International Journal of Analysis and Applications
ISSN 2291-8639
Volume 12, Number 2 (2016), 98-117
http://www.etamaths.com

MIXED PROBLEM WITH AN INTEGRAL TWO-SPACE-VARIABLES CONDITION FOR A THIRD ORDER PARABOLIC EQUATION

OUSSAEIF TAKI EDDINE* AND BOUZIANI ABDELFATAH

Abstract. This paper is concerned with the existence and uniqueness of a strong solution to a mixed problem which combine Dirichlet and integral two space variables conditions for a third order linear parabolic equation. The proof uses a functional analysis method presented, which it is based on an energy inequality and the density of the range of the operator generated by the problem.

1. Introduction

The importance of the problems with integral conditions has been pointed out by Samarskii [25].We remark that integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow, plasma physics and population dynamics.

Problems which combine local and integral condition for second order parabolic equations are investigated by the potentail method by Cannon [10] and Kamynin [19], by Fourier's method by Ionkin [15] and by the energy inequality method in [22] and [2]

Other works for mixed problems which combine local and integral conditions for second order parabolic equations were treated by Batten [23], Cannon-Esteva-van der Hoek [11], Cannon-van der Hoek [12], [13], Cahlon-Kulkarni-Shi [9] and Shi [21].

Recently, problems of this type that have non-linearity in the boundary conditions have been investigated in Jones et al. [16] and Jumahron-McKee [17], [18]. Mixed problems with only integral conditions for a second order parabolic equation have been studied by Bouziani-Benouar [7], and for a $2m$-parabolic equation in Bouziani [5].

Mixed problems with integral conditions for a third order parabolic equation have been studied by Bouziani-Benouar [1].The present paper is devoted to study the existence and the uniqueness for a strong solution of mixed problems with new integral conditions for a third order parabolic equation.

2. Formulation of the problem

In the rectangle $\Omega = (0,1) \times (0,T)$, with $T < \infty$, we consider the third order linear parabolic equation:

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t). \tag{1}$$

which can be considered as a generalization on the linearized Kortweg-de Vries equation, see for instance [24].

Condition 1. The coefficient $a(x,t)$ is a real-valued function belonging to $C^2\left(\overline{\Omega}\right)$ such that

$$c_0 \leq a(x,t) \leq c_1,\ \frac{\partial a(x,t)}{\partial t} \leq c_2.$$

In *Condition 1* and in the rest of the paper, c_i, $i = 1, ..., 6$, denote strictly positive constants.

We adjoin to (2.1) the initial condition

$$\ell u = u(x,0) = \phi\left(x\right), \qquad x \in \left(0,1\right), \tag{2}$$

2010 *Mathematics Subject Classification.* 35B45, 35D35, 35B30, 35K25, 35K35.

Key words and phrases. strong solution; integral condition; mixed problem.

with the conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=i} = 0, \text{ for } i = \{0,\alpha,\beta,1\},$$

(3)
$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=1} = 0.$$

and with integral conditions

(4)
$$\int_0^\alpha u(x,t)\,dx + \int_\beta^1 u(x,t)\,dx = 0 \qquad t\in(0,T),$$

(5)
$$\int_0^\alpha xu(x,t)\,dx + \int_\beta^1 xu(x,t)\,dx = 0 \qquad t\in(0,T),$$

where ϕ is a known function and $0<\alpha<\beta<1$, $\alpha+\beta=1$.

Condition 2. We shall assume that the function ϕ satisfies a compatibility conditions with (2.3) – (2.5).

Problem (2.1) – (2.5) arises, for instance, from the heat transfer theory. In this case, u is a temperature of a slab $0<x<1$, and the integrals in the conditions (2.4) and (2.5) are considered as the average and the weighted average temperature.

In this paper, we prove the existence and the uniqueness for a strong solution of the problem (2.1) – (2.5) as a solution of the operator equation

(6)
$$Lu = \mathcal{F}$$

where $L=(\mathcal{L},\ell)$, with domain of difinition $D(L)$ consisting of functions $u\in L^2(\Omega)$ such that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^3 u}{\partial x^3}$, $\frac{\partial^3 u}{\partial t\partial x^2}\in L^2(\Omega)$ and u satisfies conditions (2.3) – (2.5); the operator L is considered from B to F, where B is the Banach space consisting of all functions $u(x,t)$ having a finite norm

$$\|u\|_B^2 = \int_0^T\int_0^\alpha\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt + \int_0^T\int_\beta^1\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt$$
$$+\sup_{0\le\tau\le T}\left(\int_0^\alpha (5-x)\left(\frac{\partial u(x,\tau)}{\partial x}\right)^2 dx + \int_\beta^1 (\frac{5}{4}-x)\left(\frac{\partial u(x,\tau)}{\partial x}\right)^2 dx + \int_\alpha^\beta (\beta-\alpha)\left(\frac{\partial u(x,\tau)}{\partial x}\right)^2 dx\right)$$

and satisfying the conditions (2.3) – (2.5), and F is the Hilbert space consisting of all elements $\mathcal{F}=(f,\phi)$ for which the norm

$$\|\mathcal{F}\|_F^2 = \int_\Omega f^2 dxdt + \int_0^1\left(\frac{\partial\phi}{\partial x}\right)^2 dx$$

is finite. Then, we establish an energetic inequality:

(7)
$$\|u\|_B \le c\,\|Lu\|_F$$

and we show that the operator L has a closure $\overline{L}$.

Definition. A solution of the operator equation

$$\overline{L}u = \mathcal{F}$$

is called a strong solution of the problem (2.1) – (2.5).

Since points of the graph $\overline{L}$ are limits of sequences of points of the graph of L, we can extend (2.7) to apply to strong solution by taking limits, i.e.,

$$\|u\|_B \le c\left\|\overline{L}u\right\|_F, \qquad \forall u\in D(\overline{L}).$$

From this inequality we obtain the uniqueness of a strong solution if it exists, and the the equality of sets $R(\overline{L})$ and $\overline{R(L)}$. Thus , proving that the set $R(L)$ is dense in F.

3. An energety inequality and its consequences

Theorem 1. *Let* Condition 1 *be fulfilled. Then for any function* $u \in D(L)$ *we have the inequality*

$$\|u\|_B \leq c\,\|Lu\|_F \tag{8}$$

where c *is a positive constant independent of* u.

Proof.
Multiplying the equation (2.1) by the following Mu :

$$Mu = \begin{cases} Mu_1 = 4\int_x^\alpha \frac{\partial u}{\partial t} d\xi - \int_x^\alpha \left(\int_\xi^\alpha \frac{\partial u}{\partial t} d\eta - (1-\xi)\frac{\partial u}{\partial t} \right) d\xi & 0 \leq x \leq \alpha \\ Mu_2 = (x-\alpha)\int_x^\beta \frac{\partial u}{\partial t} d\xi + (\beta - x)\int_\alpha^x \frac{\partial u}{\partial t} d\xi & \alpha \leq x \leq \beta \\ Mu_3 = -\frac{1}{4}\int_\beta^x \frac{\partial u}{\partial t} d\xi - \int_\beta^x \left(\int_\beta^\xi \frac{\partial u}{\partial t} d\eta + (1-\xi)\frac{\partial u}{\partial t} \right) d\xi & \beta \leq x \leq 1 \end{cases}$$

and integrating over Ω^τ,where $\Omega^\tau = (0,1) \times (0,\tau)$,
1) on the interval $[0,\alpha]$, we denote $\Omega_\alpha^\tau = \Omega_\alpha = (0,\alpha)\times(0,\tau)$, we get

$$\begin{aligned} & \int_{\Omega_\alpha} \mathcal{L}u.Mu_1 dxdt \\ = & \int_{\Omega_\alpha} \frac{\partial u}{\partial t}.\left(4\int_x^\alpha \frac{\partial u}{\partial t} d\xi - \int_x^\alpha \left(\int_\xi^\alpha \frac{\partial u}{\partial t} d\eta - (1-\xi)\frac{\partial u}{\partial t}\right) d\xi \right) dxdt \\ & - \int_{\Omega_\alpha} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(4\int_x^\alpha \frac{\partial u}{\partial t} d\xi\right) dxdt \\ & + \int_{\Omega_\alpha} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(\int_x^\alpha \left(\int_\xi^\alpha \frac{\partial u}{\partial t} d\eta - (1-\xi)\frac{\partial u}{\partial t}\right) d\xi\right) dxdt \\ = & \int_{\Omega_\alpha} f.Mu_1 dxdt. \end{aligned} \tag{9}$$

