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# AN IMPLICIT ALGORITHM FOR A FAMILY OF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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ABSTRACT. In this paper, we establish some strong convergence theorems of an implicit algorithm for a finite family of of total asymptotically nonexpansive mappings in the setting of CAT(0) spaces. Our results extend and generalize several recent results from the current existing literatures (see, e.g., [2, 9, 14, 16, 17, 25, 29]).

## 1. Introduction and Preliminaries

A metric space $(X, d)$ is said to be a length space if any two points of $X$ are joined by a rectifiable path (i.e., a path of finite length), and the distance between any two points of $X$ is taken to be the infimum of the lengths of all rectifiable paths joining them. In this case, $d$ is said to be a length metric (otherwise known as an inner metric or intrinsic metric). In case no rectifiable path joins two points of the space, the distance between them is taken to be $\infty$.

A *geodesic* path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a mapping $c$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$, and let $d(c(t), c(t')) = |t - t'|$ for $t, t' \in [0, l]$. In particular, $c$ is an isometry, and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) *segment* joining $x$ and $y$. We say that $X$ is (i) a *geodesic space* if any two points of $X$ are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining $x$ to $y$.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [3]).

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ inequality.

Let $\triangle$ be a geodesic triangle in $X$, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{1.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [12]). If $x, y_1, y_2$ are points of a $CAT(0)$ space and $y_0$ is the mid-point of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{1.2}$$

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The inequality (1.2) is the $(CN)$ inequality of Bruhat and Tits [4]. The above inequality has been extended in [6] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha)y) &\leq \alpha d^2(z,x) + (1-\alpha)d^2(z,y) \\ &\quad -\alpha(1-\alpha)d^2(x,y) \end{aligned} \tag{1.3}$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the $(CN)$ inequality (see [[3], page 163]). Moreover, if $X$ is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z,x) + (1-\alpha)d(z,y), \tag{1.4}$$

for any $z \in X$ and $[x,y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$.

A subset $C$ of a $CAT(0)$ space $X$ is convex if for any $x, y \in C$, we have $[x,y] \subset C$.

Let $T$ be a self mapping on a nonempty subset $C$ of $X$. Denote the set of fixed points of $T$ by $F(T) = \{x \in C : T(x) = x\}$. We say that $T$ is:

(1) nonexpansive if $d(Tx, Ty) \leq d(x,y)$ for all $x, y \in C$;

(2) asymptotically nonexpansive ([10]) if there exists a sequence $\{r_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that $d(T^n x, T^n y) \leq (1+r_n)d(x,y)$ for all $x, y \in C$ and $n \geq 1$;

(3) uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L\, d(x,y)$ for all $x, y \in C$ and $n \geq 1$;

(4) semi-compact if for a sequence $\{x_n\}$ in $C$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in C$.

*Remark* 1.1. From the above definitions, it is clear that each nonexpansive mapping is an asymptotically nonexpansive mapping with the constant sequence $\{k_n\} = \{1\}$, $\forall\, n \geq 1$ and an asymptotically nonexpansive mapping is a uniformly $L$-Lipschitzian mapping with $L = \sup_{n\geq 1}\{k_n\}$.

Chang et al. [5] defined the concept of total asymptotically nonexpansive mapping as follows.

**Definition 1.2.** ([5] Definition 2.1) Let $(X,d)$ be a metric space, $K$ be its nonempty subset and let $T\colon K \to K$ be a mapping. $T$ is said to be a total asymptotically nonexpansive mapping if there exist non-negative real sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\psi\colon [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x,y) + \nu_n \psi(d(x,y)) + \mu_n$$

for all $x, y \in K$ and $n \geq 1$.

*Remark* 1.3. From the above definition, it is clear that each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\mu_n = 0$, $\nu_n = k_n - 1$ for all $n \geq 1$, $\psi(t) = t$, $t \geq 0$.

Recently, there are a lot of papers have appeared on the iterative approximation of fixed points of asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings, asymptotically nonexpansive mappings in the intermediate sense and their generalizations through Ishikawa, S-iteration, modified S-iteration, Noor iteration and implicit iterations in uniformly convex Banach spaces, convex metric spaces and CAT(0) spaces (see, e.g., [1, 2, 5, 8, 9, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24]).

Let $E$ be a Hilbert space, let $K$ be a nonempty closed convex subset of $E$ and let $\{T_i\}\colon K \to K$ $\{i = 1, 2, \ldots, N\}$ be nonexpansive mappings. In 2001, Xu and Ori [29] introduced the following implicit iteration process $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{n(mod\, N)}x_n \text{ for } n \geq 1, \tag{1.5}$$

where $x_0 \in K$ is an initial point, $\{\alpha_n\}$ is a real sequence in $(0,1)$ and proved a weak convergence of the sequence $\{x_n\}$ defined by (1.5) to a common fixed point $p \in F = \cap_{i=1}^{N} F(T_i)$.

