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PROPERTIES OF WEIGHTED COMPOSITION OPERATORS ON SOME WEIGHTED HOLOMORPHIC FUNCTION CLASSES IN THE UNIT BALL

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Abstract. In this paper, we introduce $\mathcal{N}_K$-type spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$ by the help of a non-decreasing function $K : (0, \infty) \to [0, \infty)$. Several important properties of these spaces in the unit ball are provided. The results are applied to characterize boundedness and compactness of weighted composition operators $W_{u,\phi}$ from $\mathcal{N}_K(\mathbb{B})$ spaces into Beurling-type classes. We also find the essential norm estimates for $W_{u,\phi}$ from $\mathcal{N}_K(\mathbb{B})$ spaces into Beurling-type classes.

1. Introduction

Through this paper, $\mathbb{B}$ is the unit ball of the n-dimensional complex Euclidean space $\mathbb{C}^n$, $\mathbb{S}$ is the boundary of $\mathbb{B}$. We denote the class of all holomorphic functions, with the compact-open topology on the unit ball $\mathbb{B}$ by $\mathcal{H}(\mathbb{B})$. For any $z = (z_1, z_2, \ldots, z_n), w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$, the inner product is defined by $\langle z, w\rangle = z_1\overline{w}_1 + \ldots + z_n\overline{w}_n$, and write $|z| = \sqrt{\langle z, z\rangle}$.

Two quantities A_f and B_f, both depending on a function $f \in \mathcal{H}(\mathbb{B})$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant M not depending on f, such that

$$\frac{1}{M}B_f \leq A_f \leq MB_f$$

for every $f \in \mathcal{H}(\mathbb{B})$. If the quantities A_f and B_f, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$. As usual, the letter M will denote a positive constant, possibly different on each occurrence.

Given a point $a \in \mathbb{B}$, we can associate wit it the following automorphism $\Phi_a(z) \in Aut(\mathbb{B})$:

$$\Phi_a(z) = \frac{a - P_a(z) - S_aQ_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathbb{B}, \tag{1.1}$$

where $S_a = \sqrt{1 - |a|^2}, P_a(z)$ is the orthogonal projection of $\mathbb{C}^n$ on a subspace $[a]$ generated by a, that is

$$P_a(z) = \begin{cases} 0, & \text{if } a = 0; \\ \frac{a\langle z,a\rangle}{|a|^2}, & \text{if } a \neq 0, \end{cases}$$

and $Q_a = I - P_a$ the projection on orthogonal complement $[a]$ (see, for example,[8] or [10]). The map Φ_a has the following properties that $\Phi_a(0) = a$, $\Phi_a(a) = 0$, $\Phi_a = \Phi_a^{-1}$ and

$$1 - \langle \Phi_a(z), \Phi_a(w)\rangle = \frac{(1 - |a|^2)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)}, \tag{1.2}$$

where z and w are arbitrary points in $\mathbb{B}$. In particular,

$$1 - |\Phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2}. \tag{1.3}$$

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Let V be the Lebesgue volume measure on $\mathbb{C}^n$, normalized so that $V(\mathbb{B}) \equiv 1$ and σ be the normalized surface measure on $\mathbb{S}$, so that $\sigma(\mathbb{B}) \equiv 1$. Let

$$d\tau(z) = \frac{dV(z)}{(1-|z|^2)^{n+1}},$$

which is Möbius invariant, that is for any $\psi \in Aut(\mathbb{B})$, $f \in L^1(\mathbb{B})$, we have

$$\int_{\mathbb{B}} f(z) d\tau(z) = \int_{\mathbb{B}} f \circ \psi(z) d\tau(z).$$

For $a \in \mathbb{B}$, the Möbius invariant Green function in $\mathbb{B}$ denoted by $G(z,a) = g(\Phi_a(z))$ where $g(z)$ is defined by:

$$g(z) = \frac{n+1}{2n} \int_{|z|}^{1} (1-t^2)^{n-1} t^{1-2n} dt. \tag{1.4}$$

Let $H^\infty(\mathbb{B})$ denote the Banach space of bounded functions in $\mathcal{H}(\mathbb{B})$ with the norm

$$\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|.$$

For $\alpha > 0$, the Beurling-type space (sometimes also called the Bers-type space) $H^\infty_\alpha(\mathbb{B})$ in the unit ball $\mathbb{B}$ consists of those functions $f \in \mathcal{H}(\mathbb{B})$ for which

$$\|f\|_{H^\infty_\alpha(\mathbb{B})} = \sup_{z\in\mathbb{B}} |f(z)|(1-|z|)^\alpha < \infty.$$

The Bergman space $A^2(\mathbb{B})$ consists of those functions $f \in \mathcal{H}(\mathbb{B})$ for which

$$\|f\|^2_{A^2(\mathbb{B})} = \int_{\mathbb{B}} |f(z)|^2 dV(z) < \infty.$$

Let $K : (0,\infty) \to [0,\infty)$ be a right-continuous, non-decreasing function and is not equal to zero identically. The $\mathcal{N}_K(\mathbb{B})$ space consists of all functions $f \in \mathcal{H}(\mathbb{B})$ such that

$$\|f\|^2_{\mathcal{N}_K(\mathbb{B})} = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |f(z)|^2 K(G(z,a)) d\tau(z) < \infty.$$

Moreover, $f \in \mathcal{H}(\mathbb{B})$ is said to belong to $\mathcal{N}_{K,0}(\mathbb{B})$ if

$$\lim_{|a|\to 1} \int_{\mathbb{B}} |f(z)|^2 K(G(z,a)) d\tau(z) = 0.$$

Clearly, if $K(t) = t^p$, then $\mathcal{N}_K(\mathbb{B}) = \mathcal{N}_p(\mathbb{B})$; since $G(z,a) \approx (1-|\varphi_a(z)|^2)$ (see [6]). For $K(t) = 1$ it gives the Bergman space $\mathcal{A}^2(\mathbb{B})$. If $\mathcal{N}_K(\mathbb{B})$ consists of just the constant functions, we say that it is trivial. Several important properties of the $\mathcal{N}_K(\mathbb{B})$ spaces in the unit disk in the complex plane have been characterized in [1], [4] and [9]. We assume from now that all $K : (0,\infty) \to [0,\infty)$ to appear in this paper are right-continuous, non-decreasing function and not equal to zero identically.