Integration by parts each term of (3.2) with use the conditions (2.2) – (2.5), we obtain

$$\begin{aligned} & \int_{\Omega_\alpha} \frac{\partial u}{\partial t}.\left(4\int_x^\alpha \frac{\partial u}{\partial t} d\xi - \int_x^\alpha \left(\int_\xi^\alpha \frac{\partial u}{\partial t} d\eta - (1-\xi)\frac{\partial u}{\partial t}\right) d\xi \right) dxdt \\ = & \frac{5}{2}\int_0^T \left(\int_0^\alpha \frac{\partial u}{\partial t} dx\right)^2 dt + \frac{3}{2}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t} d\xi\right)^2 dxdt \\ & -2\int_0^T \left(\left(\int_0^\alpha \frac{\partial u}{\partial t} dx\right)\left(\int_0^\alpha x\frac{\partial u}{\partial t} dx\right) dt\right), \end{aligned} \tag{10}$$

$$\begin{aligned} & -\int_{\Omega_\alpha} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(4\int_x^\alpha \frac{\partial u}{\partial t} d\xi - \int_x^\alpha \left(\int_\xi^\alpha \frac{\partial u}{\partial t} d\eta - (1-\xi)\frac{\partial u}{\partial t}\right) d\xi\right) \\ = & \int_{\Omega_\alpha} (5-x)a(x,t)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t} dxdt - \int_0^T \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(4\int_x^\alpha \frac{\partial u}{\partial t} d\xi\right)\Big|_{x=0}^{x=\alpha} dt \\ & + \int_0^T \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(\int_x^\alpha \left[\int_\xi^\alpha \frac{\partial u}{\partial t} d\eta - (1-\xi)\frac{\partial u}{\partial t}\right]\right)\Big|_{x=0}^{x=\alpha} dt \\ = & -\frac{1}{2}\int_{\Omega_\alpha} (5-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt - \frac{1}{2}\int_0^\alpha (5-x)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx \\ & + \frac{1}{2}\int_0^\alpha (5-x)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx. \end{aligned} \tag{11}$$

$$\begin{aligned}\int_{\Omega_\alpha} f.Mu_1 dxdt &= \int_{\Omega_\alpha} f.\left(4\int_x^\alpha \frac{\partial u}{\partial t}d\xi - \int_x^\alpha \left(\int_\xi^\alpha \frac{\partial u}{\partial t}d\eta - (1-\xi)\frac{\partial u}{\partial t}\right)d\xi\right)dxdt \\ &= \int_{\Omega_\alpha} f.\left(4\int_x^\alpha \frac{\partial u}{\partial t}\right)dxdt + \int_{\Omega_\alpha} f.\left((1-\xi)\int_x^\alpha \frac{\partial u}{\partial t}\right)dxdt \\ &\quad -2\int_{\Omega_\alpha} f.\left(\int_x^\alpha \int_\zeta^\alpha \frac{\partial u}{\partial t}d\eta d\xi\right)dxdt,\end{aligned}$$

where

$$2\int_{\Omega_\alpha} f.\left(\int_x^\alpha \int_\xi^\alpha \frac{\partial u}{\partial t}d\xi\right)dxdt = -2\int_0^T \left(\int_0^\alpha x\frac{\partial u}{\partial t}dx\right)\left(\int_0^\alpha fdx\right)dt + 2\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)\left(\int_x^\alpha fd\xi\right)dxdt.$$

By virtue of the Cauchy inequality and with ε,

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad a, b \in \mathbb{R}$$

We obtain

$$\begin{aligned}\int_{\Omega_\alpha} f.Mu_1 dxdt \leq\; & 4\frac{\varepsilon_1}{2}\int_{\Omega_\alpha} f^2 dxdt + \frac{4}{2\varepsilon_1}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt \\ & + \frac{1}{2\varepsilon_2}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt + \frac{\varepsilon_2}{2}\int_{\Omega_\alpha}(1-x)^2 f^2 dxdt \\ & + \frac{1}{\varepsilon_3}\int_0^T\left(\int_0^\alpha x\frac{\partial u}{\partial t}dx\right)^2 dt + \varepsilon_3\int_0^T\left(\int_0^\alpha fdx\right)^2 dt \\ & + \varepsilon_4\int_{\Omega_\alpha}\left(\int_x^\alpha fd\xi\right)^2 dxdt + \frac{1}{\varepsilon_4}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt,\end{aligned}$$

where

$$\begin{aligned}\int_{\Omega_\alpha}\left(\int_x^\alpha fd\xi\right)^2 dxdt &\leq 4\int_{\Omega_\alpha}(1-x)^2 f^2 dxdt + 2\int_0^T\left(\int_0^\alpha fdx\right)^2 dt \\ &\leq 4\int_{\Omega_\alpha} f^2 dxdt + 2\int_0^T\left(\int_0^\alpha fdx\right)^2 dt\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\int_{\Omega_\alpha} f.Mu_1 dxdt \\ \leq\; & 2\varepsilon_1\int_{\Omega_\alpha} f^2 dxdt + \frac{2}{\varepsilon_1}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt \\ & + \frac{1}{2\varepsilon_2}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt + \frac{\varepsilon_2}{2}\int_{\Omega_\alpha} f^2 dxdt \\ & + \varepsilon_3\int_0^T\left(\int_0^\alpha fdx\right)^2 dt + \frac{1}{\varepsilon_3}\int_0^T\left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)^2 dt \\ & + 4\varepsilon_4\int_{\Omega_\alpha} f^2 dxdt + 2\varepsilon_4\int_0^T\left(\int_0^\alpha fdx\right)^2 dt \\ & + \frac{1}{\varepsilon_4}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt.\end{aligned} \tag{12}$$

2) on the interval $[\beta,1]$, we denote $\Omega_\beta^\tau=\Omega_\beta=(\beta,1)\times(0,\tau)$, we get

$$
\begin{aligned}
(13)\qquad & \int_{\Omega_\beta}\mathcal{L}u.Mu_3 \\
= & \int_{\Omega_\beta}\frac{\partial u}{\partial t}.\left(-\frac{1}{4}\int_\beta^x\frac{\partial u}{\partial t}d\xi-\int_\beta^x\left(\int_\beta^\xi\frac{\partial u}{\partial t}d\eta+(1-\xi)\frac{\partial u}{\partial t}\right)d\xi\right)dxdt \\
& -\int_{\Omega_\beta}\frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(-\frac{1}{4}\int_\beta^x\frac{\partial u}{\partial t}d\xi\right)dxdt \\
& +\int_{\Omega_\beta}\frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(\int_\beta^x\left(\int_\beta^\xi\frac{\partial u}{\partial t}d\eta+(1-\xi)\frac{\partial u}{\partial t}\right)d\xi\right)dxdt \\
= & \int_{\Omega_\beta}f.Mu_3dxdt.
\end{aligned}
$$

Integration by parts each term of (3.6) with use the conditions (2.2) – (2.5), we obtain

$$
\begin{aligned}
(14)\qquad & \int_{\Omega_\beta}\frac{\partial u}{\partial t}.\left(-\frac{1}{4}\int_\beta^x\frac{\partial u}{\partial t}d\xi-\int_\beta^x\left(\int_\beta^\xi\frac{\partial u}{\partial t}d\eta+(1-\xi)\frac{\partial u}{\partial t}\right)d\xi\right)dxdt \\
= & \frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x\frac{\partial u}{\partial t}d\eta d\xi\right)^2dxdt-\frac{1}{8}\int_0^T\left(\int_\beta^1\frac{\partial u}{\partial t}dx\right)^2dt \\
& -\int_0^T\left(\left(\int_\beta^1\frac{\partial u}{\partial t}dx\right)\left(2\int_\beta^1\frac{\partial u}{\partial t}dx-2\int_\beta^1x\frac{\partial u}{\partial t}dx\right)\right)dt \\
= & \frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x\frac{\partial u}{\partial t}d\eta d\xi\right)^2dxdt-\frac{17}{8}\int_0^T\left(\int_\beta^1\frac{\partial u}{\partial t}dx\right)^2dt \\
& +2\int_0^T\left(\int_\beta^1\frac{\partial u}{\partial t}dx\right)\left(\int_\beta^1x\frac{\partial u}{\partial t}dx\right)dt
\end{aligned}
$$

