})}{\partial \bar{\zeta}}(\varphi(0)) \right| |\varphi'(0)| = 0. \end{aligned} \tag{5.3}$$

Moreover, we consider

$$\begin{aligned} & \limsup_{i\to\infty} (1 - |\zeta|^2)^\beta \left\{ \left| \frac{\partial^2[(u - u_{\delta_i}) \circ \varphi(\zeta)]}{\partial \zeta^2} \right| + \left| \frac{\partial^2[(u - u_{\delta_i}) \circ \varphi(\zeta)]}{\partial \bar{\zeta}^2} \right| \right\} \\ \leq \; & \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|\leq \delta_N} (1 - |\zeta|^2)^\beta \left\{ \left| \frac{\partial^2[(u - u_{\delta_i}) \circ \varphi(\zeta)]}{\partial \zeta^2} \right| + \left| \frac{\partial^2[(u - u_{\delta_i}) \circ \varphi(\zeta)]}{\partial \bar{\zeta}^2} \right| \right\} \\ + \; & \limsup_{i\to\infty} \sup_{|\varphi(\zeta)| > \delta_N} (1 - |\zeta|^2)^\beta \left\{ \left| \frac{\partial^2[(u - u_{\delta_i}) \circ \varphi(\zeta)]}{\partial \zeta^2} \right| + \left| \frac{\partial^2[(u - u_{\delta_i}) \circ \varphi(\zeta)]}{\partial \bar{\zeta}^2} \right| \right\} \\ = \; & I_{\varphi,i} + J_{\varphi,i}. \end{aligned} \tag{5.4}$$

Now let $N \in \mathbb{N}$ be large enough and $\delta_i \geq \frac{1}{2}$, for all $i \geq N$. Then

$$\begin{aligned} I_{\varphi,i} &\leq \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|\leq\delta_N} (1-|\zeta|^2)^\beta |\varphi''(\zeta)| \left\{ \left| \frac{\partial[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\zeta} \right| + \left| \frac{\partial[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\overline{\zeta}} \right| \right\} \\ &+ \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|\leq\delta_N} (1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2 \left\{ \left| \frac{\partial^2[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\zeta^2} \right| + \left| \frac{\partial^2[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\overline{\zeta}^2} \right| \right\}. \end{aligned}$$

Since $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$ is bounded, from Theorem 3.1, we see that

$$\sup_{\zeta\in\mathbb{D}} (1-|\zeta|^2)^\beta |\varphi''(\zeta)| < \infty,$$
$$\sup_{\zeta\in\mathbb{D}} (1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2 < \infty.$$

In addition, since the following limits are uniformly on compact subsets $\mathbb{G} \subset \mathbb{D}$,

$$\lim_{i\to\infty} \delta_i \frac{\partial u_{\delta_i}}{\partial\zeta} = \frac{\partial u}{\partial\zeta}, \quad \lim_{i\to\infty} \delta_i \frac{\partial u_{\delta_i}}{\partial\overline{\zeta}} = \frac{\partial u}{\partial\overline{\zeta}},$$
$$\lim_{i\to\infty} (\delta_i)^2 \frac{\partial^2 u_{\delta_i}}{\partial\zeta^2} = \frac{\partial^2 u}{\partial\zeta^2}, \quad \lim_{i\to\infty} (\delta_i)^2 \frac{\partial^2 u_{\delta_i}}{\partial\overline{\zeta}^2} = \frac{\partial^2 u}{\partial\overline{\zeta}^2}.$$

Then, we have

$$\limsup_{i\to\infty} \sup_{|w|\leq\delta_N} \left\{ \left| \frac{\partial u(w)}{\partial\zeta} - \frac{\partial u_{\delta_i}(w)}{\partial\zeta} \right| + \left| \frac{\partial u(w)}{\partial\overline{\zeta}} - \frac{\partial u_{\delta_i}(w)}{\partial\overline{\zeta}} \right| \right\} = 0,$$
$$\limsup_{i\to\infty} \sup_{|w|\leq\delta_N} \left\{ \left| \frac{\partial^2 u(w)}{\partial\zeta^2} - \frac{\partial^2 u_{\delta_i}(w)}{\partial\zeta^2} \right| + \left| \frac{\partial^2 u(w)}{\partial\overline{\zeta}^2} - \frac{\partial^2 u_{\delta_i}(w)}{\partial\overline{\zeta}^2} \right| \right\} = 0.$$

