

FACTS ABOUT THE FOURIER-STIELTJES TRANSFORM OF VECTOR MEASURES ON COMPACT GROUPS

YAOGAN MENSAH

ABSTRACT. This paper gives an interpretation of the Fourier-Stieltjes transform of vector measures by means of the tensor product of Hilbert spaces. It also extends the Kronecker product to some operators arising from the Fourier-Stieltjes transformation and associated with the equivalence classes of unitary representations of a compact group. We obtain among other results the effect of this product on convolution of vector measures.

1. INTRODUCTION

This paper inspects mainly two things: the Fourier-Stieltjes transformation and the Kronecker product. The importance of the Fourier transformation in mathematical science and engineering, for instance in signal processing, is well known and so we need not to lay emphasis on it. On the other hand, among the ways to bring together matrices there is the Kronecker product. It is extensively used in group theory and physics, specially in quantum information theory to determine for instance exact spin hamiltonian [9], [13]. In quantum physics the quantum states of a system is described by an hermitian positive semi-definite matrix with trace one. If X and Y represent the states of two quantum systems then the Kronecker product $X \otimes Y$ describes the joint system. Some other fundamental applications of the Kronecker product in signal processing, image processing or quantum computing can be found in [7], [11], [2], [10] and [14]. This paper aims to deepen the link between the Kronecker product and the Fourier-Stieltjes transform of vector measures.

The rest of the paper is organized as follows. In section 2 we recall the definition of vector measures on compact groups. The section 3 is divided into two parts. In the first part we give an interpretation of the Fourier-Stieltjes transform of vector measures as a bounded vector valued mappings on the tensor product of Hilbert spaces. The next part extends the Kronecker product to some operators arising from the Fourier-Stieltjes transformation and associated with the equivalence classes of unitary representations of a compact group. Here the effect of this product on the convolution of vector measures is obtained.

2. PRELIMINARIES

We summarize the definition of a vector measure on a compact group G following [3] and [6]. We denote by $\mathfrak{B}(G)$ the σ -field of Borel subsets of the compact group

2010 *Mathematics Subject Classification.* Primary: 43A77, secondary: 43A30.

Key words and phrases. vector measure, Fourier-Stieltjes transform, Kronecker product.

©2013 Authors retain the copyrights of their papers, and all open access articles are distributed under the terms of the Creative Commons Attribution License.

G and by \mathcal{A} (in the whole paper) a complex Banach algebra with topological dual \mathcal{A}^* . Also denote by $\langle x^*, x \rangle$ the duality between \mathcal{A}^* and \mathcal{A} .

A *vector measure* is a countably additive set function $\mu : \mathfrak{B}(G) \rightarrow \mathcal{A}$, that is, for any sequence (A_n) of pairwise disjoint subsets of $\mathfrak{B}(G)$, one has

$$(1) \quad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

where the second member of the above equality is convergent in the norm topology of \mathcal{A} . If μ is a vector measure on G then for each $x^* \in \mathcal{A}^*$ the measure defined by

$$(2) \quad x^* \mu(A) := \langle x^*, \mu(A) \rangle, \quad A \in \mathfrak{B}(G)$$

is a complex measure. Let f be a complex measurable Borel function f defined on G . The function f is said to be μ -integrable if the following two conditions are satisfied:

- (1) $\forall x^* \in \mathcal{A}^*$, f is $x^* \mu$ -integrable,
- (2) $\forall A \in \mathfrak{B}(G)$, $\exists y \in \mathcal{A}$, $\forall x^* \in \mathcal{A}^*$, $\langle x^*, y \rangle = \int_A f dx^* \mu$.

Then we set $y = \int_G f d\mu$.

Let $\mu : \mathfrak{B}(G) \rightarrow \mathcal{A}$ be a vector measure. The *semivariation* [3] of μ is the nonnegative function $|\mu|$ defined on $\mathfrak{B}(G)$ by

$$(3) \quad |\mu|(A) = \sup\{ |x^* \mu|(A) : x^* \in \mathcal{A}^*, \|x^*\| \leq 1 \}, \quad \forall A \in \mathfrak{B}(G)$$

where $|x^* \mu|$ is the variation of $x^* \mu$.

We recall that the variation of a scalar (real or complex) measure m is the extended nonnegative mapping $|m|$ defined on $\mathfrak{B}(G)$ by

$$(4) \quad |m|(A) = \sup_{\pi} \sum_{E \in \pi} |m(E)|$$

where the supremum is taken over all partitions π of A into finite number of pairwise disjoint members of $\mathfrak{B}(G)$ [3, page 2].