$$
\begin{aligned}
(15)\qquad & -\int_{\Omega_\beta}\frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(\int_\beta^x\left(\int_\beta^\xi\frac{\partial u}{\partial t}d\eta+(1-\xi)\frac{\partial u}{\partial t}\right)d\xi\right)dxdt \\
& +\int_{\Omega_\beta}\frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(\int_\beta^x\left(\int_\beta^\xi\frac{\partial u}{\partial t}d\eta+(1-\xi)\frac{\partial u}{\partial t}\right)d\xi\right)dxdt \\
= & \int_{\Omega_\beta}(\frac{5}{4}-x)a(x,t)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dxdt \\
& -\int_0^T\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left(-\frac{1}{4}\int_\beta^x\frac{\partial u}{\partial t}d\xi\right)\Big|_{x=\beta}^{x=1}dt \\
& +\int_0^T\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right).\left(\int_\beta^x\left(\int_\beta^\xi\frac{\partial u}{\partial t}d\xi+(1-\xi)\frac{\partial u}{\partial t}\right)d\xi\right)\Big|_{x=\beta}^{x=1}dt \\
= & -\frac{1}{2}\int_{\Omega_\beta}(\frac{5}{4}-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2dxdt-\frac{1}{2}\int_\beta^1(\frac{5}{4}-x)a(x,0)\left(\frac{\partial\phi}{\partial x}\right)^2dx \\
& +\frac{1}{2}\int_\beta^1(\frac{5}{4}-x)a(x,t)\left(\frac{\partial u}{\partial x}\right)^2dx\ ,
\end{aligned}
$$

$$\begin{aligned}\int_{\Omega_\beta} f.Mu_3dxdt &= \int_{\Omega_\beta} f.\left(-\frac{1}{4}\int_\beta^x \frac{\partial u}{\partial t}d\xi - \int_\beta^x \left(\int_\beta^\xi \frac{\partial u}{\partial t}d\eta + (1-\xi)\frac{\partial u}{\partial t}\right)d\xi\right)dxdt \\ &= \int_{\Omega_\beta} f.\left(-\frac{1}{4}\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)dxdt - \int_{\Omega_\beta} f.\left((1-x)\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)dxdt \\ &\quad -2\int_{\Omega_\beta} f.\left(\int_\beta^x\int_\beta^\xi \frac{\partial u}{\partial t}d\eta d\xi\right)dxdt,\end{aligned}$$

where

$$\begin{aligned}-2\int_{\Omega_\beta} f.\left(\int_\beta^x\int_\beta^\xi \frac{\partial u}{\partial t}d\eta d\xi\right)dxdt &= -2\int_0^T\left(\int_\beta^1 fdx\right)\left(\int_\beta^1 \frac{\partial u}{\partial t}dx - \int_\beta^1 x\frac{\partial u}{\partial t}dx\right)dt \\ &\quad +2\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)\left(\int_\beta^x fd\xi\right)dxdt.\end{aligned}$$

By virtue of the Cauchy's ε-inequality, we obtain

$$\begin{aligned}\int_{\Omega_\beta} f.Mu_3dxdt &\leq \frac{\varepsilon_5}{8}\int_{\Omega_\beta} f^2dxdt + \frac{1}{8\varepsilon_5}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2dxdt \\ &\quad +\frac{\varepsilon_6}{2}\int_{\Omega_\beta}(1-x)^2f^2dxdt + \frac{1}{2\varepsilon_6}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2dxdt \\ &\quad +\varepsilon_7\int_0^T\left(\int_\beta^1 fdx\right)^2dt + \frac{1}{\varepsilon_7}\int_0^T\left(\int_\beta^1 \frac{\partial u}{\partial t}dx\right)^2dt \\ &\quad +\varepsilon_8\int_0^T\left(\int_\beta^1 fdx\right)^2dt + \frac{1}{\varepsilon_8}\int_0^T\left(\int_\beta^1 x\frac{\partial u}{\partial t}dx\right)^2dt \\ &\quad +\frac{1}{\varepsilon_9}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2dxdt + \varepsilon_9\int_{\Omega_\beta}\left(\int_\beta^x fd\xi\right)^2dxdt,\end{aligned}$$

where

$$\int_{\Omega_\beta}\left(\int_\beta^x fd\xi\right)^2dxdt \leq 4\int_{\Omega_\beta}(x-\beta)^2f^2dxdt \leq 4\int_{\Omega_\beta} f^2dxdt,$$

Then, we obtain

$$\begin{aligned}&\int_{\Omega_\beta} f.Mu_3dxdt \\ \leq\ &\frac{\varepsilon_5}{8}\int_{\Omega_\beta} f^2dxdt + \frac{1}{8\varepsilon_5}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2dxdt \\ &+\frac{\varepsilon_6}{2}\int_{\Omega_\beta} f^2dxdt + \frac{1}{2\varepsilon_6}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2dxdt \\ &+2\varepsilon_{10}\int_0^T\left(\int_\beta^1 fdx\right)^2dt + \frac{2}{\varepsilon_{10}}\int_0^T\left(\int_\beta^1 \frac{\partial u}{\partial t}dx\right)^2dt \\ &+\frac{1}{\varepsilon_9}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2dxdt + 4\varepsilon_9\int_{\Omega_\beta} f^2dxdt,\end{aligned}\tag{16}$$

where $\varepsilon_{10} = \varepsilon_7 + \varepsilon_8$.

3) on the interval $[\alpha,\beta]$, we denote $\Omega^{\tau}_{\alpha,\beta}=\Omega_{\alpha,\beta}=(\alpha,\beta)\times(0,\tau)$, we get

$$
\begin{aligned}
(17)\qquad & \int_{\Omega_{\alpha,\beta}} \mathcal{L}u.Mu_2 dxdt \\
= & \int_{\Omega_{\alpha,\beta}} \frac{\partial u}{\partial t}\left((x-\alpha)\int_x^\beta \frac{\partial u}{\partial t}d\xi+(\beta-x)\int_\alpha^x \frac{\partial u}{\partial t}d\xi\right)dxdt \\
& -\int_{\Omega_{\alpha,\beta}} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left((x-\alpha)\int_x^\beta \frac{\partial u}{\partial t}d\xi+(\beta-x)\int_\alpha^x \frac{\partial u}{\partial t}d\xi\right)dxdt \\
= & \int_{\Omega_{\alpha,\beta}} f.Mu_2 dxdt.
\end{aligned}
$$

Integration by parts each term of (3.10) with use the conditions $(2.2)-(2.5)$, we obtain

$$
\begin{aligned}
(18)\qquad & \int_{\Omega_{\alpha,\beta}} \frac{\partial u}{\partial t}\left((x-\alpha)\int_x^\beta \frac{\partial u}{\partial t}d\xi+(\beta-x)\int_\alpha^x \frac{\partial u}{\partial t}d\xi\right)dxdt \\
= & \frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_x^\beta \frac{\partial u}{\partial t}d\xi\right)^2.
\end{aligned}
$$

$$
\begin{aligned}
(19)\qquad & -\int_{\Omega_{\alpha,\beta}} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left((x-\alpha)\int_x^\beta \frac{\partial u}{\partial t}d\xi+(\beta-x)\int_\alpha^x \frac{\partial u}{\partial t}d\xi\right)dxdt \\
= & -\int_0^T \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\left((x-\alpha)\int_x^\beta \frac{\partial u}{\partial t}d\xi+(\beta-x)\int_\alpha^x \frac{\partial u}{\partial t}d\xi\right)\Bigg|_{x=\alpha}^{x=\beta}dt \\
& -\int_0^T \left(a(x,t)\frac{\partial u}{\partial x}\right)\left(\int_\alpha^x \frac{\partial u}{\partial t}d\xi+(\beta-x)\frac{\partial u}{\partial t}\right)\Bigg|_{x=\alpha}^{x=\beta}dt \\
& +\int_{\Omega_{\alpha,\beta}}(\beta-x)a(x,t)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dxdt \\
& -\int_0^T \left(a(x,t)\frac{\partial u}{\partial x}\right)\left(\int_x^\beta \frac{\partial u}{\partial t}d\xi-(x-\alpha)\frac{\partial u}{\partial t}\right)\Bigg|_{x=\alpha}^{x=\beta}dt \\
& +\int_{\Omega_{\alpha,\beta}}(x-\alpha)a(x,t)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dxdt \\
= & -\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(\beta-\alpha)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt \\
& -\frac{1}{2}\int_\alpha^\beta(\beta-\alpha)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx+\frac{1}{2}\int_\alpha^\beta(\beta-\alpha)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
(20)\qquad & \int_{\Omega_{\alpha,\beta}} f.Mu_2 dxdt \\
= & \int_{\Omega_{\alpha,\beta}} f.\left((x-\alpha)\int_x^\beta \frac{\partial u}{\partial t}d\xi+(\beta-x)\int_\alpha^x \frac{\partial u}{\partial t}d\xi\right)dxdt \\
\leq & \left(\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_x^\beta \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(x-\alpha)^2 f^2 dxdt\right) \\
& +\left(\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^x \frac{\partial u}{\partial t}\right)^2 dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(\beta-x)^2 f^2 dxdt\right).
\end{aligned}
$$