Given $u \in \mathcal{H}(\mathbb{B})$ and ϕ a holomorphic self-map of $\mathbb{B}$. The weighted composition operator $W_{u,\phi} : \mathcal{H}(\mathbb{B}) \to \mathcal{H}(\mathbb{B})$ is defined by

$$W_{u,\phi}(f)(z) = u(z)(f \circ \phi)(z), \quad z \in \mathbb{B}.$$

It is obvious that $W_{u,\phi}$ can be regarded as a generalization of the multiplication operator $M_u f = u \cdot f$ and composition operator $C_\phi f = f \circ \phi$. Weighted composition operators are a general class of operators and appear naturally in the study of isometries on most of the function spaces. Operators of this kind also appear in many branches of analysis; the theory of dynamical systems, semigroups, the theory of operator algebras, the theory of solubility of equations with deviating argument and so on. The behavior of those operators is studied extensively on various spaces of holomorphic functions (see for example [2], [5], [6] and [9]).

Recall that the pseudohyperbolic metric in the ball is defined as:

$$\rho(z,w) = |\Phi_w(z)|, \quad z \in \mathbb{B}. \tag{1.5}$$

It is true metric (see [3]). Also it is easy to verify, in particular, that $\rho(0,w) = |w|$ and $\rho(\Phi_a(z), w) = |z|$.

The following lemma was proved in [6]:

Lemma 1.1. *For $z, w \in \mathbb{B}$, if $\rho(z,w) \leq \frac{1}{2}$. Then*

$$\frac{1}{6} \leq \frac{1-|z|^2}{1-|w|^2} \leq 6. \tag{1.6}$$

The following proposition was proved as part of Lemma 2.3 in [6], and hence, we omit the details.

Proposition 1.1. *For $z, w \in \mathbb{B}$, if $\rho(z,w) \leq \frac{1}{4}$. Then*

$$|f(z) - f(w)| \leq 2\sqrt{n}|f(\Phi_a(z))|\rho(z,w).$$

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $M > 0$ such that $\|T(f)\|_Y \leq M\|f\|_X$ for all maps $f \in X$. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For a Complex Banach spaces X and Y of $\mathcal{H}(\mathbb{B})$, $T : X \to Y$ is compact (respectively weakly compact) if it maps the closed unit ball of X onto a relatively compact (respectively relatively weakly compact) set in Y.

In this paper, we introduce $\mathcal{N}_K(\mathbb{B})$ spaces, in terms of the right continuous and non-decreasing function $K : (0,\infty) \to [0,\infty)$ on the unit ball $\mathbb{B}$. We prove that $\mathcal{N}_K(\mathbb{B})$ contained in Beurling-type space $H^\infty_\alpha(\mathbb{B}), \alpha = \frac{n+1}{2}$. A sufficient and necessary condition for $\mathcal{N}_K(\mathbb{B})$ non-trivial is given. We discus the nesting property of $\mathcal{N}_K(\mathbb{B})$. We obtain the complete characterizations of the boundedness and compactness of weigted composition operators from $\mathcal{N}_K(\mathbb{B})$ spaces into Beurling-type classes. We also find the essential norm estimates for these operators. Our results contain the results in the unit disk as particular cases (for example [4], [6] and [9]).

2. $\mathcal{N}_K(\mathbb{B})$ SPACES IN THE UNIT BALL

The following results play an important role in the proof of our main result. They also have their own interest.

Proposition 2.1. *Let $K : (0,\infty) \to [0,\infty)$ be non-decreasing function. Then $\mathcal{N}_K(\mathbb{B}) \subset H^\infty_{\frac{n+1}{2}}(\mathbb{B})$.*

Proof. For $a \in \mathbb{B}$, let $\mathbb{B}_{\frac{1}{2}} = \{z \in \mathbb{B} : |z| < \frac{1}{2}\}$. Without loss of generality, assume that $K(\frac{3}{4}) > 0$. If $f \in \mathcal{N}_K(\mathbb{B})$, then

$$\|f\|^2_{\mathcal{N}_K(\mathbb{B})} \geq K(3/4) \int_{\mathbb{B}_{\frac{1}{2}}} |f(z)|^2 d\tau(z).$$

By By the subharmonicity of $|f(z)|^2$ and hence by ([7], Theorem 2.1.4), we have

$$|f(0)|^2 \leq \frac{1}{V(\mathbb{B}_{\frac{1}{2}})} \int_{\mathbb{B}_{\frac{1}{2}}} |f(z)|^2 d\tau(z) = 4^n \int_{\mathbb{B}_{\frac{1}{2}}} |f(z)|^2 d\tau(z).$$

Thus

$$|f(0)|^2 \leq \frac{4^n}{K(3/4)} \|f\|^2_{\mathcal{N}_K(\mathbb{B})}, \qquad f \in \mathcal{N}_K(\mathbb{B}).$$

For every fixed $z \in \mathbb{B}$, we put

$$F(w) = \frac{(1-|z|^2)^{\frac{n+1}{2}}|f(\Phi_a(w))|}{(1-\langle w, z\rangle)^{n+1}}, \quad w \in \mathbb{B}, \tag{2.1}$$

which is clearly $F \in \mathcal{H}(\mathbb{B})$.