A vector measure μ is said to be of *bounded semivariation* if $|\mu|(G) < \infty$. The range of a vector measure is bounded if and only if it is of bounded semivariation [3, page 4]. So a vector measure is said to be *bounded* if it is of bounded semivariation. Denote by $M^1(G, \mathcal{A})$ the set of bounded \mathcal{A} -valued measures on G . The set $M^1(G, \mathcal{A})$ is equipped with the norm

$$(5) \quad \|\mu\| := \int_G \chi_G d|\mu| = |\mu|(G)$$

where χ_G is the characteristic function of G .

Let μ, ν be in $M^1(G, \mathcal{A})$. The *convolution product* of μ with ν is given by :

$$(6) \quad \mu * \nu(f) = \int \int_G f(gh) d\mu(g) d\nu(h), \quad f \in \mathcal{C}(G, \mathcal{A}).$$

where $\mathcal{C}(G, \mathcal{A})$ is the set of \mathcal{A} -valued continuous functions on G . Equipped with this product and the above norm, the space $M^1(G, \mathcal{A})$ is a complex Banach algebra.

3. MAIN RESULTS

3.1. Fourier-Stieltjes transformation: a tensor product interpretation.

The Fourier(-Stieltjes) transform of a complex valued function (or measure) on a compact group is well-known [5]. For a given function f , the Fourier transform of f is a collection of bounded operators on some Hilbert spaces. In [1], Assiamoua defined the Fourier-Stieltjes transform of a Banach algebra valued measure on a compact group by interpreting it as a collection of some continuous sesquilinear mappings. In this section, we use the modern technique of tensor product to keep the interpretation of the Fourier-Stieltjes transform of a vector measure as a collection of operators.

Let \widehat{G} be the unitary dual of the compact group G *i.e.* the set of equivalence classes of unitary irreducible representations of G . In any equivalence class σ belonging to \widehat{G} , we choose an element U^σ and denoted its hilbertian representation space by H_σ . Since G is compact, the Hilbert space H_σ is finite dimensional [4], and we denote its dimension by d_σ . We fix definitively a canonical basis $(\xi_1^\sigma, \dots, \xi_{d_\sigma}^\sigma)$ of H_σ . For $g \in G$, we set

$$(7) \quad u_{ij}^\sigma(g) = \langle U_g^\sigma \xi_j^\sigma, \xi_i^\sigma \rangle$$

where \langle, \rangle is the inner product in H_σ , and denote by $\overline{U^\sigma}$ the contragredient of the representation U^σ , that is the representation of G on H_σ which matrix elements are given by

$$(8) \quad \langle \overline{U_g^\sigma} \xi_j^\sigma, \xi_i^\sigma \rangle = \overline{u_{ij}^\sigma(g)},$$

where $\overline{u_{ij}^\sigma(g)}$ is the complex conjugate of $u_{ij}^\sigma(g)$. Let us denote by \overline{H}_σ the conjugate Hilbert space to H_σ . We denote by $H_\sigma \widehat{\otimes} \overline{H}_\sigma$ the tensor product of Hilbert spaces $H_\sigma \otimes \overline{H}_\sigma$ equipped with the projective tensor product norm. A basis of $H_\sigma \widehat{\otimes} \overline{H}_\sigma$ is $(\xi_i^\sigma \otimes \xi_j^\sigma)_{1 \leq i, j \leq d_\sigma}$. Then the Fourier-Stieltjes transform of a bounded \mathcal{A} -valued measure μ is the collection of linear operators $(\widehat{\mu}(\sigma))_{\sigma \in \widehat{G}}$ from $H_\sigma \widehat{\otimes} \overline{H}_\sigma$ into \mathcal{A} defined by :

$$(9) \quad \widehat{\mu}(\sigma)(\xi \otimes \eta) = \int_G \langle \overline{U_g^\sigma} \xi, \eta \rangle d\mu(g), \quad \xi, \eta \in H_\sigma.$$

Now, let $\mathfrak{B}(H_\sigma \widehat{\otimes} \overline{H}_\sigma, \mathcal{A})$ be the space of bounded linear mappings from $H_\sigma \widehat{\otimes} \overline{H}_\sigma$ into the Banach algebra \mathcal{A} . Let $\mathfrak{B}(\widehat{G}, \mathcal{A})$ be the bundle over \widehat{G} whose fiber at σ is $\mathfrak{B}(H_\sigma \widehat{\otimes} \overline{H}_\sigma, \mathcal{A})$, that is

$$(10) \quad \mathfrak{B}(\widehat{G}, \mathcal{A}) = \prod_{\sigma \in \widehat{G}} \mathfrak{B}(H_\sigma \widehat{\otimes} \overline{H}_\sigma, \mathcal{A}).$$