In 2003, Sun [27] introduced the following implicit iterative sequence $\{x_n\}$

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} x_n \text{ for } n \geq 1, \tag{1.6}$$

for a finite family of asymptotically quasi-nonexpansive self-mappings on a bounded closed convex subset $K$ of a Hilbert space $E$ with $\{\alpha_n\}$ a real sequence in $(0,1)$ and an initial point $x_0 \in K$, where $n = (k(n)-1)N + i(n)$, $1(n) \in \{1, 2, \dots, N\}$, and proved a strong convergence of the sequence $\{x_n\}$ defined (1.6) to a common fixed point $p \in F = \cap_{i=1}^{N} F(T_i)$. The result of Sun [27] generalized and extended the corresponding main result of Wittmann [28] and Xu and Ori [29].

Inspired and motivated by [28, 29], we now define a modified implicit iteration process for a finite family of total asymptotically nonexpansive mappings as below.

**Modified implicit iterative process in CAT(0) space**

Let $C$ be a nonempty closed convex subset of a CAT(0) space $X$, and $\{T_1, T_2, \dots, T_N\}$ be a finite family of $N$ $(\{\mu_{i,n}\}, \{\nu_{i,n}\}, \psi_i)$-total asymptotically nonexpansive self mappings on $C$. From an arbitrary $x_0 \in C$, we define the sequence $\{x_n\}$ by:

$$\begin{aligned} x_1 &= (1-\alpha_1)x_0 \oplus \alpha_1 T_1 x_1, \\ x_2 &= (1-\alpha_2)x_1 \oplus \alpha_2 T_2 x_2, \\ &\vdots \\ x_N &= (1-\alpha_N)x_{N-1} \oplus \alpha_N T_N x_N, \\ x_{N+1} &= (1-\alpha_{N+1})x_N \oplus \alpha_{N+1} T_1^2 x_{N+1}, \\ &\vdots \\ x_{2N} &= (1-\alpha_{2N})x_{2N} \oplus \alpha_{2N} T_N^2 x_{2N}, \\ x_{2N+1} &= (1-\alpha_{2N+1})x_{2N+1} \oplus \alpha_{2N+1} T_1^3 x_{2N+1}, \\ &\vdots \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ is an appropriate sequence in $(0,1)$.

The above iteration can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} \oplus (1-\alpha_n) T_{i(n)}^{k(n)} x_n, \text{ for } n \geq 1 \tag{1.8}$$

where $n = \big(k(n)-1\big)N + i(n)$, $k(n) > 1$ is a positive integer such that $k(n) \to \infty$ as $n \to \infty$.

Let $X$ be a CAT(0) space. Then, the following inequality holds:

$$d(\lambda\, x \oplus (1-\lambda)z, \lambda\, y \oplus (1-\lambda)w) \leq \lambda\, d(x,y) + (1-\lambda) d(z,w), \tag{1.9}$$

for all $x, y, z, w \in X$ (see [6]).

Let $\{T_i : i \in I = \{1, 2, \dots, N\}\}$ be the set of $N$ uniformly $L_i$ $(i = 1, 2, \dots, N)$-Lipschitzian self mappings of $C$. We show that (1.8) exists. Let $x_0 \in C$ and $x_1 = \alpha_1 x_0 \oplus (1-\alpha_1) T_1 x_1$. Define $W : C \to C$ by $W(x) = \alpha_1 x_0 \oplus (1-\alpha_1) T_1 x$ for $x \in C$. The existence of $x_1$ is guaranteed if $W$ has a fixed point. For any $x, y \in C$, we have

$$\begin{aligned} d(Wx, Wy) &\leq (1-\alpha_1) d(T_1 x, T_1 y) \leq (1-\alpha_1) L_1\, d(x,y) \\ &\leq (1-\alpha_1) L\, d(x,y) \end{aligned} \tag{1.10}$$

where $L = \max\{L_i : i \in I\}$. Now, $W$ is a contraction if $(1-\alpha_1)L < 1$ or $L > 1/(1-\alpha_1)$. As $\alpha_1 \in (0,1)$, therefore $W$ is a contraction if $1 < L < 2$. By the Banach contraction principle $W$ has a unique fixed point. Thus, the existence of $x_1$ is established. Thus, the implicit algorithm (1.8) is well defined.

The goal of this paper is to study strong convergence of iterative algorithm (1.8) for the class of uniformly $L_i$-Lipschitzian and $(\{\mu_{i,n}\},\{\nu_{i,n}\},\psi_i)$-total asymptotically nonexpansive mappings (for $i=1,2,\dots,N$) in the setting of CAT(0) spaces. Our results extend, improve and generalize several results from the current existing literature.

We need the following useful notion and lemmas for the development of our main results.