Putting (3.3), (3.4) and (3.5) into (3.2), on Ω_α,

$$\int_{\Omega_\alpha} \mathcal{L}u.Mu_1 dxdt = \int_{\Omega_\alpha} f.Mu_1 dxdt$$

We obtain

$$\begin{aligned}
& \frac{5}{2}\int_0^T \left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)^2 dt + \frac{3}{2}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt - 2\int_0^T \left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)\left(\int_0^\alpha x\frac{\partial u}{\partial t}dx\right)dt \\
& -\frac{1}{2}\int_{\Omega_\alpha}(5-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt - \frac{1}{2}\int_0^\alpha (5-x)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx \\
& +\frac{1}{2}\int_0^\alpha \left(\frac{\partial u}{\partial x}\right)^2 a(x,\tau)(5-x)dx \\
\leq\ & 2\varepsilon_1\int_{\Omega_\alpha} f^2 dxdt + \frac{2}{\varepsilon_1}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt + \frac{1}{2\varepsilon_2}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt + \frac{\varepsilon_2}{2}\int_{\Omega_\alpha} f^2 dxdt \\
& +\varepsilon_3\int_0^T\left(\int_0^\alpha f dx\right)^2 dt + \frac{1}{\varepsilon_3}\int_0^T \left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)^2 dt \\
& +4\varepsilon_4\int_{\Omega_\alpha} f^2 dxdt + 2\varepsilon_4\int_0^T\left(\int_0^\alpha f dx\right)^2 dt + \frac{1}{\varepsilon_4}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\ ,
\end{aligned}$$

Then

$$\begin{aligned}
& \frac{5}{2}\int_0^T \left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)^2 dt + \frac{3}{2}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt \\
& -2\int_0^T \left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)\left(\int_0^\alpha x\frac{\partial u}{\partial t}dx\right)dt \\
& -\frac{1}{2}\int_{\Omega_\alpha}(5-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt - \frac{1}{2}\int_0^\alpha (5-x)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx \\
& +\frac{1}{2}\int_0^\alpha (5-x)a(x,t)\left(\frac{\partial u}{\partial x}\right)^2 dx \\
\leq\ & \left(2\varepsilon_1 + \frac{\varepsilon_2}{2} + 4\varepsilon_4\right)\int_{\Omega_\alpha} f^2 dxdt \\
& + \left(\frac{2}{\varepsilon_1} + \frac{1}{2\varepsilon_2} + \frac{1}{\varepsilon_4}\right)\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt \\
& + (\varepsilon_3 + 2\varepsilon_4)\int_0^T\left(\int_0^\alpha f dx\right)^2 dt + \frac{1}{\varepsilon_3}\int_0^T\left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)^2 dt
\end{aligned} \tag{21}$$

Putting (3.7), (3.8) and (3.9) into (3.6), on Ω_β,

$$\int_{\Omega_\beta} \mathcal{L}u.Mu_3 dxdt = \int_{\Omega_\beta} f.Mu_3 dxdt$$

We obtain

$$\begin{aligned}
&\frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt-\frac{17}{8}\int_0^T\left(\int_\beta^1 \frac{\partial u}{\partial t}dx\right)^2 dt+2\int_0^T\left(\int_\beta^1 \frac{\partial u}{\partial t}dx\right)\left(\int_\beta^1 x\frac{\partial u}{\partial t}dx\right)dt\\
&-\frac{1}{2}\int_{\Omega_\beta}(\frac{5}{4}-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt\\
&-\frac{1}{2}\int_\beta^1(\frac{5}{4}-x)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx+\frac{1}{2}\int_\beta^1(\frac{5}{4}-x)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx\\
\leq\ &\frac{\varepsilon_5}{8}\int_{\Omega_\beta} f^2 dxdt+\frac{1}{8\varepsilon_5}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\\
&+\frac{\varepsilon_6}{2}\int_{\Omega_\beta} f^2 dxdt+\frac{1}{2\varepsilon_6}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\\
&+2\varepsilon_{10}\int_0^T\left(\int_\beta^1 f dx\right)^2 dt+\frac{2}{\varepsilon_{10}}\int_0^T\left(\int_\beta^1 \frac{\partial u}{\partial t}dx\right)^2 dt\\
&+\frac{1}{\varepsilon_9}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+4\varepsilon_9\int_{\Omega_\beta} f^2 dxdt.
\end{aligned}$$

Then

$$\begin{aligned}
(22)\qquad &\frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt-\frac{17}{8}\int_0^T\left(\int_\beta^1 \frac{\partial u}{\partial t}dx\right)^2 dt\\
&+2\int_0^T\left(\int_\beta^1 \frac{\partial u}{\partial t}dx\right)\left(\int_\beta^1 x\frac{\partial u}{\partial t}dx\right)dt\\
&-\frac{1}{2}\int_{\Omega_\beta}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial a(x,t)}{\partial t}(\frac{5}{4}-x)dxdt\\
&-\frac{1}{2}\int_\beta^1\left(\frac{\partial \phi}{\partial x}\right)^2 a(x,0)(\frac{5}{4}-x)dx+\frac{1}{2}\int_\beta^1\left(\frac{\partial u}{\partial x}\right)^2 a(x,t)(\frac{5}{4}-x)dx\\
\leq\ &\left(\frac{\varepsilon_5}{8}+\frac{\varepsilon_6}{2}+4\varepsilon_9\right)\int_{\Omega_\beta} f^2 dxdt\\
&+\left(\frac{1}{8\varepsilon_5}+\frac{1}{2\varepsilon_6}+\frac{1}{\varepsilon_9}\right)\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\\
&+2\varepsilon_{10}\int_0^T\left(\int_\beta^1 f dx\right)^2 dt+\frac{2}{\varepsilon_{10}}\int_0^T\left(\int_\beta^1 \frac{\partial u}{\partial t}dx\right)^2 dt.
\end{aligned}$$

Putting (3.11) , (3.12) and (3.13) into (3.10), on $\Omega_{\alpha,\beta}$:

$$\int_{\Omega_{\alpha,\beta}}\mathcal{L}u.Mu_2 dxdt=\int_{\Omega_{\alpha,\beta}} f.Mu_2 dxdt$$

We obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_{\alpha}^{x}\frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_{x}^{\beta}\frac{\partial u}{\partial t}d\xi\right)^2\\
&-\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(\beta-\alpha)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt\\
&-\frac{1}{2}\int_{\alpha}^{\beta}(\beta-\alpha)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx+\frac{1}{2}\int_{\alpha}^{\beta}(\beta-\alpha)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx\\
\leq\ &\left(\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_{\alpha}^{x}\frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(\beta-x)^2 f^2 dxdt\right)\\
&+\left(\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_{x}^{\beta}\frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(x-\alpha)^2 f^2 dxdt\right),
\end{aligned}
$$

that implies:

$$
\begin{aligned}
&-\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(\beta-\alpha)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt\\
&-\frac{1}{2}\int_{\alpha}^{\beta}(\beta-\alpha)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx+\frac{1}{2}\int_{\alpha}^{\beta}(\beta-\alpha)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx\\
\leq\ &\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(\beta-x)^2 f^2 dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(x-\alpha)^2 f^2 dxdt,\\
\leq\ &\int_{\Omega_{\alpha,\beta}} f^2 dxdt,
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{c_0}{2}\int_{\alpha}^{\beta}(\beta-\alpha)\left(\frac{\partial u}{\partial x}\right)^2 dx\\
\leq\ &\frac{c_2}{2}\int_{\Omega_{\alpha,\beta}}(\beta-\alpha)\left(\frac{\partial u}{\partial x}\right)^2 dxdt\ +\frac{c_1}{2}\int_{\alpha}^{\beta}\left(\frac{\partial \phi}{\partial x}\right)^2 dx\\
&+\int_{\Omega_{\alpha,\beta}} f^2 dxdt\ .
\end{aligned}
\tag{23}
$$

According to the condition (1.4) we have:

$$
\left(\int_{0}^{\alpha}\frac{\partial u}{\partial t}dx\right)^2=\left(\int_{\beta}^{1}\frac{\partial u}{\partial t}dx\right)^2.
$$