For $\varphi \in \mathfrak{B}(\widehat{G}, \mathcal{A})$, set

$$(11) \quad \|\varphi\|_\infty = \sup\{\|\varphi(\sigma)\| : \sigma \in \widehat{G}\}$$

where $\|\varphi(\sigma)\|$ denotes the operator norm of $\varphi(\sigma)$. Define

$$(12) \quad \mathfrak{B}_\infty(\widehat{G}, \mathcal{A}) = \{\varphi \in \mathfrak{B}(\widehat{G}, \mathcal{A}) : \|\varphi\|_\infty < \infty\}.$$

Theorem 3.1. *For $\mu \in M^1(G, \mathcal{A})$, we have $\widehat{\mu} \in \mathfrak{B}_\infty(\widehat{G}, \mathcal{A})$.*

Proof. It is clear that for each σ , the object $\widehat{\mu}(\sigma)$ is linear from $H_\sigma \widehat{\otimes} \overline{H}_\sigma$ into \mathcal{A} . Now

$$\begin{aligned} \|\widehat{\mu}(\sigma)(\xi \otimes \eta)\| &= \left\| \int_G \langle \overline{U}_g^\sigma \xi, \eta \rangle d\mu(g) \right\| \\ &\leq \int_G |\langle \overline{U}_g^\sigma \xi, \eta \rangle| |d\mu|(g) \\ &\leq \|\xi\| \|\eta\| \|\mu\| = \|\xi \otimes \eta\| \|\mu\|. \end{aligned}$$

Thus $\|\widehat{\mu}(\sigma)\| \leq \|\mu\|$ for each σ . Therefore $\|\widehat{\mu}\|_\infty \leq \|\mu\| < \infty$. \square

Let $\mu \in M^1(G, \mathcal{A})$ and $\sigma \in \widehat{G}$. We associate with the operator $\widehat{u}(\sigma)$ the matrix denoted by $\mathcal{M}(\widehat{u}(\sigma))$ which (i, j) -entry belongs to the Banach algebra \mathcal{A} and is given by

$$(13) \quad [\mathcal{M}(\widehat{u}(\sigma))]_{ij} = \widehat{\mu}(\sigma)(\xi_j^\sigma \otimes \xi_i^\sigma).$$

For vector measures μ and ν we denote by $(\widehat{\mu} \times \widehat{\nu})(\sigma)$ the operator from $H_\sigma \widehat{\otimes} \overline{H}_\sigma$ into \mathcal{A} associated with the product of matrices $\mathcal{M}(\widehat{\nu}(\sigma))\mathcal{M}(\widehat{\mu}(\sigma))$, that is

$$(14) \quad (\widehat{\mu} \times \widehat{\nu})(\sigma)(\xi_j^\sigma \otimes \xi_i^\sigma) = \sum_{k=1}^{d_\sigma} \widehat{\nu}(\sigma)(\xi_k^\sigma \otimes \xi_i^\sigma) \widehat{\mu}(\sigma)(\xi_j^\sigma \otimes \xi_k^\sigma).$$

The following theorem is the analogue of the *convolution theorem*.

Theorem 3.2. For $\mu, \nu \in M^1(G, \mathcal{A})$, we have $\widehat{\mu * \nu} = \widehat{\mu} \times \widehat{\nu}$.

Proof.

$$\begin{aligned} (\widehat{\mu * \nu})(\sigma)(\xi_j^\sigma \otimes \xi_i^\sigma) &= \int_G \langle \overline{U}_t^\sigma \xi_j^\sigma, \xi_i^\sigma \rangle d(\mu * \nu)(t) \\ &= \int_G \int_G \langle \overline{U}_{st}^\sigma \xi_j^\sigma, \xi_i^\sigma \rangle d\mu(s) d\nu(t) \\ &= \int_G \int_G \langle \overline{U}_t^\sigma \overline{U}_s^\sigma \xi_j^\sigma, \xi_i^\sigma \rangle d\mu(s) d\nu(t). \end{aligned}$$

Now, we express $\overline{U}_s^\sigma \xi_j^\sigma$ in the canonical basis of H_σ :

$$(15) \quad \overline{U}_s^\sigma \xi_j^\sigma = \sum_{k=1}^{d_\sigma} \langle \overline{U}_s^\sigma \xi_j^\sigma, \xi_k^\sigma \rangle \xi_k^\sigma.$$