Let $\{T_i : i \in I\}$ be the set of $N$ self mappings of $C$. A mapping $T\colon C \to C$ is said to satisfy condition $(A)$ if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0)=0$, $f(r)>0$ for all $r \in (0,\infty)$ such that $d(x.p) \geq f(d(x,\mathcal{F}(\mathcal{T})))$ for $x \in C$ where $d(x,F(T)) = \inf\{d(x,p) : p \in F(T) \neq \emptyset\}$. Condition $(A)$ was introduced by Senter and Dotson [26].

**Lemma 1.4.** *([6]) Let $X$ be a CAT(0) space.*
*(i) For $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that*

$$d(x,z) = t\,d(x,y) \quad \text{and} \quad d(y,z) = (1-t)\,d(x,y). \quad (A)$$

*We use the notation $(1-t)x \oplus ty$ for the unique point $z$ satisfying (A).*
*(ii) For $x,y,z \in X$ and $t \in [0,1]$, we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)\,d(x,z) + t\,d(y,z).$$

**Lemma 1.5.** *([18]) Suppose that $\{a_n\}$, $\{b_n\}$ and $\{r_n\}$ be sequences of nonnegative numbers such that $a_{n+1} \leq (1+b_n)a_n + r_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

## 2. Main Results

In this section, we establish strong convergence theorems using implicit iteration scheme (1.8) for $(\{\mu_{i,n}\},\{\nu_{i,n}\},\psi_i)$-total asymptotically nonexpansive mappings (for $i=1,2,\dots,N$) in the setting of CAT(0) spaces.

**Lemma 2.1.** *Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$. Let $\{T_i : i \in I\}$ be $N$ uniformly $L_i$-Lipschitzian and $(\{\mu_{i,n}\},\{\nu_{i,n}\},\psi_i)$-total asymptotically nonexpansive mappings with $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by the algorithm (1.8), where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1/2)$. If the following conditions are satisfied:*

*(i) $\sum_{n=1}^{\infty} \mu_{i,n} < \infty$, $\sum_{n=1}^{\infty} \nu_{i,n} < \infty$ for $i \in I$;*

*(ii) there exists a constant $M>0$ such that $\psi(t) \leq Mt$, $t \geq 0$, where $\psi(a) = \max\{\psi_i(a) : i \in I\}$, $a \geq 0$.*

*Then $\lim_{n\to\infty} d(x_n,p)$ and $\lim_{n\to\infty} d(x_n,\mathcal{F})$ exist for $p \in \mathcal{F}$.*

*Proof.* Let $p \in \mathcal{F}$. Then, from (1.8) and Lemma 1.4(ii), we have

$$\begin{aligned}
d(x_n,p) &= d(\alpha_n x_{n-1} \oplus (1-\alpha_n)T_{i(n)}^{k(n)} x_n, p)\\
&\leq \alpha_n d(x_{n-1},p) + (1-\alpha_n) d(T_{i(n)}^{k(n)} x_n, p)\\
&\leq \alpha_n d(x_{n-1},p) + (1-\alpha_n)[d(x_n,p) + \nu_{i,k(n)}\psi(d(x_n,p)) + \mu_{i,k(n)}]\\
&\leq \alpha_n d(x_{n-1},p) + (1-\alpha_n)[d(x_n,p) + M\nu_{i,k(n)}\, d(x_n,p) + \mu_{i,k(n)}]\\
&= \alpha_n d(x_{n-1},p) + (1-\alpha_n)[(1+M\nu_{i,k(n)})\, d(x_n,p) + \mu_{i,k(n)}] \qquad (2.1)\\
&\leq \alpha_n d(x_{n-1},p) + (1-\alpha_n + M\nu_{i,k(n)})\, d(x_n,p) + (1-\alpha_n)\mu_{i,k(n)}.
\end{aligned}$$

Since $\alpha_n \in [\delta, 1-\delta]$, the above inequality gives that

$$d(x_n,p) \leq d(x_{n-1},p) + \frac{M\nu_{i,k(n)}}{\delta} d(x_n,p) + \Big(\frac{1}{\delta} - 1\Big)\mu_{i,k(n)}. \qquad (2.2)$$