We can prove that $\|F\|^2_{\mathcal{N}_K(\mathbb{B})} \leq \|f\|^2_{\mathcal{N}_K(\mathbb{B})}$, and so $F \in \mathcal{N}_K(\mathbb{B})$. Then, we have

$$|f(a)|^2(1-|a|^2)^{n+1} = |F(0)|^2 \leq \frac{4^n}{K(3/4)} \|f\|^2_{\mathcal{N}_K(\mathbb{B})},$$

for all $z \in \mathbb{B}$, which implies that:

$$\|f\|^2_{H^\infty_{\frac{n+1}{2}}(\mathbb{B})} = \sup_{a\in\mathbb{B}} |f(a)|^2(1-|a|^2)^{n+1} \leq \frac{4^n}{K(3/4)} \|f\|^2_{\mathcal{N}_K(\mathbb{B})}.$$

That is, $\mathcal{N}_K(\mathbb{B}) \subset H^\infty_{\frac{n+1}{2}}(\mathbb{B})$.

Proposition 2.2. *For $z, w \in \mathbb{B}$ and $f \in \mathcal{N}_K(\mathbb{B})$, we have*

$$|f(z) - f(w)| \leq M\|f\|_{\mathcal{N}_K(\mathbb{B})} \max\{(1-|z|^2)^{-\frac{n+1}{2}}, (1-|w|^2)^{-\frac{n+1}{2}}\}\rho(z,w). \tag{2.2}$$

Here, $M = \frac{6^{\frac{n+1}{2}} 2^{2(n+1)}\sqrt{n}(3+2\sqrt{3})}{K(3/4)}$.

Proof. We consider two cases:
Case 1: $\rho(z,w) \geq \frac{1}{4}$.
Since $|f(z) - f(w)| \leq |f(z)| + |f(w)|$, by Proposition 2.1, we have

$$\begin{aligned} & \min\{(1-|z|^2)^{\frac{n+1}{2}}, (1-|w|^2)^{\frac{n+1}{2}}\}|f(z) - f(w)| \\ \leq\ & (1-|z|^2)^{\frac{n+1}{2}}|f(z)| + (1-|w|^2)^{\frac{n+1}{2}}|f(w)| \\ \leq\ & 2\|f\|_{H^\infty_{\frac{n+1}{2}}(\mathbb{B})} \leq \frac{2^{2n+1}}{K(3/4)}\|f\|^2_{\mathcal{N}_K(\mathbb{B})} \\ \leq\ & \frac{2^{2n+3}\rho(z,w)}{K(3/4)}\|f\|^2_{\mathcal{N}_K(\mathbb{B})}. \end{aligned}$$

Which implies that

$$|f(z) - f(w)| \leq \frac{2^{2n+3}}{K(3/4)}\|f\|_{\mathcal{N}_K(\mathbb{B})} \max\{(1-|z|^2)^{-\frac{n+1}{2}}, (1-|w|^2)^{-\frac{n+1}{2}}\}\rho(z,w).$$

Case 2: $\rho(z,w) > \frac{1}{4}$.
Take and fix $w \in \mathbb{B}$, from $\rho(\Phi_a(z), w) = |z|$ it follows that if $z \in \overline{\mathbb{B}}_{\frac{1}{2}}$, $(\overline{\mathbb{B}} = \mathbb{B} \cup \mathbb{S})$, then $\rho(\Phi_a(z), w) < \frac{1}{2}$.
In this case, by Proposition 2.1 and Lemma 1.1, we have

$$\begin{aligned} |f(\Phi_w(z))| &\leq \frac{\|f\|_{H^\infty_{\frac{n+1}{2}}(\mathbb{B})}}{(1-|\Phi_w(z)|^2)^{\frac{n+1}{2}}} \\ &\leq \frac{2^{2n+1}}{K(3/4)} \frac{\|f\|^2_{\mathcal{N}_K(\mathbb{B})}}{(1-|\Phi_w(z)|^2)^{\frac{n+1}{2}}} \\ &= \frac{2^{2n+1}}{K(3/4)} \frac{\|f\|^2_{\mathcal{N}_K(\mathbb{B})}}{(1-|w|^2)^{\frac{n+1}{2}}} \left[\frac{1-|w|^2}{1-|\Phi_w(z)|^2}\right]^{\frac{n+1}{2}} \\ &\leq \frac{6^{\frac{n+1}{2}} 2^{2n+1}}{K(3/4)} \frac{\|f\|^2_{\mathcal{N}_K(\mathbb{B})}}{(1-|w|^2)^{\frac{n+1}{2}}}. \end{aligned}$$

By Proposition 1.1, we have

$$|f(z) - f(w)| \leq \frac{6^{\frac{n+1}{2}} 2^{2(n+1)}\sqrt{n}(3+2\sqrt{3})}{K(3/4)} \frac{\|f\|^2_{\mathcal{N}_K(\mathbb{B})}}{(1-|w|^2)^{\frac{n+1}{2}}}\rho(z,w).$$

Combining the results of the two cases yields

$$|f(z) - f(w)| \leq M\|f\|_{\mathcal{N}_K(\mathbb{B})} \max\{(1-|z|^2)^{-\frac{n+1}{2}}, (1-|w|^2)^{-\frac{n+1}{2}}\}\rho(z,w),$$

where $M = \frac{6^{\frac{n+1}{2}} 2^{2(n+1)}\sqrt{n}(3+2\sqrt{3})}{K(3/4)}$.

Lemma 2.1. *For $a \in \mathbb{B}, 0 < \delta < 1$ and $f \in \mathcal{N}_K(\mathbb{B})$, we have*

$$|f(a) - f(\delta a)| \leq \frac{M}{(1-|a|^2)^{\frac{n+1}{2}}}\|f\|^2_{\mathcal{N}_K(\mathbb{B})}. \tag{2.3}$$

Consequently, for any $0 < r < 1$, we have

$$\sup_{|a| \leq r} |f(a) - f(\delta a)| \leq \frac{M}{(1-r^2)^{\frac{n+1}{2}}}\|f\|^2_{\mathcal{N}_K(\mathbb{B})}. \tag{2.4}$$

Here, M is the constant from Proposition 2.2.