So, we are adding between (3.14) and (3.15) , we obtain

$$
\begin{aligned}
&\frac{3}{2}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\left(\frac{3}{8}\right)\int_0^T\left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)^2 dt\\
&-\frac{1}{2}\int_{\Omega_\alpha}(5-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt-\frac{1}{2}\int_0^\alpha (5-x)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx\\
&+\frac{1}{2}\int_0^\alpha (5-x)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx\\
&-\frac{1}{2}\int_{\Omega_\beta}(\frac{5}{4}-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt-\frac{1}{2}\int_\beta^1 (\frac{5}{4}-x)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx\\
&+\frac{1}{2}\int_\beta^1 (\frac{5}{4}-x)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx\\
\leq\ &\left(2\varepsilon_1+\frac{\varepsilon_2}{2}+4\varepsilon_4\right)\int_{\Omega_\alpha} f^2 dxdt+\left(\frac{2}{\varepsilon_1}+\frac{1}{2\varepsilon_2}+\frac{1}{\varepsilon_4}\right)\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\\
&+(\varepsilon_3+2\varepsilon_4)\int_0^T\left(\int_0^\alpha f dx\right)^2 dt+\left(\frac{1}{\varepsilon_3}+\frac{2}{\varepsilon_{10}}\right)\int_0^T\left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)^2 dt\\
&\left(\frac{\varepsilon_5}{8}+\frac{\varepsilon_6}{2}+4\varepsilon_9\right)\int_{\Omega_\beta} f^2 dxdt+\left(\frac{1}{8\varepsilon_5}+\frac{1}{2\varepsilon_6}+\frac{1}{\varepsilon_9}\right)\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\\
&+2\varepsilon_{10}\int_0^T\left(\int_\beta^1 f dx\right)^2 dt,
\end{aligned}
$$

If we put $\varepsilon_1=4$, $\varepsilon_2=2$, $\varepsilon_3=8$, $\varepsilon_4=4$, $\varepsilon_5=1$, $\varepsilon_6=2$, $\varepsilon_9=2$, $\varepsilon_{10}=8$. we have

$$
\begin{aligned}
&\frac{3}{2}\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\left(\frac{3}{8}\right)\int_0^T\left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)^2 dt\\
&-\frac{1}{2}\int_{\Omega_\alpha}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial a(x,t)}{\partial t}(5-x)dxdt-\frac{1}{2}\int_0^\alpha\left(\frac{\partial \phi}{\partial x}\right)^2 a(x,0)(5-x)dx\\
&+\frac{1}{2}\int_0^\alpha\left(\frac{\partial u}{\partial x}\right)^2 a(x,T)(5-x)dx\\
&-\frac{1}{2}\int_{\Omega_\beta}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial a(x,t)}{\partial t}(\frac{5}{4}-x)dxdt-\frac{1}{2}\int_\beta^1\left(\frac{\partial \phi}{\partial x}\right)^2 a(x,0)(\frac{5}{4}-x)dx\\
&+\frac{1}{2}\int_\beta^1\left(\frac{\partial u}{\partial x}\right)^2 a(x,T)(\frac{5}{4}-x)dx\\
\leq\ &25\int_{\Omega_\alpha} f^2 dxdt+\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\\
&+16\int_0^T\left(\int_0^\alpha f dx\right)^2 dt+\left(\frac{3}{8}\right)\int_0^T\left(\int_0^\alpha \frac{\partial u}{\partial t}dx\right)^2 dt\\
&\frac{73}{8}\int_{\Omega_\beta} f^2 dxdt+\frac{7}{8}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+16\int_0^T\left(\int_\beta^1 f dx\right)^2 dt,
\end{aligned}
$$

Then

$$\begin{aligned}
&\left(\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt + \frac{5}{8}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\right) \\
&+\left(\frac{1}{2}\int_0^\alpha (5-x)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx \; + \frac{1}{2}\int_\beta^1 (\frac{5}{4}-x)a(x,\tau)\left(\frac{\partial u}{\partial x}\right)^2 dx\right) \\
\leq \; &\left(\begin{array}{c} 25\int_{\Omega_\alpha} f^2 dxdt + 16\int_0^T \left(\int_0^\alpha f dx\right)^2 dt \\ +\frac{1}{2}\int_{\Omega_\alpha}(5-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt \; + \frac{1}{2}\int_0^\alpha (5-x)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx \end{array}\right) \\
&+\left(\begin{array}{c} \frac{73}{8}\int_{\Omega_\beta} f^2 dxdt + 16\int_0^T \left(\int_\beta^1 f dx\right)^2 dt \\ +\frac{1}{2}\int_{\Omega_\beta}(\frac{5}{4}-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt \; + \frac{1}{2}\int_\beta^1 (\frac{5}{4}-x)a(x,0)\left(\frac{\partial \phi}{\partial x}\right)^2 dx \end{array}\right).
\end{aligned}$$

According to *Condition 1,* we then get

$$\begin{aligned}
&\left(\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt + \frac{5}{8}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\right) \\
&+\left(\frac{c_0}{2}\int_0^\alpha (5-x)\left(\frac{\partial u}{\partial x}\right)^2 dx \; + \frac{c_0}{2}\int_\beta^1 (\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dx\right) \\
\leq \; &\left(\begin{array}{c} 25\int_{\Omega_\alpha} f^2 dxdt + 16\int_0^T \left(\int_0^\alpha f dx\right)^2 dt \\ +\frac{c_2}{2}\int_{\Omega_\alpha}(5-x)\left(\frac{\partial u}{\partial x}\right)^2 dxdt \; + \frac{c_1}{2}\int_0^\alpha (5-x)\left(\frac{\partial \phi}{\partial x}\right)^2 dx \end{array}\right) \\
&+\left(\begin{array}{c} \frac{73}{8}\int_{\Omega_\beta} f^2 dxdt + 16\int_0^T \left(\int_\beta^1 f dx\right)^2 dt \\ +\frac{c_2}{2}\int_{\Omega_\beta}(\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dxdt \; + \frac{c_1}{2}\int_\beta^1 (\frac{5}{4}-x)\left(\frac{\partial \phi}{\partial x}\right)^2 dx \end{array}\right),
\end{aligned}$$

Then

$$\begin{aligned}
&\left(\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt + \frac{5}{8}\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\right) \\
&+\left(\frac{c_0}{2}\int_0^\alpha (5-x)\left(\frac{\partial u}{\partial x}\right)^2 dx \; + \frac{c_0}{2}\int_\beta^1 (\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dx\right) \\
\leq \; &\left(\begin{array}{c} 25\int_{\Omega_\alpha} f^2 dxdt + 16\int_0^T \left(\int_0^\alpha f dx\right)^2 dt \\ +\frac{c_2}{2}\int_{\Omega_\alpha}(5-x)\left(\frac{\partial u}{\partial x}\right)^2 dxdt \; + \frac{5c_1}{2}\int_0^\alpha \left(\frac{\partial \phi}{\partial x}\right)^2 dx \end{array}\right) \\
&+\left(\begin{array}{c} \frac{73}{8}\int_{\Omega_\beta} f^2 dxdt + 16\int_0^T \left(\int_\beta^1 f dx\right)^2 dt \\ +\frac{c_2}{2}\int_{\Omega_\beta}(\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dxdt \; + \frac{5c_1}{8}\int_\beta^1 \left(\frac{\partial \phi}{\partial x}\right)^2 dx \end{array}\right) \\
\leq \; & 25\left(\int_{\Omega_\alpha} f^2 dxdt + \int_{\Omega_\beta} f^2 dxdt + \int_0^T \left(\int_0^\alpha f dx\right)^2 dt + \int_0^T \left(\int_\beta^1 f dx\right)^2 dt\right) \\
&+\frac{5c_1}{2}\left(\int_0^\alpha \left(\frac{\partial \phi}{\partial x}\right)^2 dx + \int_\beta^1 \left(\frac{\partial \phi}{\partial x}\right)^2 dx\right) \\
&+\frac{c_2}{2}\left(\int_{\Omega_\alpha}(5-x)\left(\frac{\partial u}{\partial x}\right)^2 dxdt \; + \int_{\Omega_\beta}(\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dxdt\right).
\end{aligned}$$

that implies

$$\begin{aligned}
&\frac{5}{8}\left(\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\right)\\
&+\frac{c_0}{2}\left[\int_0^\alpha(5-x)\left(\frac{\partial u}{\partial x}\right)^2 dx\ +\int_\beta^1(\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dx\right]\\
\leq\ & 25\left(\int_{\Omega_\alpha} f^2dxdt+\int_{\Omega_\beta} f^2dxdt+\int_0^T\left(\int_0^\alpha f dx\right)^2 dt+\int_0^T\left(\int_\beta^1 f dx\right)^2 dt\right)\\
&+\frac{5c_1}{2}\left(\int_0^\alpha\left(\frac{\partial \phi}{\partial x}\right)^2 dx+\int_\beta^1\left(\frac{\partial \phi}{\partial x}\right)^2 dx\right)\\
&+\frac{c_2}{2}\left(\int_{\Omega_\alpha}(5-x)\left(\frac{\partial u}{\partial x}\right)^2 dxdt\ +\int_{\Omega_\beta}(\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dxdt\right)
\end{aligned}$$