On simplification, we get that

$$\begin{aligned} d(x_n, p) &\le \Big(\frac{\delta}{\delta - M\nu_{i,k(n)}}\Big) d(x_{n-1}, p) + \Big(\frac{1}{\delta} - 1\Big)\Big(\frac{\delta}{\delta - M\nu_{i,k(n)}}\Big)\mu_{i,k(n)} \\ &= \Big(1 + \frac{M\nu_{i,k(n)}}{\delta - M\nu_{i,k(n)}}\Big) d(x_{n-1}, p) + \Big(\frac{1}{\delta} - 1\Big)\Big(\frac{\delta}{\delta - M\nu_{i,k(n)}}\Big)\mu_{i,k(n)} \\ &= (1 + A_{i,k(n)}) d(x_{n-1}, p) + B_{i,k(n)} \qquad (2.3) \end{aligned}$$

where $A_{i,k(n)} = \frac{M\nu_{i,k(n)}}{\delta - M\nu_{i,k(n)}}$ and $B_{i,k(n)} = \Big(\frac{1}{\delta} - 1\Big)\Big(\frac{\delta}{\delta - M\nu_{i,k(n)}}\Big)\mu_{i,k(n)}$. Since $\sum_{k(n)=1}^{\infty} \nu_{i,k(n)} < \infty$ for $i \in I$ therefore $\lim_{k(n)\to\infty} \nu_{i,k(n)} = 0$, and hence, there exists a natural number $n_1$ such that $\nu_{ik(n)} < \delta/2$ for $k(n) \ge n_1/N + 1$ or $n > n_1$. Then, we have that $\sum_{k(n)=1}^{\infty} A_{i,k(n)} < \Big(\frac{2M}{\delta(2-M)}\Big) \sum_{k(n)=1}^{\infty} \nu_{ik(n)} < \infty$. Similarly, $\sum_{k(n)=1}^{\infty} w_{i,k(n)} < \infty$. Similarly, $\sum_{k(n)=1}^{\infty} B_{i,k(n)} < \infty$.

Now, for any $p \in \mathcal{F}$, from (2.3), for $k(n) \ge n_1/N + 1$, we have

$$d(x_n, \mathcal{F}) \le (1 + A_{i,k(n)})\, d(x_{n-1}, \mathcal{F}) + B_{i,k(n)}, \qquad (2.4)$$

By Lemma 1.5, (2.3) and (2.4), we obtain $\lim_{n\to\infty} d(x_n, p)$ and $\lim_{n\to\infty} d(x_n, \mathcal{F})$ both exist. This completes the proof. □

**Theorem 2.2.** *Let $C$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$. Let $\{T_i : i \in I\}$ be $N$ uniformly $L_i$-Lipschitzian and $(\{\mu_{i,n}\}, \{\nu_{i,n}\}, \psi_i)$-total asymptotically nonexpansive mappings with $\mathcal{F} = \cap_{i=1}^{N} F(T_i)$ is nonempty and closed. Suppose that the sequence $\{x_n\}$ is defined by the algorithm (1.8), where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$. If the following conditions are satisfied:*

*(i) $\sum_{n=1}^{\infty} \mu_{i,n} < \infty$, $\sum_{n=1}^{\infty} \nu_{i,n} < \infty$ for $i \in I$;*

*(ii) there exists a constant $M > 0$ such that $\psi(t) \le Mt$, $t \ge 0$, where $\psi(a) = \max\{\psi_i(a) : i \in I\}$, $a \ge 0$.*

*Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in I\}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

*Proof.* If $x_n \to p$ as $n \to \infty$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \le d(x_n, \mathcal{F}) \le d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By Lemma 1.5, we have that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. Further, by assumption $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, we conclude that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence.

Since $x \le exp(x-1)$ for $x \ge 1$, therefore from (2.3), we have

$$\begin{aligned} d(x_{n+m}, p) &\le (1 + A_{i,k(n)}) d(x_{n-1}, p) + B_{i,k(n)} \\ &\le \Big(e^{\sum_{i=1}^{N}\sum_{k(n)=1}^{\infty} A_{i,k(n)}}\Big)\, d(x_n, p) + \sum_{i=1}^{N}\sum_{k(n)=1}^{\infty} B_{i,k(n)} \\ &< R\, d(x_n, p) + R \sum_{i=1}^{N}\sum_{k(n)=1}^{\infty} B_{i,k(n)} \qquad (2.5) \end{aligned}$$

for all natural numbers $m, n$, where $R = \Big(e^{\sum_{i=1}^{N}\sum_{k(n)=1}^{\infty} A_{i,k(n)}}\Big) + 1 < \infty$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\} \subset \mathcal{F}$

such that $d(x_{n_k}, p_{n_k}) \to 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$d(x_{n_k}, p_{n_k}) < \frac{\varepsilon}{4R} \quad \text{and} \quad \sum_{i=1}^{N} \sum_{j=n_{k_\varepsilon}}^{\infty} B_{i,j} < \frac{\varepsilon}{4R} \tag{2.6}$$

for $k \geq k_\varepsilon$.