Proof. Proposition 2.2 shows that

$$|f(a)-f(\delta a)| \le M\|f\|_{\mathcal{N}_K(\mathbb{B})} \max\{(1-|a|^2)^{-\frac{n+1}{2}}, (1-|\delta a|^2)^{-\frac{n+1}{2}}\}\rho(a,\delta a).$$

The well-known formula

$$1-|\rho(a,\delta a)|^2 = 1-|\Phi_a(\delta a)|^2 = \frac{(1-|a|^2)(1-|\delta a|^2)}{|1-\langle a,\delta a\rangle|^2},$$

together with simple calculations gives

$$\rho(a,\delta a) = \frac{(1-|a|^2)|a|}{1-\delta|a|^2} \le 1.$$

On the other hand, $(1-|\delta a|^2)^{-\frac{n+1}{2}} \le (1-|a|^2)^{-\frac{n+1}{2}}$. The inequalities in (2.3) now follow.

If $|a| \le r$, then $1-\delta|a|^2 \ge 1-r^2$. Taking supremum of (2.3) in a yields (2.4).

Theorem 2.1. *If*

$$\int_0^1 \frac{r^{2n-1}}{(1-r^2)^{n+1}} K(g(r))dr < \infty, \tag{2.5}$$

then $\mathcal{N}_K(\mathbb{B})$ contains all polynomials; otherwise, $\mathcal{N}_K(\mathbb{B})$ contains only constant functions.

Proof. First assume that (2.5) holds. Let $f(z)$ be a polynomial i.e. (there exists a $M>0$ such that $|f(z)|^2 \le M$ for all $z \in \overline{\mathbb{B}}$. Then,

$$\begin{aligned}\int_{\mathbb{B}} |f(z)|^2 K(G(z,a))d\tau(z) &= \int_{\mathbb{B}} |f(\Phi_a(z))|^2 K(g(z)) \frac{dV(z)}{(1-|z|^2)^{n+1}} \\ &= 2n\int_0^1 \frac{r^{2n-1}}{(1-r^2)^{n+1}} K(g(r))dr \int_{\mathbb{S}} |f\circ\varphi_a(r\zeta)|^2 d\sigma(\zeta) \\ &\le 2nM\int_0^1 \frac{r^{2n-1}}{(1-r^2)^{n+1}} K(g(r))dr.\end{aligned}$$

Since a is arbitrary, it follows that

$$\|f\|_{\mathcal{N}_K(\mathbb{B})} \le 2nM\int_0^1 \frac{r^{2n-1}}{(1-r^2)^{n+1}} K(g(r))dr < \infty.$$

Thus, $f \in \mathcal{N}_K(\mathbb{B})$ and the first half of the theorem is proved.

Now, we assume that the integral in (2.5) is divergent. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of non-negative integers, $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n \ge 1$, $f(z) = z^\alpha$. Then, we have $|f(r\zeta)|^2 = r^{2|\alpha|}|\zeta^\alpha|^2$, and

$$\int_{\mathbb{S}} |(r\zeta)^\alpha|^2 d\sigma(r\zeta) \ge r^{2|\alpha|}\frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} \ge Mr^{2|\alpha|}.$$

Thus,

$$\|f\|_{\mathcal{N}_K(\mathbb{B})} \ge \frac{nM}{2^{2|\alpha|-1}}\int_{\frac{1}{2}}^1 \frac{r^{2n-1}}{(1-r^2)^{n+1}} K(g(r))dr. \tag{2.6}$$

There exists $a \in \mathbb{B}$ such that $f(a) \ne 0$, by the subharmonicity of $|f\circ\Phi_a(r\zeta)|^2$,

$$\|f\|_{\mathcal{N}_K(\mathbb{B})} \ge \frac{3n}{2}|f(a)|^2\int_0^{\frac{1}{2}} \frac{r^{2n-1}}{(1-r^2)^{n+1}} K(g(r))dr. \tag{2.7}$$

Combining (2.6) and (2.7), we see that (2.5) implies that $\|f\|_{\mathcal{N}_K(\mathbb{B})} = \infty$. It is proved that $f \notin \mathcal{N}_K(\mathbb{B})$ and, since α is arbitrary, any non-constant polynomial is not contained in $\mathcal{N}_K(\mathbb{B})$. We conclude that $\mathcal{N}_K(\mathbb{B})$ contains only constant functions. The theorem is proved.

Lemma 2.2. *For $w \in \mathbb{B}$ we define the probe function in $\mathcal{N}_K(\mathbb{B})$ as*

$$h_w(z) = \frac{(1-|w|^2)^{n+1}}{(1-\langle z,w\rangle)^{\frac{3}{2}(n+1)}}.$$

Suppose that condition (2.5) is satisfied. Then $h_w \in \mathcal{N}_K(\mathbb{B})$ and $\|h_w\|_{\mathcal{N}_K(\mathbb{B})} \leq 1$.

Proof. Trivially $h_w \in \mathcal{N}_K(\mathbb{B})$. It is also easy to see that

$$\|h_w\|^2_{\mathcal{N}_K(\mathbb{B})} = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \left| \frac{(1-|w|^2)^{n+1}}{(1-\langle z,w\rangle)^{\frac{3}{2}(n+1)}} \right|^2 K(G(z,a))d\tau(z) \leq 1,$$

this by a change of variables and since condition (2.5) is satisfied.

Such h_w is a normalized reproducing kernel function in the Bergman space $A^2(\mathbb{B})$. Also note that

$$h_w(w) = \left(\frac{1}{1-|w|^2}\right)^{\frac{n+1}{2}}, \quad \forall w \in \mathbb{B}.$$

In Section 4, we will discuss the estimation for the lower bounded of $\|W_{u,\phi}\|_e$ We will make use of weakly convergent sequences in the Bergman space $A^2(\mathbb{B})$. The following lemma plays an important role.