Hence, by the above equations, we have

$$I_{\varphi,i} = 0. \tag{5.5}$$

Next, considering $|\varphi(\zeta)| > \delta_N$, we obtain

$$\begin{aligned} J_{\varphi,i} &\leq \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)^\beta |\varphi''(\zeta)| \left\{ \left| \frac{\partial[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\zeta} \right| + \left| \frac{\partial[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\overline{\zeta}} \right| \right\} \\ &+ \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2 \left\{ \left| \frac{\partial^2[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\zeta^2} \right| + \left| \frac{\partial^2[(u-u_{\delta_i})(\varphi(\zeta))]}{\partial\overline{\zeta}^2} \right| \right\} \\ &\leq \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)^\beta |\varphi''(\zeta)| \left\{ \left| \frac{\partial u(\varphi(\zeta))}{\partial\zeta} \right| + \left| \frac{\partial u(\varphi(\zeta))}{\partial\overline{\zeta}} \right| \right\} \\ &+ \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)^\beta |\varphi''(\zeta)| \delta_i \left\{ \left| \frac{\partial u(\delta_i\varphi(\zeta))}{\partial\zeta} \right| + \left| \frac{\partial u(\delta_i\varphi(\zeta))}{\partial\overline{\zeta}} \right| \right\} \\ &+ \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2 \left\{ \left| \frac{\partial^2 u(\varphi(\zeta))}{\partial\zeta^2} \right| + \left| \frac{\partial^2 u(\varphi(\zeta))}{\partial\zeta^2} \right| \right\} \\ &+ \limsup_{i\to\infty} \sup_{|\varphi(\zeta)|>\delta_N} (1-|\zeta|^2)^\beta |\varphi'(\zeta)|^2 (\delta_i)^2 \left\{ \left| \frac{\partial^2 u(\delta_i\varphi(\zeta))}{\partial\overline{\zeta}^2} \right| + \left| \frac{\partial^2 u(\delta_i\varphi(\zeta))]}{\partial\overline{\zeta}^2} \right| \right\} \\ &= \sum_{j=1}^4 R_j. \end{aligned}$$

Now we estimate the quantities R_j, where $j = 1, 2, 3$. We define

$$\begin{aligned} G_b(\zeta) &= F^{\alpha}_{b,1}(\zeta) - \frac{5-2\alpha}{3-\alpha}F^{\alpha}_{b,2}(\zeta) + \frac{2-\alpha}{3-\alpha}F^{\alpha}_{b,3}(\zeta), \\ K_b(\zeta) &= F^{\alpha}_{b,1}(\zeta) - 2F^{\alpha}_{b,2}(\zeta) + F^{\alpha}_{b,3}(\zeta). \end{aligned}$$