Hence, for any $m \in \mathbb{N}$ and for $n \geq n_{k_\varepsilon}$, by (2.5) we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_{n_k}) + d(x_n, p_{n_k}) \\ &\leq R\, d(x_n, p_{n_k}) + R \sum_{i=1}^{N} \sum_{j=n_{k_\varepsilon}}^{\infty} B_{i,j} \\ &\quad + R\, d(x_n, p_{n_k}) + R \sum_{i=1}^{N} \sum_{j=n_{k_\varepsilon}}^{\infty} B_{i,j} \\ &= 2R\, d(x_n, p_{n_k}) + 2R \sum_{i=1}^{N} \sum_{j=n_{k_\varepsilon}}^{\infty} B_{i,j} \\ &< 2R.\frac{\varepsilon}{4R} + 2R.\frac{\varepsilon}{4R} = \varepsilon. \end{aligned} \tag{2.7}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $C$. By the completeness of $C$, we can assume that $\lim_{n\to\infty} x_n = q$. We will prove that $q$ is a common fixed point of $\{T_i : i \in I\}$, that is, we will show that $q \in \mathcal{F}$. Since $C$ is closed, therefore $q \in C$. Next, we show that $q \in \mathcal{F}$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, gives that $d(q, \mathcal{F}) = 0$. Since $\mathcal{F}$ is closed, $q \in \mathcal{F}$. Thus $q$ is a common fixed point of $\{T_i : i \in I\}$. This completes the proof. □

**Theorem 2.3.** *Let $C$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$. Let $\{T_i : i \in I\}$ be $N$ uniformly $L_i$-Lipschitzian and $(\{\mu_{i,n}\}, \{\nu_{i,n}\}, \psi_i)$-total asymptotically nonexpansive mappings with $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ defined by the algorithm (1.8), where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$. If the following conditions are satisfied:*

*(i) $\sum_{n=1}^{\infty} \mu_{i,n} < \infty$, $\sum_{n=1}^{\infty} \nu_{i,n} < \infty$ for $i \in I$;*

*(ii) there exists a constant $M > 0$ such that $\psi(t) \leq Mt$, $t \geq 0$, where $\psi(a) = \max\{\psi_i(a) : i \in I\}$, $a \geq 0$.*

*Then $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = \limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0$ if $\{x_n\}$ converges to a unique point in $\mathcal{F}$.*

*Proof.* Let $p \in \mathcal{F}$. Since $\{x_n\}$ converges to $p$, $\lim_{n\to\infty} d(x_n, p) = 0$. So, for a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, p) < \varepsilon \ \text{ for } n \geq n_0.$$

Taking the infimum over $p \in \mathcal{F}$, we obtain that

$$d(x_n, \mathcal{F}) < \varepsilon \ \text{ for } n \geq n_0.$$

This means that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Thus we obtain that

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = \limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0.$$

This completes the proof. □

As shown in the preceding proof, the property needed to assure that $p \in \mathcal{F}$ is exactly the following one. Given any sequence $\{u_n\}$ of real numbers there is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\lim_{j\to\infty} u_{n_j} = \liminf_{n\to\infty} u_n$. In general, if $\{u_{m_j}\}$ is a convergent subsequence of $\{u_n\}$, then $\liminf_{n\to\infty} u_n \leq \lim_{j\to\infty} u_{m_j}$. This immediately gives the following result.

**Corollary 2.4.** *Let $C$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$. Let $\{T_i : i \in I\}$ be $N$ uniformly $L_i$-Lipschitzian and $(\{\mu_{i,n}\}, \{\nu_{i,n}\}, \psi_i)$-total asymptotically nonexpansive mappings with $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ defined by the algorithm (1.8), where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$. If the following conditions are satisfied:*

*(i) $\sum_{n=1}^{\infty} \mu_{i,n} < \infty$, $\sum_{n=1}^{\infty} \nu_{i,n} < \infty$ for $i \in I$;*

*(ii) there exists a constant $M > 0$ such that $\psi(t) \leq Mt$, $t \geq 0$, where $\psi(a) = \max\{\psi_i(a) : i \in I\}$, $a \geq 0$.*

*Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in I\}$ if and only if there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $p \in \mathcal{F}$.*

**Corollary 2.5.** *Let $C$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$. Let $\{T_i : i \in I\}$ be $N$ asymptotically nonexpansive mappings of $C$ with $\{k_{i,n}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_{i,n} - 1) < \infty$ for all $i \in I$. Suppose that $\mathcal{F} = \cap_{i=1}^{N} F(T_i)$ is nonempty and closed. Starting from arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by the algorithm (1.8), where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in I\}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

*Proof.* Follows from Theorem 2.2 with $\mu_{i,n} = 0$, $\nu_{i,n} = (k_{i,n} - 1)$ for all $i \in I$ and $\psi(t) = t$, $t \geq 0$. This completes the proof. □

**Lemma 2.6.** *Let $C$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$. Let $\{T_i : i \in I\}$ be $N$ uniformly $L_i$-Lipschitzian and $(\{\mu_{i,n}\}, \{\nu_{i,n}\}, \psi_i)$-total asymptotically nonexpansive mappings with $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by the algorithm (1.8), where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$. If the following conditions are satisfied:*