Lemma 2.3. *Suppose $\{f_m\}_{m\geq 1} \in A^2(\mathbb{B})$ is a sequence that converges weakly to zero in $A^2(\mathbb{B})$. Then $\{f_m\}_{m\geq 1}$ converges weakly to zero in $\mathcal{N}_K(\mathbb{B})$ as well.*

Proof. Let $\Gamma \in \overline{\mathcal{N}_K}(\mathbb{B})$ be a bounded liner functional on $\mathcal{N}_K(\mathbb{B})$. By the fact that $\|f\|_{\mathcal{N}_K(\mathbb{B})} \leq \|f\|_{A^2(\mathbb{B})}$, then

$$\begin{aligned} \|\Gamma\|_{\overline{A^2}(\mathbb{B})} &= \sup_{f\in A^2(\mathbb{B})} \frac{|\Gamma(f)|}{\|f\|_{A^2(\mathbb{B})}} \leq \sup_{f\in A^2(\mathbb{B})} \frac{|\Gamma(f)|}{\|f\|_{\mathcal{N}_K(\mathbb{B})}} \\ &\leq \sup_{f\in \mathcal{N}_K(\mathbb{B})} \frac{|\Gamma(f)|}{\|f\|_{\mathcal{N}_K(\mathbb{B})}} = \|\Gamma\|_{\overline{\mathcal{N}_K}(\mathbb{B})}, \end{aligned}$$

which implies Γ is also a bounded linear functional on $A^2(\mathbb{B})$. Since $f_m \to 0$ weakly in $A^2(\mathbb{B})$, we conclude that $\Gamma(f_m) \to 0$. Therefore, $f_m \to 0$ weakly in $\mathcal{N}_K(\mathbb{B})$ as well.

Corollary 2.1. *Let $\{w_m\}_{m\in\mathbb{N}} \subset \mathbb{B}$ and $|w_m| \to 1$ as $m \to \infty$, then $\{h_{w_m}\}$ converges weakly to zero in $\mathcal{N}_K(\mathbb{B})$.*

Proof. It is well known that $h_{w_m} \to 0$ weakly in $A^2(\mathbb{B})$ as $m \to \infty$. Indeed, for any $f \in A^2(\mathbb{B})$, using the reproducing property, we have

$$\langle f, h_{w_m}\rangle = (1-|w_m|^2)^{\frac{n+1}{2}} f(w_m),$$

which converges to zero as $m \to \infty$, because the set of polynomials is dense in $A^2(\mathbb{B})$ (see [10], Proposition 2.6). The conclusion of the corollary follows immediately from Lemma 2.3.

3. Weighted composition operators from $\mathcal{N}_K(\mathbb{B})$ into $H^\infty_\alpha(\mathbb{B})$

In this section, we will consider the operator $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$.

Theorem 3.1. *Let $\phi : \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping and $u \in \mathcal{H}(\mathbb{B})$. For $0 < \alpha < \infty$, then $W_{u,\phi} : H^\infty_\alpha(\mathbb{B}) \to \mathcal{N}_K(\mathbb{B})$ is a bounded operator if and only if*

$$\sup_{a\in\mathbb{B}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} < \infty. \tag{3.1}$$

Proof. First assume that condition (3.1) holds, by Proposition 1.1, we have

$$\begin{aligned} \|W_{u,\phi}(f)\|_{H^\infty_\alpha(\mathbb{B})} &= \sup_{z\in\mathbb{B}} |u(z)||f(\phi(z))|(1-|z|^2)^\alpha \\ &\leq \|f\|_{H^\infty_{\frac{n+1}{2}}} \sup_{z\in\mathbb{B}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} \\ &\leq C\|f\|_{\mathcal{N}_K(\mathbb{B})}. \end{aligned}$$

This implies that $W_{u,\phi} : H^\infty_\alpha(\mathbb{B}) \to \mathcal{N}_K(\mathbb{B})$ is a bounded operator.
Conversely, assume that $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$ is bounded, then $\|W_{u,\phi}(f)\|_{H^\infty_\alpha(\mathbb{B})} \leq \|f\|_{\mathcal{N}_K(\mathbb{B})}$. Let h_w be the test function in Lemma 2.2 with $w = \phi(z)$, then we get

$$h_{\phi(z)}(\phi(z)) = \left(\frac{1}{1-|\phi(z)|^2}\right)^{\frac{n+1}{2}}.$$

Hence, there exist a positive constant M such that:

$$M \geq \|h_w\|_{\mathcal{N}_K(\mathbb{B})} \geq \|W_{u,\phi}(h_w)\|_{H^\infty_\alpha(\mathbb{B})} \geq \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}.$$

This completes the proof of the theorem.

Using the standard arguments similar to those outlined in proposition 3.11 of [2], we have the following lemma:

Lemma 3.1. *Let $\phi : \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping and $u \in \mathcal{H}(\mathbb{B})$. For $0 < \alpha < \infty$, then $W_{u,\phi} : H^\infty_\alpha(\mathbb{B}) \to \mathcal{N}_K(\mathbb{B})$ is compact if and only if*

$$\lim_{m\to\infty} \|W_{u,\phi}(f_m)\|_{\mathcal{N}_K(\mathbb{B})} = 0,$$

for every bounded sequence $\{f_m\} \subset \mathcal{N}_K(\mathbb{B})$ which converges to 0 uniformly on any compact subsets of $\mathbb{B}$ as $m \to \infty$.