Using Lemme 1 in [9] , we have

$$\begin{aligned}
&\left(\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\right)\\
&+\left(\int_0^\alpha(5-x)\left(\frac{\partial u}{\partial x}\right)^2 dx\ +\int_\beta^1(\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dx\right)\\
\leq\ & c_3\left(\begin{array}{c}\left(\int_{\Omega_\alpha} f^2dxdt+\int_{\Omega_\beta} f^2dxdt\right)+\int_0^T\left(\left(\int_0^\alpha f dx\right)^2+\left(\int_\beta^1 f dx\right)^2\right)dt\\ +\left(\int_0^\alpha\left(\frac{\partial \phi}{\partial x}\right)^2 dx+\int_\beta^1\left(\frac{\partial \phi}{\partial x}\right)^2 dx\right)\end{array}\right),
\end{aligned}\tag{24}$$

where

$$c_3=\frac{\max\left(25,\frac{5c_1}{2}\right)}{\min\left(\frac{5}{8},\frac{c_0}{2}\right)}\exp\left(\frac{c_2}{2}T\right)$$

according to (3.16) and by using Lemme 1 in [9], we get

$$\begin{aligned}
&\int_\alpha^\beta(\beta-\alpha)\ \left(\frac{\partial u\left(x,\tau\right)}{\partial x}\right)^2 dx\\
\leq\ & c_4\left(\int_{\Omega_{\alpha,\beta}} f^2dxdt+\int_\alpha^\beta\left(\frac{\partial \phi}{\partial x}\right)^2 dx\right)
\end{aligned}\tag{25}$$

where

$$c_4=\frac{\max\left(1,\frac{c_1}{2}\right)}{\frac{c_0}{2}}\exp\left(\frac{c_2}{2}T\right)$$

we are adding between (3.17) and (3.18) , we obtain

$$\begin{aligned}
&\int_{\Omega_\alpha}\left(\int_x^\alpha \frac{\partial u}{\partial t}d\xi\right)^2 dxdt+\int_{\Omega_\beta}\left(\int_\beta^x \frac{\partial u}{\partial t}d\xi\right)^2 dxdt\\
&+\left(\begin{array}{c}\int_0^\alpha(5-x)\left(\frac{\partial u}{\partial x}\right)^2 dx\ +\int_\beta^1(\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dx\\ +\int_\alpha^\beta(\beta-\alpha)\left(\frac{\partial (x,t)}{\partial x}\right)^2 dx\end{array}\right)\\
\leq\ & \max\left(c_3,c_4\right)\left(\begin{array}{c}\int_\Omega f^2dxdt+\int_0^T\left(\left(\int_0^\alpha f dx\right)^2+\left(\int_\beta^1 f dx\right)^2\right)dt\\ +\left(\int_0^1\left(\frac{\partial \phi}{\partial x}\right)^2 dx\right)\end{array}\right)\\
\leq\ & c_5\left(\int_\Omega f^2dxdt+\int_0^1\left(\frac{\partial \phi}{\partial x}\right)^2 dx\right)
\end{aligned}\tag{26}$$

where

$$c_5 = 1 + \max(c_3, c_4).$$

The right-hand side of (3.19) is independent of τ, hence replacing the left-hand side by its upper bound with respect to τ from 0 to T, we obtain the desired inequality, where $c = (c_5)^{\frac{1}{2}}$. □

Proposition 2. *The operator L from B to F admits a closure.*

Proof. Suppose that $\{u_n\} \in D(L)$ is a sequence such that

(27) $$u_n \to 0 \qquad \text{in } B$$

and

$$Lu_n \to (f, \phi) \qquad \text{in } F;$$

we must show that

$$f \equiv 0 and\ \phi \equiv 0.$$

According to (3.20) we get

$$u_n \to 0 \qquad \text{in } D'(\Omega)$$

By virtue of the continuity of derivation of $D'(\Omega)$ in $D'(\Omega)$, we deduce that

(28) $$\mathcal{L}u_n \to 0 \qquad \text{in } D'(\Omega).$$

Further, according to (3.21), we have

(29) $$\mathcal{L}u_n \to f \qquad \text{in } L^2(\Omega),$$

thus we have

(30) $$\mathcal{L}u_n \to f \qquad \text{in } D'(\Omega).$$

Then by of the uniqueness of the limit in $D'(\Omega)$ we see that $f \equiv 0$.

On the other hand, (3.21) implies that

(31) $$\frac{d\ell u_n}{dx} \to \frac{d\phi}{dx} \qquad \text{in } L^2(0,1).$$

Moreover, since by virtue of (3.20) and the fact that

$$\int_0^\alpha (5-x)\left(\frac{\partial u}{\partial x}\right)^2 dx \ + \int_\beta^1 (\frac{5}{4}-x)\left(\frac{\partial u}{\partial x}\right)^2 dx + \int_\alpha^\beta (\beta-\alpha)\left(\frac{\partial (x,t)}{\partial x}\right)^2 dx \le \|u_n\|_B^2, \quad \forall n,$$

we have

(32) $$\frac{d\ell u_n}{dx} \to 0 \qquad \text{in } L^2(0,1).$$

Now the uniqueness of the limit in $L^2(0,1)$ implies that $\phi \equiv 0$. □

Theorem 1 is valid for strong solution, i.e., we have the inequality

(33) $$\|u\|_B \le c\left\|\overline{L}u\right\|_F, \qquad \forall u \in D(\overline{L}).$$

Hence we obtain

Corollary 3. *A strong solution of the problem* (2.1) – (2.5) *is unique if it exists, and depends continuously on $\mathcal{F} = (f, \phi) \in F$.*

Corollary 4. *The range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F, and $R(\overline{L}) = \overline{R(L)}$.*

4. Existence of solutions

To show the existance of solutions, we prove that $R(L)$ is dense in F for all $u \in D(L)$ and for arbitrary $\mathcal{F} = (f, \phi) \in F$.

Theorem 5. *Suppose the conditions of theorem 1 are satisfied. Then the problem* $(2.1)-(2.5)$ *admits a unique strong solution* $u = \overline{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$.

Proof. First we prove that $R(L)$ is dense in F for the special case where $D(L)$ is reduced to $D_0(L)$, where $D_0(L) = \{u,\ u \in D(L) : \ell u = 0\}$. □

Proposition 6. *Let the conditions of theorem 2 be satisfied, if, for* $\omega \in L^2(\Omega)$ *and for all* $u \in D_0(L)$, *we have*

$$\int_\Omega \mathcal{L}u.\omega \ dxdt = 0, \tag{34}$$

then ω *vanishes almost everywhere in* Ω.

Proof. The scalar product of F is defined by

$$(Lu, \omega)_F = \int_\Omega \mathcal{L}u.\omega \ dxdt + \int_0^1 \left(\frac{\partial \ell u}{\partial x}\right)\left(\frac{\partial \omega_0}{\partial x}\right) dx. \tag{35}$$

the equality (4.1) can be written as follows:

$$\int_\Omega \frac{\partial u}{\partial t}\omega dxdt = \int_\Omega \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right)\omega dxdt. \tag{36}$$

If we put

$$u = \Im_t\left(e^{c_6 t}z\right) = \int_0^t e^{c_6 t}z(x,\tau)\, d\tau,$$

where c_6 is a constant such that $c_6 c_0 - c_2 \geq 0$, and z, $\frac{\partial z}{\partial x}$, $\frac{\partial}{\partial x}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)$, $\frac{\partial^2}{\partial x^2}(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}) \in L^2(\Omega)$, then, u satisfies the conditions $(2.3)-(2.5)$. As a result of (4.3), we obtain the equality

$$\int_\Omega e^{c_6 t}z\omega dxdt = \int_\Omega \frac{\partial^2}{\partial x^2}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\omega dxdt. \tag{37}$$

In terms of the given function ω, and from the equality (4.4) we give the function ω in terms of z as follows:

$$\omega = \begin{cases} \omega_1 = (1-x)\int_x^\alpha z d\xi - 2\int_x^\alpha \int_\xi^\alpha z d\eta d\xi & 0 \leq x \leq \alpha \\ \omega_2 = -(\beta - x)\int_\alpha^x z d\xi - (x-\alpha)\int_x^\beta z d\xi & \alpha \leq x \leq \beta \\ \omega_3 = -(1-x)\int_\beta^x z d\xi - 2\int_\beta^x \int_\beta^\xi z d\eta d\xi & \beta \leq x \leq 1 \end{cases} \tag{38}$$

So, $\omega \in L^2(\Omega)$, and z satisfies the same conditions of the function u and

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{x=\alpha} = 0, \quad \left.\frac{\partial^2 z}{\partial x^2}\right|_{x=\beta} = 0. \tag{39}$$

Replacing ω in (4.4) by its representation (4.5) and integrating by parts each term of (4.4) with the use of conditions of z, we obtain