*(i) $\sum_{n=1}^{\infty} \mu_{i,n} < \infty$, $\sum_{n=1}^{\infty} \nu_{i,n} < \infty$ for $i \in I$;*

*(ii) there exists a constant $M > 0$ such that $\psi(t) \leq Mt$, $t \geq 0$, where $\psi(a) = \max\{\psi_i(a) : i \in I\}$, $a \geq 0$.*

*Then $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$ for each $l \in I$.*

*Proof.* Let $L = \max\{L_i : i \in I\}$. Note that $\{x_n\}$ is bounded as $\lim_{n\to\infty} d(x_n, p)$ exists by Lemma 2.1. So, there exists $R' > 0$ and $x_0 \in X$ such that $x_n \in B'_R(x_0) = \{x : d(x, x_0) < R'\}$ for $n \geq 1$. Denote $d(x_{n-1}, T_{i(n)}^{k(n)})$ by $\rho_n$.

We claim that $\lim_{n\to\infty} \rho_n = 0$.

For any $p \in \mathcal{F}$, apply (1.3) to (1.8), we have

$$\begin{aligned}
d^2(x_n, p) &= d^2(\alpha_n x_{n-1} \oplus (1-\alpha_n) T_{i(n)}^{k(n)} x_n, p) \\
&\leq \alpha_n d^2(x_{n-1}, p) + (1-\alpha_n) d^2(T_{i(n)}^{k(n)} x_n, p) \\
&\quad -\alpha_n(1-\alpha_n) d^2(x_{n-1}, T_{i(n)}^{k(n)} x_n) \\
&\leq \alpha_n d^2(x_{n-1}, p) + (1-\alpha_n)[d(x_n, p) + \nu_{i,k(n)} \psi(d(x_n, p)) + \mu_{i,k(n)}]^2 \\
&\quad -\alpha_n(1-\alpha_n) d^2(x_{n-1}, T_{i(n)}^{k(n)} x_n) \\
&\leq \alpha_n d^2(x_{n-1}, p) + (1-\alpha_n)[d(x_n, p) + M\nu_{i,k(n)} d(x_n, p) + \mu_{i,k(n)}]^2 \\
&\quad -\alpha_n(1-\alpha_n) d^2(x_{n-1}, T_{i(n)}^{k(n)} x_n) \\
&= \alpha_n d^2(x_{n-1}, p) + (1-\alpha_n)[(1 + M\nu_{i,k(n)}) d(x_n, p) + \mu_{i,k(n)}]^2 \\
&\quad -\alpha_n(1-\alpha_n) d^2(x_{n-1}, T_{i(n)}^{k(n)} x_n).
\end{aligned} \tag{2.8}$$

Now, using (2.3), we get

$$\begin{aligned}\alpha_n(1-\alpha_n)\rho_n^2 &\le \alpha_n d^2(x_{n-1},p)-d^2(x_n,p)+(1-\alpha_n)[(1+M\nu_{i,k(n)})\\ &\quad\times\{(1+A_{i,k(n)})\,d(x_{n-1},p)+B_{i,k(n)}\}+\mu_{i,k(n)}]^2\\ &= \alpha_n d^2(x_{n-1},p)-d^2(x_n,p)+(1-\alpha_n)[(1+M\nu_{i,k(n)})(1+A_{i,k(n)})\times\\ &\quad d(x_{n-1},p)+(1+M\nu_{i,k(n)})B_{i,k(n)}+\mu_{i,k(n)}]^2\\ &= \alpha_n d^2(x_{n-1},p)-d^2(x_n,p)+(1-\alpha_n)[(1+f_{i,k(n)})d(x_{n-1},p)\times\\ &\quad +g_{i,k(n)}]^2\end{aligned} \tag{2.9}$$

where $f_{i,k(n)}=M\nu_{i,k(n)}+A_{i,k(n)}+MA_{i,k(n)}\nu_{i,k(n)}$ and $g_{i,k(n)}=(1+M\nu_{i,k(n)})B_{i,k(n)}+\mu_{i,k(n)}$. Since $\sum_{k(n)=1}^{\infty}\mu_{i,k(n)}<\infty$, $\sum_{k(n)=1}^{\infty}\nu_{i,k(n)}<\infty$ and $\sum_{k(n)=1}^{\infty}B_{i,k(n)}<\infty$, it follows that $\sum_{k(n)=1}^{\infty}f_{i,k(n)}<\infty$ and $\sum_{k(n)=1}^{\infty}g_{i,k(n)}<\infty$. Again, note that

$$\begin{aligned}\alpha_n(1-\alpha_n)\rho_n^2 &\le \alpha_n d^2(x_{n-1},p)-d^2(x_n,p)+(1-\alpha_n)[d(x_{n-1},p)+l_{i,k(n)}]^2\\ &= d^2(x_{n-1},p)-d^2(x_n,p)+(1-\alpha_n)q_{i,k(n)},\end{aligned} \tag{2.10}$$