Theorem 3.2. *Let $\phi : \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping and $u \in \mathcal{H}(\mathbb{B})$. For $0 < \alpha < \infty$, then $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$ is compact if and only if*

$$\lim_{r\to 1^-} \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} = 0. \tag{3.2}$$

Proof. First assume that $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$ is compact, then it is bounded. By Theorem theorem 3.1, we have

$$L = \sup_{a\in\mathbb{B}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} < \infty,$$

Note that $\lim\limits_{r\to 1^-} L(r)$ always exists, where:

$$L(r) = \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}.$$

Now, we show that (3.4) holds.
Assume on the contrary that there exists $\varepsilon_0 > 0$ such that

$$\lim_{r\to 1^-} \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} = \varepsilon_0.$$

There exists an $r_0 \in (0,1)$ such that $r_0 < r < 1$, we have $L(r) > \frac{\varepsilon_0}{2}$. Then, by the standard diagonal process, we can construct a sequence $\{z_m\} \subset \mathbb{B}$ such that $|\phi(z)| \to 1$ as$m \to \infty$, and also for each $m \in \mathbb{N}$,

$$\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} \geq \frac{\varepsilon_0}{4}.$$

Clearly, we can assume that $w_n = \phi(z_m)$ tends to $w_0 \in \partial\mathbb{B}$ as $m \to \infty$. Let $h_{w_m} = \frac{(1-|w_m|^2)^{n+1}}{(1-\langle z_m, w_m\rangle)^{\frac{3}{2}(n+1)}}$ be the function in Lemma 2.2 with $w_n = \phi(z_m)$. Then $h_{w_n} \to h_{w_0}$ with respect to the compact-open topology. Define $f_m = h_{w_m} - h_{w_0}$. Then $\|f_m\|_{\mathcal{N}_K(\mathbb{B})} \leq 1$ and $f_m \to 0$ uniformly on compact subsets of $\mathbb{B}$. Thus, $f_m \circ \phi \to 0$ in $H^\infty_\alpha(\mathbb{B})$ by assumption. But, for m big enough,

$$\|W_{u,\phi}(f_n)\|_{H^\infty_\alpha(\mathbb{B})} \geq \frac{|u(z_m)|(1-|z_m|^2)^\alpha}{(1-|\phi(z_m)|^2)^{\frac{n+1}{2}}} \geq \frac{\varepsilon_0}{4},$$

which is a contradiction.
Conversely, if (3.4) holds, we assume that $\{f_m\}$ is a bounded sequence in $\mathcal{N}_K(\mathbb{B})$ norm which converges

to zero uniformly on every compact subset of $mathbbB$, then for all $\varepsilon > 0$ there exists $\delta \in (0,1)$ and $m_\varepsilon < m$ such that for $\delta < r < 1$, we have

$$\begin{aligned}\|W_{u,\phi}(f_n)\|_{H^\infty_\alpha(\mathbb{B})} &\leq \sup_{|\phi(z)|>r} |u(z)||f_m(\phi(z))|(1-|z|^2)^\alpha \\ &\quad + \sup_{|\phi(z)|\leq r} |f_m(\phi(z))|(1-|z|^2)^\alpha \leq \varepsilon.\end{aligned}$$

From this $\|W_{u,\phi}(f_n)\|_{H^\infty_\alpha(\mathbb{B})} \to 0$ as $m \to \infty$, it follows that $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$ is a compact operator. This completes the proof of the theorem.

As a corollary of Theorems 3.1 and 3.2, we have:

Corollary 3.1. *Let $\phi : \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping and $0 < \alpha < \infty$. Then composition operator $C_\phi : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$*

- *is bounded if and only if*

$$\sup_{z\in\mathbb{B}} \frac{(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} < \infty; \tag{3.3}$$

- *is compact if and only if*

$$\lim_{r\to 1^-} \sup_{|\phi(z)|>r} \frac{(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} = 0. \tag{3.4}$$

4. Essential norms of weighted composition operators from $\mathcal{N}_K(\mathbb{B})$ into $H^\infty_\alpha(\mathbb{B})$

In this section, we study the essential norm of weighted composition operator $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$. Let us denote by $\mathcal{C} := \mathcal{C}(\mathcal{N}_K(\mathbb{B}), H^\infty_\alpha(\mathbb{B}))$ the set of all compact operators acting from $\mathcal{N}_K(\mathbb{B})$ into $H^\infty_\alpha(\mathbb{B})$. Then the essential norm of $W_{u,\phi}$ is defined as follows:

$$\|W_{u,\phi}\|_e = \inf_{\mathcal{O}\in\mathcal{C}} \{\|W_{u,\phi} - \mathcal{O}\|\}. \tag{4.1}$$

Obviously, the essential norm of a compact operator is zero.

Note that by using the standard argument, it can be shown that a composition operator $C_\phi : \mathcal{N}_K(\mathbb{B}) \to \mathcal{N}_K(\mathbb{B})$ is compact if and only if for any bounded sequence $\{f_m\} \subset \mathcal{N}_K(\mathbb{B})$ converging to zero uniformly on every compact subset of $\mathbb{B}$, the sequence $\{\|f_m \circ \phi\|\}$ converges to zero as $m \to \infty$.

Lemma 4.1. *Suppose $\phi : \mathbb{B} \to \mathbb{B}$ is a holomorphic mapping such that $\|\phi\|_\infty < 1$, and $u \in \mathcal{H}(\mathbb{B})$. For $0 < \alpha < \infty$, then $W_{u,\phi} : H^\infty_\alpha(\mathbb{B}) \to \mathcal{N}_K(\mathbb{B})$ is compact if and only if*

$$\lim_{m\to\infty} \|W_{u,\phi}(f_m)\|_{\mathcal{N}_K(\mathbb{B})} = 0,$$

Proof. Let $r = \|\phi\|_\infty$ and take an arbitrary $f \in \mathcal{N}_K(\mathbb{B})$. Then we have

$$\|W_{u,\phi}(f)\|_{\mathcal{N}_K(\mathbb{B})} \leq \|f\circ\phi\|_\infty \|u\|_{\mathcal{N}_K(\mathbb{B})} \leq \left(\sup_{\{z:|z|\leq r\}} |f(z)|\right) \|u\|_{\mathcal{N}_K(\mathbb{B})} < \infty.$$

This show that $W_{u,\phi}$ maps $\mathcal{N}_K(\mathbb{B})$ into itself.

Now suppose that $\{f_m\}$ is a bounded sequence in $\mathcal{N}_K(\mathbb{B})$ that converges to zero uniformly on every compact subset of $\mathbb{B}$. Applying the above estimate with $f = f_m$, we have

$$\|W_{u,\phi}(f_m)\|_{\mathcal{N}_K(\mathbb{B})} \leq \left(\sup_{\{z:|z|\leq r\}} |f_m(z)|\right) \|u\|_{\mathcal{N}_K(\mathbb{B})} < \infty.$$

Since the set $z : |z| \leq r$ is compact, the right-hand side of the last quantity converges to 0 as $m \to \infty$, hence so does the sequence $\{\|W_{u,\phi}(f_m)\|_{\mathcal{N}_K(\mathbb{B})}\}$. This means that $W_{u,\phi}$ is compact.