Because $\|u\|_{Lip^{\alpha}_H} \leq 1$, we have

$$\begin{aligned} & \sup_{|\varphi(\zeta)|>\delta_N}(1-|\zeta|^2)^{\beta}|\varphi''(\zeta)|\left\{\left|\frac{\partial u(\varphi(\zeta))}{\partial \zeta}\right| + \left|\frac{\partial u(\varphi(\zeta))}{\partial \overline{\zeta}}\right|\right\}, \\ \preceq\; & \frac{1}{\delta_N}\|u\|_{Lip^{\alpha}_H}\sup_{|\varphi(\zeta)|>\delta_N}(1-|\zeta|^2)^{\beta}|\varphi''(\zeta)|\frac{(3-\alpha)^{-1}|\varphi(\zeta)|}{(1-|\varphi(\zeta)|^2)^{1-\alpha}} \\ \preceq\; & \sup_{|b|>\delta_N}\|C_{\varphi}G_b\|_{\mathcal{Z}^{\beta}_H} \\ \preceq\; & \sup_{|b|>\delta_N}\|C_{\varphi}F^{\alpha}_{b,1}\|_{\mathcal{Z}^{\beta}_H} + \frac{5-2\alpha}{3-\alpha}\sup_{|b|>\delta_N}\|C_{\varphi}F^{\alpha}_{b,2}\|_{\mathcal{Z}^{\beta}_H} \\ & + \frac{2-\alpha}{3-\alpha}\sup_{|b|>\delta_N}\|C_{\varphi}F^{\alpha}_{b,3}\|_{\mathcal{Z}^{\beta}_H}. \end{aligned} \tag{5.6}$$

Consequently,

$$R_1 \preceq \sum_{k=1}^{3}\limsup_{|b|\to 1}\|C_{\varphi}F^{\alpha}_{b,k}\|_{\mathcal{Z}^{\beta}_H}. \tag{5.7}$$

Similarly, we see that

$$R_2 \preceq \sum_{k=1}^{3}\limsup_{|b|\to 1}\|C_{\varphi}F^{\alpha}_{b,k}\|_{\mathcal{Z}^{\beta}_H}. \tag{5.8}$$

Because $\|u\|_{Lip^{\alpha}_H} \leq 1$, for all $u \in Lip^{\alpha}_H$, we have

$$\begin{aligned} & \sup_{|\varphi(\zeta)|>\delta_N}(1-|\zeta|^2)^{\beta}|\varphi'(\zeta)|^2\left\{\left|\frac{\partial^2 u(\varphi(\zeta))}{\partial \zeta^2}\right| + \left|\frac{\partial^2 u(\varphi(\zeta))}{\partial \zeta^2}\right|\right\}, \\ \preceq\; & \|u\|_{Lip^{\alpha}_H}\sup_{|\varphi(\zeta)|>\delta_N}(1-|\zeta|^2)^{\beta}|\varphi'(\zeta)|^2\frac{2|\varphi(\zeta)|^2}{3(1-|\varphi(\zeta)|^2)^{2-\alpha}} \\ \preceq\; & \sup_{|b|>\delta_N}\|C_{\varphi}K_b\|_{\mathcal{Z}^{\beta}_H} \\ \preceq\; & \sup_{|b|>\delta_N}\|C_{\varphi}F^{\alpha}_{b,1}\|_{\mathcal{Z}^{\beta}_H} + 2\sup_{|b|>\delta_N}\|C_{\varphi}F^{\alpha}_{b,2}\|_{\mathcal{Z}^{\beta}_H} + \sup_{|b|>\delta_N}\|C_{\varphi}F^{\alpha}_{b,3}\|_{\mathcal{Z}^{\beta}_H}. \end{aligned} \tag{5.9}$$

Thus, we obtain

$$R_3 \preceq \sum_{k=1}^{3}\limsup_{|b|\to 1}\|C_{\varphi}F^{\alpha}_{b,k}\|_{\mathcal{Z}^{\beta}_H}. \tag{5.10}$$

Similarly, we see that

$$R_4 \preceq \sum_{k=1}^{3}\limsup_{|b|\to 1}\|C_{\varphi}F^{\alpha}_{b,k}\|_{\mathcal{Z}^{\beta}_H}. \tag{5.11}$$

By the inequalities (5.7)-(5.11), we obtain

$$J_{\varphi,i} \preceq \max_{1\le k\le 3}\Big\{\limsup_{|b|\to 1}\|C_\varphi F^{\alpha}_{b,k}\|_{\mathcal{Z}^{\beta}_{H}}\Big\}. \tag{5.12}$$