where $l_{i,k(n)}=f_{i,k(n)}d(x_{n-1},p)+g_{i,k(n)}$ and $q_{i,k(n)}=l_{i,k(n)}^2+2l_{i,k(n)}d(x_{n-1},p)$. Since $\{d(x_{n-1},p)\}$ is convergent, $\sum_{k(n)=1}^{\infty}f_{i,k(n)}<\infty$ and $\sum_{k(n)=1}^{\infty}g_{i,k(n)}<\infty$, it follows that $\sum_{k(n)=1}^{\infty}l_{i,k(n)}<\infty$ and $\sum_{k(n)=1}^{\infty}q_{i,k(n)}<\infty$. This implies that

$$\begin{aligned}\rho_n^2 &\le \frac{1}{\alpha_n(1-\alpha_n)}[d^2(x_{n-1},p)-d^2(x_n,p)]+\frac{q_{i,k(n)}}{\alpha_n}\\ &\le \frac{1}{\delta^2}[d^2(x_{n-1},p)-d^2(x_n,p)]+\frac{q_{i,k(n)}}{\delta}.\end{aligned} \tag{2.11}$$

Since $\sum_{k(n)=1}^{\infty}q_{i,k(n)}<\infty$, $\{d(x_n,p)\}$ is convergent and $\delta>0$, therefore on taking limit as $n\to\infty$ in (2.11), we get

$$\lim_{n\to\infty}\rho_n=0. \tag{2.12}$$

Further,

$$\begin{aligned}d(x_n,x_{n-1}) &\le (1-\alpha_n)d\big(T_{i(n)}^{k(n)}x_n,x_{n-1}\big)\\ &= (1-\alpha_n)\rho_n\le(1-\delta)\rho_n,\end{aligned} \tag{2.13}$$

which implies that $\lim_{n\to\infty}d(x_n,x_{n-1})=0$.

For a fixed $j\in I$, we have $d(x_{n+j},x_n)\le d(x_{n+j},x_{n+j-1})+\cdots+d(x_n,x_{n-1})$, and hence

$$\lim_{n\to\infty}d(x_{n+j},x_n)=0 \text{ for } j\in I. \tag{2.14}$$

For $n>N$, $n=(n-N)(mod\,N)$. Also, $n=(k(n)-1)N+i(n)$. Hence, $n-N=((k(n)-1)-1)N+i(n)=(k(n-N))N+i(n-N)$. That is, $k(n-N)=k(n)-1$ and $i(n-N)=i(n)$.

Therefore, we have

$$\begin{aligned}d(x_{n-1},T_nx_n) &\le d\big(x_{n-1},T_{i(n)}^{k(n)}x_n\big)+d\big(T_{i(n)}^{k(n)}x_n,T_nx_n\big)\\ &\le \rho_n+L\,d\big(T_{i(n)}^{k(n)-1}x_n,x_n\big)\\ &\le \rho_n+L^2\,d(x_n,x_{n-N})+L\,d\big(T_{i(n-N)}^{k(n-N)}x_{n-N},x_{(n-N)-1}\big)\\ &\quad +L\,d(x_{(n-N)-1},x_n)\\ &\le \rho_n+L^2\,d(x_n,x_{n-N})+L\,\rho_{n-N}\\ &\quad +L\,d(x_{(n-N)-1},x_n).\end{aligned} \tag{2.15}$$

Using (2.12) and (2.14) in (2.15), we get

$$\lim_{n\to\infty}d(x_{n-1},T_nx_n)=0. \tag{2.16}$$

Since

$$d(x_n, T_n x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n x_n), \tag{2.17}$$

using (2.13) and (2.16) in (2.17), we have

$$\lim_{n\to\infty} d(x_n, T_n x_n) = 0. \tag{2.18}$$

Hence, for all $l \in I$, we have

$$\begin{aligned} d(x_n, T_{n+l}x_n) &\leq d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l}x_{n+l}) \\ &\quad + d(T_{n+l}x_{n+l}, T_{n+l}x_n) \\ &\leq (1+L)\, d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l}x_{n+l}). \end{aligned} \tag{2.19}$$

Using (2.14) and (2.18) in (2.19), we obtain

$$\lim_{n\to\infty} d(x_n, T_{n+l}x_n) = 0, \; \forall\, l \in I. \tag{2.20}$$

Thus, $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$ for $l \in I$. This completes the proof. □

As an application of Theorem 2.2, we establish some strong convergence results as follows.