In the following theorem we formulate and prove an estimate for the upper bound of the essential norm of $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$.

Theorem 4.1. *Let $\phi : \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping and $u \in \mathcal{H}(\mathbb{B})$. For $0 < \alpha < \infty$, suppose that $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$ is a bounded operator. Then*

$$\|W_{u,\phi}\|_e \leq M \lim_{r\to 1^-} \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}, \tag{4.2}$$

where $M = \frac{6^{\frac{n+1}{2}} 2^{2(n+1)} \sqrt{n}(3+2\sqrt{3})}{K(3/4)}$ is the constant from Proposition 2.2.

Proof. Since $W_{u,\phi}$ is bounded, we see that $u \in H^\infty_\alpha(\mathbb{B})$, and Theorems 3.1, 3.2 shows that

$$\lim_{r\to 1^-} \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}$$

exists and is a real number.
First, we prove that for any $r \in [0,1)$,

$$\|W_{u,\phi}\|_e \leq M \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}. \tag{4.3}$$

For each $k \in \mathbb{N}$, set $\phi_k(z) = \frac{kz}{k+1}$ for all $z \in \mathbb{B}$. By Lemma 4.1, C_{ϕ_k} is compact on $\mathcal{N}_K(\mathbb{B})$, and hence, $\mathcal{O} = W_{u,\phi} \circ C_{\phi_k} \in \mathcal{C}$ is compact acting from $\mathcal{N}_K(\mathbb{B})$ into $H^\infty_\alpha(\mathbb{B})$. Then for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \|W_{u,\phi}\|_e &= \inf_{\mathcal{O}\in\mathcal{C}} \{\|W_{u,\phi} - \mathcal{O}\|\} \leq \|W_{u,\phi} - W_{u,\phi} \circ C_{\phi_k}\|_e \\ &= \sup_{\|f\|_{\mathcal{N}_K(\mathbb{B})}\leq 1} \|(W_{u,\phi} - W_{u,\phi} \circ C_{\phi_k})(f)\|_{H^\infty_\alpha(\mathbb{B})}, \end{aligned}$$

which implies that

$$\|W_{u,\phi}\|_e \leq \inf_{k\in\mathbb{N}} \left\{ \sup_{\|f\|_{\mathcal{N}_K(\mathbb{B})}\leq 1} \|(W_{u,\phi} - W_{u,\phi} \circ C_{\phi_k})(f)\|_{H^\infty_\alpha(\mathbb{B})} \right\}. \tag{4.4}$$

For $f \in \mathcal{N}_K(\mathbb{B})$, we estimate

$$\begin{aligned} &\|(W_{u,\phi} - W_{u,\phi} \circ C_{\phi_k})(f)\|_{H^\infty_\alpha(\mathbb{B})} \\ = &\sup_{z\in\mathbb{B}} \left\{ |u(z)| \left| f(\phi(z)) - f\left(\frac{k\phi(z)}{k+1}\right) \right| (1-|z|^2)^\alpha \right\} \\ \leq &\sup_{|\phi(z)|>r} \left\{ |u(z)| \left| f(\phi(z)) - f\left(\frac{k\phi(z)}{k+1}\right) \right| (1-|z|^2)^\alpha \right\} \\ &+ \sup_{|\phi(z)|\leq r} \left\{ |u(z)| \left| f(\phi(z)) - f\left(\frac{k\phi(z)}{k+1}\right) \right| (1-|z|^2)^\alpha \right\}. \end{aligned}$$

On the one hand, by Lemma 2.1, equation (2.3), we have

$$\begin{aligned} &\sup_{|\phi(z)|>r} \left\{ |u(z)| \left| f(\phi(z)) - f\left(\frac{k\phi(z)}{k+1}\right) \right| (1-|z|^2)^\alpha \right\} \\ \leq &\sup_{|\phi(z)|>r} \left| f(\phi(z)) - f\left(\frac{k\phi(z)}{k+1}\right) \right| \sup_{z\in\mathbb{B}} \left\{ |u(z)|(1-|z|^2)^\alpha \right\} \\ \leq &\left(\sup_{|\phi(z)|>r} \frac{M|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} \right) \|f\|_{\mathcal{N}_K(\mathbb{B})}. \end{aligned}$$

On the one hand, by Lemma 2.1, equation (2.4), we have

$$\begin{aligned} &\sup_{|\phi(z)|\leq r} \left\{ |u(z)| \left| f(\phi(z)) - f\left(\frac{k\phi(z)}{k+1}\right) \right| (1-|z|^2)^\alpha \right\} \\ \leq &\left(\frac{Mr\|u\|_{H^\infty_\alpha(\mathbb{B})}}{(k+1)(1-r^2)^\alpha} \right) \|f\|_{\mathcal{N}_K(\mathbb{B})}. \end{aligned}$$

Therefore, if $\|f\|_{\mathcal{N}_K(\mathbb{B})} \leq 1$, then

$$\begin{aligned} & \|(W_{u,\phi} - W_{u,\phi} \circ C_{\phi_k})(f)\|_{H^\infty_\alpha(\mathbb{B})} \\ \leq \ & \sup_{z\in\mathbb{B}} \Big\{ |u(z)| \Big| f(\phi(z)) - f\Big(\frac{k\phi(z)}{k+1}\Big) \Big| (1-|z|^2)^\alpha \Big\} \\ \leq \ & M \Big\{ \Big(\sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} \Big) + \frac{Mr\|u\|_{H^\infty_\alpha(\mathbb{B})}}{(k+1)(1-r^2)^\alpha} \Big\}. \end{aligned}$$