Hence, by applying (5.5) and (5.12), we determine that

$$\limsup_{i\to\infty}\|(C_\varphi - C_\varphi\mathcal{T}_{\delta_i}\|_{Lip^{\alpha}_{H}\to\mathcal{Z}^{\beta}_{H}} \preceq \max_{1\le k\le 3}\Big\{\limsup_{|b|\to 1}\|C_\varphi F^{\alpha}_{b,k}\|_{\mathcal{Z}^{\beta}_{H}}\Big\}.$$

Finally, we prove that

$$\|C_\varphi\|_{e,Lip^{\alpha}_{H}\to\mathcal{Z}^{\beta}_{H}} \preceq \max\{B_1, B_2\}.$$

According to the definition of the essential norm, we only need to prove that

$$\limsup_{i\to\infty}\|C_\varphi - C_\varphi\mathcal{T}_{\delta_i}\|_{Lip^{\alpha}_{H}\to\mathcal{Z}^{\beta}_{H}} \preceq \max\{B_1, B_2\}.$$

From (5.6), we see that

$$R_1 \preceq \limsup_{|\varphi(\zeta)|\to 1}(1-|\zeta|^2)^{\beta}|\varphi''(\zeta)|\frac{|\varphi(\zeta)|}{(1-|\varphi(\zeta)|^2)^{1-\alpha}} = B_2. \tag{5.13}$$

Similarly,

$$R_2 \preceq B_2. \tag{5.14}$$

Moreover, for (5.9), we see that

$$R_3 \preceq \limsup_{|\varphi(\zeta)|\to 1}(1-|\zeta|^2)^{\beta}|\varphi'(\zeta)|^2\frac{2|\varphi(\zeta)|^2}{3(1-|\varphi(\zeta)|^2)^{2-\alpha}} = B_1. \tag{5.15}$$

Similarly,

$$R_4 \preceq B_1. \tag{5.16}$$

Hence, by the inequalities (5.13)-(5.16), we obtain

$$\|C_\varphi\|_{e,Lip^{\alpha}_{H}\to\mathcal{Z}^{\beta}_{H}} \preceq \max\{B_1, B_2\}.$$

The proof now is complete. □

Theorem 5.2. *Let $\varphi \in S(\mathbb{D})$ such that $C_\varphi : Lip^{\alpha}_{H} \to \mathcal{Z}^{\beta}_{H}$ is bounded. Then*

$$\|C_\varphi\|_{e,Lip^{\alpha}_{H}\to\mathcal{Z}^{\beta}_{H}} \approx \limsup_{j\to\infty}\|C_\varphi p_j\|_{\mathcal{Z}^{\beta}_{H}}.$$

Proof. First, we prove that

$$\|C_\varphi\|_{e,Lip^{\alpha}_{H}\to\mathcal{Z}^{\beta}_{H}} \succeq \limsup_{j\to\infty}\|C_\varphi p_j\|_{\mathcal{Z}^{\beta}_{H}}.$$

Recall that, the sequence $p_j(w) = j^{-\alpha}(w^j + \overline{w}^j)$, for $w \in \mathbb{D}$ and when $j \geq 0$ is an integer. Then $\|p_j\|_{Lip_H^\alpha} \asymp 1$ and p_j converges uniformly to 0 on compact subsets $\mathbb{G} \subset \mathbb{D}$. Therefore, by Lemma 4.1 we see that

$$\lim_{j\to\infty} \|\mathcal{T} p_j\|_{\mathcal{Z}_H^\beta} = 0.$$

Hence,

$$\|C_\varphi - \mathcal{T}\|_{Lip_H^\alpha \to \mathcal{Z}_H^\beta} \succeq \limsup_{j\to\infty} \|(C_\varphi - \mathcal{T})p_j\|_{\mathcal{Z}_H^\beta} \succeq \limsup_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}.$$

Therefore,

$$\|C_\varphi\|_{e,Lip_H^\alpha \to \mathcal{Z}_H^\beta} \succeq \limsup_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}. \tag{5.17}$$

Next, we prove that

$$\|C_\varphi\|_{e,Lip_H^\alpha \to \mathcal{Z}_H^\beta} \preceq \limsup_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}.$$