**Theorem 2.7.** *Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$. Let $\{T_i : i \in I\}$ be $N$ uniformly $L_i$-Lipschitzian and $(\{\mu_{i,n}\}, \{\nu_{i,n}\}, \psi_i)$-total asymptotically nonexpansive mappings with $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists one member $T$ in $\{T_i : i \in I\}$ which is either semicompact or satisfies condition $(A)$. Suppose that the sequence $\{x_n\}$ is defined by the algorithm (1.8), where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$. If the following conditions are satisfied:*

*(i) $\sum_{n=1}^{\infty} \mu_{i,n} < \infty$, $\sum_{n=1}^{\infty} \nu_{i,n} < \infty$ for $i \in I$;*

*(ii) there exists a constant $M > 0$ such that $\psi(t) \leq Mt$, $t \geq 0$, where $\psi(a) = \max\{\psi_i(a) : i \in I\}$, $a \geq 0$.*

*Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in I\}$.*

*Proof.* By Lemma 2.1, we see that

$$\lim_{n\to\infty} d(x_n, x^*) \quad \text{and} \quad \lim_{n\to\infty} d(x_n, \mathcal{F}) \text{ exist.}$$

Let one of $T_i's$, say, $T_s$, $s \in I$ is either semicompact or satisfies condition $(A)$. If $T_s$ is semicompact, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to z \in C$ as $j \to \infty$. Now, Lemma 2.6 guarantees that $\lim_{n\to\infty} d(x_{n_j}, T_s x_{n_j}) = 0$ for $s \in I$ and so $d(z, T_s z) = 0$ for $s \in I$. This implies that $z \in \mathcal{F}$. Therefore, $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. If $T_s$ satisfies condition $(A)$, then we also have $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Now, Theorem 2.2 implies that $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. This completes the proof. □

**Theorem 2.8.** *Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$. Let $\{T_i : i \in I\}$ be $N$ $(\{\mu_{i,n}\}, \{\nu_{i,n}\}, \psi_i)$-total asymptotically nonexpansive mappings. Suppose that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ ($T_i, i = 1, 2, \dots, N$, need not to be continuous). Starting from arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by the algorithm (1.8), where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$. Assume that $(i')$ $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ if the sequence $\{z_n\}$ in $C$ satisfies $(ii')$ $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$, then $\liminf_{n\to\infty} d(z_n, \mathcal{F}) = 0$ or $\limsup_{n\to\infty} d(z_n, \mathcal{F}) = 0$. If the following conditions are satisfied:*

*(i) $\sum_{n=1}^{\infty} \mu_{i,n} < \infty$, $\sum_{n=1}^{\infty} \nu_{i,n} < \infty$ for all $i \in I$;*

*(ii) there exists a constant $M > 0$ such that $\psi(t) \leq Mt$, $t \geq 0$, where $\psi(a) = \max\{\psi_i(a) : i \in I\}$, $a \geq 0$.*

*Then $\{x_n\}$ converges to a unique point in $\mathcal{F}$.*

*Proof.* By hypothesis $(i')$ and $(ii')$, we have that

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0 \text{ or } \limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0.$$

Therefore, we obtain from Theorem 2.2 that the sequence $\{x_n\}$ converges to a unique point in $\mathcal{F}$. This completes the proof. □

Finally, we obtain the following result from Theorem 2.7 as corollary.

**Corollary 2.9.** *Let $C$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$. Let $\{T_i : i \in I\}$ be $N$ asymptotically nonexpansive mappings of $C$ with $\{h_{i,n}\} \subset [1,\infty)$ for $i \in I$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, where $h_n = \max\{h_{i,n} : i \in I\}$. Suppose that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists one member $T$ in $\{T_i : i \in I\}$ which is either semicompact or satisfies condition $(A)$. From an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by algorithm (1.8), where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$.. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in I\}$.*

*Remark* 2.10. Our results extend, generalize and improve several corresponding approximation results from the current existing literature to the case of implicit iteration process and more general class of nonexpansive and asymptotically nonexpansive mappings considered in this paper (see, e.g., [2, 7, 14, 16, 17, 29] and many others).

*Remark* 2.11. Our results also extend the corresponding results [25] to the case of finite family of mappings and implicit iteration process considered in this paper.

**Example 2.12.** *(*[11]*, Example 3.1) Let $\mathbb{R}$ be the real line with the usual norm $\|.\|$ and $C = [-1, 1]$. Define a mapping $T : C \to C$ by*

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0). \end{cases}$$

*Then $T$ is an asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for $n \geq 1$ and uniformly L-Lipschtzian mapping with $L = \sup_{n\geq 1}\{k_n\}$ and hence it is a total asymptotically nonexpansive mapping by remark 1.3. Also the fixed point of $T$, that is, $F(T) = \{0\}$.*

## 3. Conclusion

In this paper, we establish some strong convergence theorems using implicit algorithm (1.8) for a finite family of $(\{\mu_{i,n}\}, \{\nu_{i,n}\}, \psi_i)$-total asymptotically nonexpansive mappings which is more general than the class of nonexpansive and asymptotically nonexpansive mappings in the framework of CAT(0) spaces.

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