1) on the interval $\Omega_\alpha = (0,\alpha) \times (0,\tau)$, we obtain

$$\int_{\Omega_\alpha} e^{c_6 t}z\omega_1 dxdt = \int_\Omega \frac{\partial^2}{\partial x^2}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\omega_1 dxdt. \tag{40}$$

Integrating by parts each term of (4.7) with respect to x and t by taking the conditions of the function z yields

$$\begin{aligned}
&\int_{\Omega_\alpha} \frac{\partial^2}{\partial x^2}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left((1-x)\int_x^\alpha zd\xi - 2\int_x^\alpha\int_\xi^\alpha zd\eta d\xi\right)dxdt \\
=\ &\int_0^T \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left((1-x)\int_x^\alpha zd\xi - 2\int_x^\alpha\int_\xi^\alpha zd\eta d\xi\right)\Bigg|_{x=0}^{x=\alpha} dt \\
&-\int_{\Omega_\alpha} \frac{\partial}{\partial x}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left(-(1-x)z + \int_x^\alpha zd\xi\right)dxdt \\
=\ &\int_0^T \left(a(x,t)\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left(-(1-x)z + \int_x^\alpha zd\xi\right)\Bigg|_{x=0}^{x=\alpha} dt \\
&+\int_{\Omega_\alpha} \left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left(-(1-x)z + \int_x^\alpha zd\xi\right)dxdt \\
=\ &-\int_{\Omega_\alpha} a\left(x,t\right)\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}(1-x)\frac{\partial z}{\partial x}\ dxdt \\
=\ &-\frac{1}{2}\int_0^\alpha e^{-c_6 t}(1-x)a\left(x,t\right)\left(\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)^2\Bigg|_{t=0}^{t=T} dx \\
&-\frac{1}{2}\int_{\Omega_\alpha} e^{-c_6 t}(1-x)\left(c_6 a\left(x,t\right) - \frac{\partial a\left(x,t\right)}{\partial t}\right)\left(\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)^2 dxdt
\end{aligned}$$

By using the conditions of z, we obtain

$$-\frac{1}{2}\left(c_6 c_0 - c_2\right)\int_{\Omega_\alpha} e^{-c_6 t}(1-x)\left(\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)^2 dxdt \leq 0. \tag{41}$$

and

$$\begin{aligned}
&\int_{\Omega_\alpha} e^{c_6 t}z\omega_1 dxdt \\
=\ &\frac{3}{2}\int_{\Omega_\alpha} e^{c_6 t}\left(\int_x^\alpha zd\xi\right)^2 dxdt + \frac{1}{2}\int_0^T e^{c_6 t}\left(\int_0^\alpha zdx\right)^2 dt \\
&-2\int_0^T e^{c_6 t}\left(\int_0^\alpha zdx\right)\left(\int_0^\alpha \xi zd\xi\right)dt.
\end{aligned} \tag{42}$$

2) on the interval $\Omega_\beta = (\beta,1)\times(0,\tau)$,we obtain

$$\int_{\Omega_\beta} e^{c_6 t}z\omega_3 dxdt = \int_{\Omega_\beta} \frac{\partial^2}{\partial x^2}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\omega_3 dxdt. \tag{43}$$

Integrating by parts each term of (4.10) with respect to x and t by taking the conditions of the function z yields

$$\begin{aligned}
&\int_{\Omega_\beta}\frac{\partial^2}{\partial x^2}\left(a\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}\right)\left[-(1-x)\int_\beta^x zd\xi-2\int_\beta^x\int_\beta^\xi zd\eta d\xi\right]dxdt\\
=&\int_0^T\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}\right)\left(-(1-x)\int_\beta^x zd\xi-2\int_\beta^x\int_\beta^\xi zd\eta d\xi\right)\Bigg|_{x=\beta}^{x=1}dt\\
&-\int_{\Omega_\beta}\frac{\partial}{\partial x}\left(a\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}\right)\left[-(1-x)z-\int_\beta^x zd\xi\right]dxdt\\
=&\int_0^T\left(a(x,t)\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}\right)\left(-(1-x)z-\int_\beta^x zd\xi\right)\Bigg|_{x=\beta}^{x=1}dt\\
&+\int_{\Omega_\beta}\left(a\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}\right)\left[-(1-x)z-\int_\beta^x zd\xi\right]dxdt\\
=&-\int_{\Omega_\beta}a\left(x,t\right)\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}(1-x)\frac{\partial z}{\partial x}\;dxdt\\
=&-\frac{1}{2}\int_\beta^1 e^{-c_6t}(1-x)a\left(x,t\right)\left(\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}\right)^2\Bigg|_{t=0}^{t=T}dx\\
&-\frac{1}{2}\int_{\Omega_\beta}e^{-c_6t}(1-x)\left(c_6a\left(x,t\right)-\frac{\partial a\left(x,t\right)}{\partial t}\right)\left(\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}\right)^2dxdt
\end{aligned}$$

By using the conditions of z, we obtain

$$-\frac{1}{2}\left(c_6c_0-c_2\right)\int_{\Omega_\beta}e^{-c_6t}(1-x)\left(\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}\right)^2dxdt\leq 0. \tag{44}$$

and

$$\begin{aligned}
&\int_{\Omega_\beta}e^{c_6t}z\omega_3dxdt\\
=&\;\frac{3}{2}\int_{\Omega_\beta}e^{c_6t}\left(\int_\beta^1 zdx\right)^2dxdt\\
&-2\int_0^T e^{c_6t}\left(\int_\beta^1 zdx\right)\left(\left(\int_\beta^1 zdx\right)-\left(\int_\beta^1 xzdx\right)\right)dt\\
=&\;\frac{3}{2}\int_{\Omega_\beta}e^{c_6t}\left(\int_\beta^x zd\xi\right)^2dxdt\\
&+2e^{c_6t}\left(\int_\beta^1 zdx\right)\left(\int_\beta^1 xzdx\right)dt.
\end{aligned} \tag{45}$$

3) on the interval $\Omega_{\alpha,\beta}=(\alpha,\beta)\times(0,\tau)$,we obtain

$$\int_{\Omega_{\alpha,\beta}}e^{c_6t}z\omega_2dxdt=\int_{\Omega\alpha,\beta}\frac{\partial^2}{\partial x^2}\left(a\frac{\partial\Im_t\left(e^{c_6t}z\right)}{\partial x}\right)\omega_2dxdt \tag{46}$$

Integrating by parts each term of (4.13) with respect to x and t by taking the conditions of the function z yields

$$\begin{aligned}
&\int_{\Omega_{\alpha,\beta}} \frac{\partial^2}{\partial x^2}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left[-(\beta-x)\int_\alpha^x zd\xi-(x-\alpha)\int_x^\beta zd\xi\right]dxdt \\
=\;& \int_0^T \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left(-(\beta-x)\int_\alpha^x zd\xi-(x-\alpha)\int_x^\beta zd\xi\right)\Bigg|_{x=\alpha}^{x=\beta}dt \\
&+\int_{\Omega_{\alpha,\beta}} \frac{\partial}{\partial x}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left((\beta-x)z-\int_\alpha^x zd\xi\right)dxdt \\
&+\int_{\Omega_{\alpha,\beta}} \frac{\partial}{\partial x}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left(-(x-\alpha)z+\int_x^\beta zd\xi\right)dxdt \\
=\;& \left(\begin{array}{c}\int_0^T\left(a(x,t)\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left((\beta-x)z-\int_\alpha^x zd\xi\right)\Big|_{x=\alpha}^{x=\beta}dt \\ +\int_0^T\left(a(x,t)\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left(-(x-\alpha)z+\int_x^\beta zd\xi\right)\Big|_{x=\alpha}^{x=\beta}dt\end{array}\right) \\
&-\left(\begin{array}{c}\int_{\Omega_{\alpha,\beta}}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left((\beta-x)\frac{\partial z}{\partial x}+2z\right)dxdt \\ +\int_{\Omega_{\alpha,\beta}}\left(a\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)\left((x-\alpha)\frac{\partial z}{\partial x}-2z\right)dxdt\end{array}\right) \\
=\;& -\int_{\Omega_{\alpha,\beta}} a\left(x,t\right)\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}(\beta-\alpha)\frac{\partial z}{\partial x}\;dxdt \\
=\;& -\frac{1}{2}\int_\alpha^\beta e^{-c_6 t}(\beta-\alpha)a\left(x,t\right)\left(\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)^2\Bigg|_{t=0}^{t=T}dx \\
&-\frac{1}{2}\int_{\Omega_{\alpha,\beta}} e^{-c_6 t}(\beta-\alpha)\left(c_6 a\left(x,t\right)-\frac{\partial a\left(x,t\right)}{\partial t}\right)\left(\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)^2dxdt
\end{aligned}$$