It then follows that

$$\begin{aligned} & \inf_{k\in\mathbb{N}} \Big\{ \sup_{\|f\|_{\mathcal{N}_K(\mathbb{B})}\leq 1} \|(W_{u,\phi} - W_{u,\phi} \circ C_{\phi_k})(f)\|_{H^\infty_\alpha(\mathbb{B})} \Big\} \\ \leq \ & M \inf_{k\in\mathbb{N}} \Big\{ \Big(\sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} \Big) + \frac{Mr\|u\|_{H^\infty_\alpha(\mathbb{B})}}{(k+1)(1-r^2)^\alpha} \Big\} \\ \leq \ & M \Big(\sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} \Big). \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we obtain (4.3).
Now letting $r \to 1$ in (4.3), we arrive at the desired inequality

$$\|W_{u,\phi}\|_e \leq M \lim_{r\to 1} \Big(\sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} \Big). \tag{4.6}$$

This completes the proof of the theorem.

We now discuss the estimation for the lower bound of the essential norm of $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$.

Theorem 4.2. *Let $\phi : \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping and $u \in \mathcal{H}(\mathbb{B})$. For $0 < \alpha < \infty$, suppose that $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B})$ is a bounded operator. Then*

$$\|W_{u,\phi}\|_e \geq \lim_{r\to 1^-} \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}. \tag{4.7}$$

Proof. The case $\|\phi\|_\infty < 1$ is obvious since the right hand side is zero. Now assume that $\|\phi\|_\infty = 1$. For any $r \in (0;1)$, the set $S_r := \{z \in \mathbb{B} : |\phi(z)| > r$ is not empty. For each $z \in \mathbb{B}$, consider the probe function h_w in Lemma 2.2 with $w = \phi(z)$. Then for any compact operator $\mathcal{O} \in \mathcal{C}$ we have

$$\begin{aligned} \|W_{u,\phi} - \mathcal{O}\| &= \sup_{\|f\|_{\mathcal{N}_K(\mathbb{B})}\leq 1} \|(W_{u,\phi} - \mathcal{O})(f)\| H^\infty_\alpha(\mathbb{B}) \\ &\geq \|(W_{u,\phi} - \mathcal{O})(h_{\phi(z)})\|_{H^\infty_\alpha(\mathbb{B})} \\ &\geq \|W_{u,\phi}(h_{\phi(z)})\|_{H^\infty_\alpha(\mathbb{B})} - \|\mathcal{O}(h_{\phi(z)})\|_{H^\infty_\alpha(\mathbb{B})} \\ &\geq \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}} - \|\mathcal{O}(h_{\phi(z)})\|_{H^\infty_\alpha(\mathbb{B})}, \end{aligned}$$

which is equivalent to

$$\|W_{u,\phi} - \mathcal{O}\| + \|\mathcal{O}(h_{\phi(z)})\|_{H^\infty_\alpha(\mathbb{B})} \geq \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}. \tag{4.8}$$

Taking the supremum on z over the set S_r on both sides of (4.8) yields

$$\|W_{u,\phi} - \mathcal{O}\| + \sup_{z\in S_r} \|\mathcal{O}(h_{\phi(z)})\|_{H^\infty_\alpha(\mathbb{B})} \geq \sup_{z\in S_r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}.$$

which is

$$\|W_{u,\phi} - \mathcal{O}\| + \sup_{|\phi(z)|>r} \|\mathcal{O}(h_{\phi(z)})\|_{H^\infty_\alpha(\mathbb{B})} \geq \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}. \tag{4.9}$$

Denote $H(r) = \sup_{|\phi(z)|>r} \|\mathcal{O}(h_{\phi(z)})\|_{H^\infty_\alpha(\mathbb{B})}$. Since $H(r)$ decreases as r increases, $\lim_{r\to 1} H(r)$ exists. We claim that this limit is necessarily zero. For the purpose of obtaining a contradiction, assume that

$\lim_{r\to 1} H(r) = L > 0$. Then there is a sequence $\{z_m\} \subset \mathbb{B}$ satisfying $|\phi(z_m)| \to 1$ as $m \to \infty$, and for each $m \in \mathbb{N}$,

$$\|\mathcal{O}(h_{\phi(z)})\|_{H_\alpha^\infty(\mathbb{B})} \geq \frac{1}{2}L. \tag{4.10}$$

By Corollary 3.1, $\{h_{\phi(z_m)}\}$ converges weakly to zero in $\mathcal{N}_K(\mathbb{B})$. Since $\mathcal{O}$ is compact, we have $\{\|\mathcal{O}(h_{\phi(z)})\|_{H_\alpha^\infty(\mathbb{B})}\}$ converges to zero as $m \to \infty$, which contradicts (4.10). Therefore,

$$\lim_{r\to 1^-} \sup_{|\phi(z)|>r} \|\mathcal{O}(h_{\phi(z)})\|_{H_\alpha^\infty(\mathbb{B})} = 0.$$

Letting $r \to 1^-$ on both sides of (4.9), we conclude that for any compact operator $\mathcal{O} \in \mathcal{C}$,

$$\|W_{u,\phi} - \mathcal{O}\| \geq \lim_{r\to 1^-} \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}.$$

From this, it follows that

$$\|W_{u,\phi}\|_e = \inf_{\mathcal{O}\in\mathcal{C}} \|W_{u,\phi} - \mathcal{O}\| \geq \lim_{r\to 1^-} \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}.$$

This completes the proof of the theorem.

In conclusion, combining Theorems 4.1 and 4.2, we obtain a full description of the essential norm of $W_{u,\phi}$.

Theorem 4.3. *Let $\phi : \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping and $u \in \mathcal{H}(\mathbb{B})$. For $0 < \alpha < \infty$, suppose that $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H_\alpha^\infty(\mathbb{B})$ is a bounded operator. Then*

$$\|W_{u,\phi}\|_e \approx \lim_{r\to 1^-} \sup_{|\phi(z)|>r} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^{\frac{n+1}{2}}}. \tag{4.11}$$

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