Since $C_\varphi : Lip_H^\alpha \to \mathcal{Z}_H^\beta$ is bounded, then by Theorem 3.1

$$L := \sup_{j\geq 0} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta} < \infty.$$

Now consider the test function $F_{b,k}^\alpha$ with $b \in \mathbb{D}$ in (2.6), for $k = 1, 2, 3$. By the linearity of C_φ, for any fixed positive integer $n \geq 2$, we have

$$\begin{aligned}
& \|C_\varphi F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta} \\
\leq \quad & (1 - |b|^2)^k \sum_{j=0}^{\infty} \frac{\Gamma(j + k - \alpha)}{j!\Gamma(k - \alpha)} |b|^j \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta} \\
\leq \quad & (1 - |b|^2)^k \left\{ \left[\sum_{j=k-1}^{n+k-2} + \sum_{j=n+k-1}^{\infty} \right] \frac{\Gamma(j + k - \alpha)}{j!\Gamma(k - \alpha)} |b|^j \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta} \right. \\
\preceq \quad & (1 - |b|^2)^k L + \sup_{j\geq n} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}.
\end{aligned}$$

Then, for every positive integer $n \geq 2$ and $k = 1, 2, 3$, we obtain

$$\begin{aligned}
\limsup_{|b|\to 1} \|C_\varphi F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta} \quad &\leq \quad \sup_{j\geq n} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta} \\
&\preceq \quad \limsup_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}.
\end{aligned}$$

Hence,

$$\max_{1\leq k\leq 3} \left\{ \limsup_{|b|\to 1} \|C_\varphi F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta} \right\} \quad \preceq \quad \limsup_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}.$$

By Theorem 5.1, we obtain

$$\|C_\varphi\|_{e,Lip_H^\alpha \to \mathcal{Z}_H^\beta} \preceq \max_{1\leq k\leq 3} \left\{ \limsup_{|b|\to 1} \|C_\varphi F_{b,k}^\alpha\|_{\mathcal{Z}_H^\beta} \right\} \preceq \sup_{j\to\infty} \|C_\varphi p_j\|_{\mathcal{Z}_H^\beta}. \tag{5.18}$$

By (5.17) and (5.18), we have achieved the desired result. □

6. Conclusions

In this work, an interesting result in harmonic mappings about the operator-theoretic properties of composition operators between harmonic Lipschitz spaces $Lip_H^\alpha, (0 < \alpha < 1)$ and weighted harmonic Zygmund spaces $\mathcal{Z}_H^\beta, (0 < \beta < \infty)$ has been obtained. It is well known that the existing similar results in spaces of analytic functions have been applied many times to the composition operators between $Lip_H^\alpha, (0 < \alpha < 1)$ and weighted harmonic Zygmund spaces $\mathcal{Z}_H^\beta, (0 < \beta < \infty)$. We hope that this study can attract people's attention to the operator theory on harmonic mappings.