By using the conditions of z, we obtain

$$-\frac{1}{2}\left(c_6 c_0-c_2\right)\int_{\Omega_{\alpha,\beta}} e^{-c_6 t}(\beta-\alpha)\left(\frac{\partial \Im_t\left(e^{c_6 t}z\right)}{\partial x}\right)^2dxdt\le 0. \tag{47}$$

and

$$\begin{aligned}
&\int_{\Omega_{\alpha,\beta}} e^{c_6 t}z\omega_2 dxdt \\
=\;& \frac{1}{2}\int_{\Omega_{\alpha,\beta}} e^{c_6 t}\left(\int_\alpha^x zd\xi\right)^2dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}} e^{c_6 t}\left(\int_x^\beta zd\xi\right)^2dxdt
\end{aligned} \tag{48}$$

Putting and using the results of (4.8),(4.9) , (4.11),(4.12) and (4.14),(4.15) into (4.4) , we obtain

$$\begin{aligned} & \frac{3}{2}\int_{\Omega_\alpha} e^{c_6 t}\left(\int_x^\alpha z d\xi\right)^2 dxdt + \frac{3}{2}\int_{\Omega_\beta} e^{c_6 t}\left(\int_\beta^x z d\xi\right)^2 dxdt \\ & +\frac{1}{2}\int_{\Omega_{\alpha,\beta}} e^{c_6 t}\left(\int_\alpha^x z d\xi\right)^2 dxdt + \frac{1}{2}\int_{\Omega_{\alpha,\beta}} e^{c_6 t}\left(\int_x^\beta z d\xi\right)^2 dxdt \\ \leq\ & -\frac{1}{2}(c_6 c_0 - c_2)\int_{\Omega_\alpha} e^{-c_6 t}(1-x)\left(\frac{\partial \Im_t\left(e^{c_6 t} z\right)}{\partial x}\right)^2 dxdt \\ & -\frac{1}{2}(c_6 c_0 - c_2)\int_{\Omega_\beta} e^{-c_6 t}(1-x)\left(\frac{\partial \Im_t\left(e^{c_6 t} z\right)}{\partial x}\right)^2 dxdt \\ & -\frac{1}{2}(c_6 c_0 - c_2)\int_{\Omega_{\alpha,\beta}} e^{-c_6 t}(\beta-\alpha)\left(\frac{\partial \Im_t\left(e^{c_6 t} z\right)}{\partial x}\right)^2 dxdt \\ \leq\ & 0. \end{aligned}$$

and thus $z = 0$ in Ω, then $\omega = 0$ in Ω. This proves Proposition 2. □

We return to the proof of Theorem 2. We have already noted that it is sufficient to prove that the set $R(L)$ dense in F.

Suppose that, for some $W = (\omega, \omega_0) \in R(L)^\perp$ and for all $u \in D(L)$, it holds

$$(Lu, \omega)_F = \int_\Omega \mathcal{L}u.\omega dxdt + \int_0^1 \left(\frac{\partial \ell u}{\partial x}\right)\left(\frac{\partial \omega_0}{\partial x}\right) dx = 0. \tag{49}$$

Then we must prove that $W = 0$. Putting $u \in D_0(L)$ in (4.16) , we have

$$\int_\Omega \mathcal{L}u.\omega \ dxdt = 0, \quad u \in D_0(L).$$

Hence Proposition 2 implies that $\omega = 0$. Thus (4.16) takes the form

$$\int_0^1 \left(\frac{\partial \ell u}{\partial x}\right)\left(\frac{\partial \omega_0}{\partial x}\right) dx = 0, \quad u \in D(L). \tag{50}$$

Since the range of the trace operator ℓ is dense in the Hilbert F space with the norm

$$\left(\int_0^1 \left(\frac{\partial \ell u}{\partial x}\right)^2 dx\right)^{\frac{1}{2}},$$

the equality (4.17) implies that $\omega_0 = 0$ (we recall satisfies a compatibility conditions). Hence $W = 0$. This completes the proof of Theorem 2.

References

[1] A. Bouziani and N.-E. Benouar, Mixed problem with integral conditions for a third order parabolic equation, Kobe J. Math. 15 (1998), no. 1, 47–58.
[2] N.E. Benouar and N.I. Yurchuk, Mixed problem with an integral condition for parabolic equations with the Bessel operator, Differentsial'nye Uravneniya, 27 (1991), 2094-2098.
[3] A. Bouziani, Mixed problem for certain nonclassical equations with a small parameter, Bulletin de la Classe des Sciences, Académie Royale de Belgique, 5 (1994), 389-400.
[4] A. Bouziani, Solution forte d'un problème de transmission parabolique-hyperbolique pour une structure pluridemensionnelle, Bulletin de la Classe des Sciences, Académie Royale de Belgique, 7 (1996), 369-386.
[5] A. Bouziani, Mixed problem with integral conditions for a certain parabolic equation, J. of Appl. Math. and Stoch. Anal. 9 (1996), 323-330.
[6] A. Bouziani, Mixed problem with nonlocal condition for certain pluriparabolic equations, Hiroshima Math. J. 27 (1997), 373-390.
[7] A. Bouziani and N.E. Benouar, Problème mixte avec conditions intégrales pour une classe d'équations paraboliques, Comptes Rendus de l'Académie des Sciences, Paris t.321, Série I, (1995), 1177-1182.
[8] A. Bouziani and N.E. Benouar, Problèmes aux limites pour use classe d'équations de type non classique pour use structure pluri-dimensionnelle, Bull. of the Polish Acad. of Sciences- Mathematics 43,(1995), 317-328.

[9] B. Cahlon, D.M. Kulkarni and P.Shi, Stewise stability for the heat equation with a nonlocal constraint, SIAM l. Nurner. Anal., 32 (1995), 571-593.

[10] J. R. Cannon, The solution of the heat equation subject to the specification of energy, Quart. Appl. Math. 21 (1963), 155–160.

[11] J. R. Cannon, S. Pérez Esteva, and J. van der Hoek, A Galerkin procedure for the diffusionequation subject to the specification of mass, SIAM J., Numer. Anal. 24 (1987), no. 3, 499–515.

[12] J.R. Cannon and J. Van der Hoek, The existence and the continuous dependence for the solution of the heat equation subject to the specification of energy, Boll. Uni. Math. Ital. Suppl. 1 (1981), 253-282.

[13] J.R. Cannon and J. Van der Hoek, An implicit finite difference scheme for the diffusion of mass in a portion of the domain, Numer. Solutions of PDEs (ed. by J. Noye), North-Holland, Amsterdam (1982), 527 539.

[14] L. Garding, , Cauchy's Problem for Hyperbolic Equations, Univ. of Chicago Lecture Notes 1957.

[15] N.I. Ionkin, Solution of boundary value problem in heat conduction theory with nonlocl boundary conditions, Differentsial'nye Uravneniya, 13 (1977), 294-304.

[16] S. Jones, B. Jumarhon, S. McKee, and J.A.Scott, A mathematical model of biosensor, J. Eng. Math. 30 (1996), 312-337.

[17] B. Jumarhon, and S. McKee, On the heat equation with nonlinear and nonlocal boundary conditions, J. Math. Anal. Appl. 190 (1995), 806-820.

[18] B. Jumarhon, and S. McKee, Product integration methods for solving a system of nonlinear Volterra integral equations, J. Comput. Appl. Math. 69 (1996), 285-301.

[19] N.I. Kamynin, A boundary value problem in the theory of the heat conduction with non-classical boundary condition, Th. Vychisl. Mat. Mat. Fiz. 43 (1964),1006-1024.

[20] K. Rektorys, Variational Methods in Mathematics, Sciences and Engineering, 2nd ed., Dordrecht, Boston, Reidel 1979.

[21] P. Shi, Weak solution to an evolution problem with a nonlocal constraint, SIAM. J. Math. Anal. 24 (1993), 46-58.

[22] N.I. Yurchuk, Mixed problem with an integral condition for certain parabolic equations, Differentsial'nye Uravneniya 22 (1986), 2117-2126.

[23] G. W. Batten, Jr., Second order correct boundary conditions of the numerical solution of the mixed boundary problem for parabolic equations, Math. Compt., 17 (1963), 405–413.

[24] N.I. Yurchuk, Mixed problems for linearized Kortweg-de-Vries equations degenerating in time into parabolic equation, Soviet Math., 33 (1986), 435-437.

[25] A.A. Samarskii, Some problems in differential equations theory, Differents. Uravn. 16 (1980), 1925-1935.

Department of Mathematics and Informatics; The Larbi Ben M'hidi University, Oum El Bouaghi, Algeria

*Corresponding author: taki_maths@live.fr