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References

[1] M. Aljuaid, M.A. Bakhit, On Characterizations of Weighted Harmonic Bloch Mappings and Its Carleson Measure Criteria, J. Funct. Spaces. 2023 (2023), 8500633. https://doi.org/10.1155/2023/8500633.
[2] M. Aljuaid, M.A. Bakhit, Composition Operators From Harmonic $\mathcal{H}^\infty$ Space Into Harmonic Zygmund Space, AIMS Math. 8 (2023), 23087–23107. https://doi.org/10.3934/math.20231175.
[3] M. Aljuaid, F. Colonna, Characterizations of Bloch-Type Spaces of Harmonic Mappings, J. Funct. Spaces. 2019 (2019), 5687343. https://doi.org/10.1155/2019/5687343.
[4] M. Aljuaid, F. Colonna, Composition Operators on Some Banach Spaces of Harmonic Mappings, J. Funct. Spaces. 2020 (2020), 9034387. https://doi.org/10.1155/2020/9034387.
[5] M. Aljuaid, F. Colonna, On the Harmonic Zygmund Spaces, Bull. Aust. Math. Soc. 101 (2020), 466–476. https://doi.org/10.1017/s0004972720000180.
[6] S. Axler, P. Bourdon, W. Ramey, Harmonic function theory, Springer, New York, 2001. https://doi.org/10.1007/978-1-4757-8137-3.
[7] M.A. Bakhit, Essential Norms of Stević–Sharma Operators from General Banach Spaces into Zygmund-Type Spaces, J. Math. 2022 (2022), 1230127. https://doi.org/10.1155/2022/1230127.
[8] M.A. Bakhit, A. Kamal, On Stevic-Sharma Operators from General Class of Analytic Function Spaces into Zygmund-Type Spaces, J. Funct. Spaces. 2022 (2022), 6467750. https://doi.org/10.1155/2022/6467750.
[9] C. Boyd, P. Rueda, Isometries of Weighted Spaces of Harmonic Functions, Potent. Anal. 29 (2008), 37–48. https://doi.org/10.1007/s11118-008-9086-4.
[10] R.E. Castillo, J.C. Ramos-Fernández, E.M. Rojas, A New Essential Norm Estimate of Composition Operators from Weighted Bloch Space into μ-Bloch Spaces, J. Funct. Spaces. 2013 (2013), 817278. https://doi.org/10.1155/2013/817278.
[11] J.S. Choa, K.J. Izuchi, S. Ohno, Composition Operators on the Space of Bounded Harmonic Functions, Integr. Equ. Oper. Theory. 61 (2008), 167–186. https://doi.org/10.1007/s00020-008-1579-4.
[12] F. Colonna, The Bloch Constant of Bounded Harmonic Mappings, Indiana Univ. Math. J. 38 (1989), 829–840. https://www.jstor.org/stable/24895370.
[13] C. Cowen, B. MacCluer, Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[14] Y. Estaremi, A. Ebadian, S. Esmaeili, Essential norm of composition operators on Harmonic Bloch spaces, Filomat. 36 (2022), 3105–3118. https://doi.org/10.2298/fil2209105e.

[15] E. Jorda, A.M. Zarco, Isomorphisms on Weighed Banach Spaces of Harmonic and Holomorphic Functions, J. Funct. Spaces. 2013 (2013), 178460. https://doi.org/10.1155/2013/178460.

[16] E. Jorda, A.M. Zarco, Weighted Banach spaces of harmonic functions, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM. 108 (2012), 405–418. https://doi.org/10.1007/s13398-012-0109-z.

[17] A. Kamal, S.A. Abd-Elhafeez, M. Hamza Eissa, On Product-Type Operators Between H^∞ and Zygmund Spaces, Appl. Math. Inf. Sci. 16 (2022), 623–633. https://doi.org/10.18576/amis/160416.

[18] J. Laitila, H.O. Tylli, Composition Operators on Vector-valued Harmonic Functions and Cauchy Transforms, Indiana Univ. Math. J. 55 (2006), 719–746. https://www.jstor.org/stable/24902369.

[19] W. Lusky, On Weighted Spaces of Harmonic and Holomorphic Functions, J. London Math. Soc. 51 (1995), 309–320. https://doi.org/10.1112/jlms/51.2.309.

[20] W. Lusky, On the Isomorphism Classes of Weighted Spaces of Harmonic and Holomorphic Functions, Stud. Math. 175 (2006), 19–45. https://eudml.org/doc/285212.

[21] R. Yoneda, A Characterization of the Harmonic Bloch Space and the Harmonic Besov Spaces by an Oscillation, Proc. Edinburgh Math. Soc. 45 (2002), 229–239. https://doi.org/10.1017/s001309159900142x.

[22] C. Zhang, Compact Composition Operators on Besov Spaces on the Unit Ball, Bull. Korean Math. Soc. 60 (2023), 649–657. https://doi.org/10.4134/BKMS.B220137.