So

$$\begin{aligned} (\widehat{\mu * \nu})(\sigma)(\xi_j^\sigma \otimes \xi_i^\sigma) &= \sum_{k=1}^{d_\sigma} \int_G \int_G \langle \overline{U}_s^\sigma \xi_j^\sigma, \xi_k^\sigma \rangle \langle \overline{U}_t^\sigma \xi_k^\sigma, \xi_i^\sigma \rangle d\mu(s) d\nu(t) \\ &= \sum_{k=1}^{d_\sigma} \int_G \langle \overline{U}_t^\sigma \xi_k^\sigma, \xi_i^\sigma \rangle d\nu(t) \int_G \langle \overline{U}_s^\sigma \xi_j^\sigma, \xi_k^\sigma \rangle d\mu(s) \\ &= \sum_{k=1}^{d_\sigma} \widehat{\nu}(\sigma)(\xi_k^\sigma \otimes \xi_i^\sigma) \widehat{\mu}(\sigma)(\xi_j^\sigma \otimes \xi_k^\sigma) \\ &= (\widehat{\mu} \times \widehat{\nu})(\sigma)(\xi_j^\sigma \otimes \xi_i^\sigma). \end{aligned}$$

\square

Remark: Application to convolution equations

The above result can be useful in the resolution of the convolution equation

$$(16) \quad f * h = g$$

where f , g and h (the unknown function) are \mathcal{A} -valued functions on G . Here f , g and h can be viewed as the vector measures $f dx$, $g dx$ and $h dx$ respectively where dx denotes the normalized Haar measure of G . For each $\sigma \in \widehat{G}$, the equation (16) is transformed into

$$(17) \quad \widehat{f}(\sigma) \times \widehat{h}(\sigma) = \widehat{g}(\sigma).$$

Therefore if the operator $\widehat{f}(\sigma)$ is invertible, $\widehat{h}(\sigma)$ can be derived from (17) and h is recovered by the following reconstruction formula

$$(18) \quad h = \sum_{\sigma \in \widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \widehat{h}(\sigma)(\xi_j^\sigma \otimes \xi_i^\sigma) u_{ij}^\sigma.$$

3.2. The mappings \mathfrak{T}_σ and their Kronecker product. Now we turn our attention over some matrix valued mappings acting on the set of bounded vector measures. Let $\mathcal{M}_{d_\sigma}(\mathcal{A})$ be the set of $d_\sigma \times d_\sigma$ matrices with entries in the Banach algebra \mathcal{A} .

We denote by \mathfrak{T}_σ the linear mapping

$$\begin{aligned} \mathfrak{T}_\sigma : M^1(G, \mathcal{A}) &\rightarrow \mathcal{M}_{d_\sigma}(\mathcal{A}) \\ \mu &\mapsto \mathfrak{T}_\sigma(\mu) = \mathcal{M}(\widehat{\mu}(\sigma)). \end{aligned}$$

We set

$$\Delta(\widehat{G}) = \{\mathfrak{T}_\sigma : \sigma \in \widehat{G}\}.$$

The following result shows the effect of each \mathfrak{T}_σ on convolution of vector measures.

Theorem 3.3. *For $\mathfrak{T}_\sigma \in \Delta(\widehat{G})$, $\mu, \nu \in M^1(G, \mathcal{A})$, we have:*

$$(19) \quad \mathfrak{T}_\sigma(\mu * \nu) = \mathfrak{T}_\sigma(\nu) \mathfrak{T}_\sigma(\mu).$$

Proof.

$$\begin{aligned} \mathfrak{T}_\sigma(\mu * \nu) &= \mathcal{M}(\widehat{\mu * \nu}(\sigma)) \\ &= \mathcal{M}((\widehat{\mu} \times \widehat{\nu})(\sigma)) \quad \text{according to Theorem 3.2.} \\ &= \mathcal{M}(\widehat{\nu}(\sigma)) \mathcal{M}(\widehat{\mu}(\sigma)) \\ &= \mathfrak{T}_\sigma(\nu) \mathfrak{T}_\sigma(\mu). \end{aligned}$$

□

Now we are going to extend the Kronecker product to the mappings \mathfrak{T}_σ . We also give some basic properties of this extension. The following definition of Kronecker product of matrices can be found in [8].

Definition 3.4. *Let $X = (x_{ij})$ and Y be two matrices. The Kronecker product (also called tensor product) of X by Y is the matrix*

$$(20) \quad X \otimes Y = \begin{pmatrix} x_{11}Y & x_{12}Y & \cdots & x_{1n}Y \\ x_{21}Y & \cdots & & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1}Y & x_{m2}Y & \cdots & x_{mn}Y \end{pmatrix}.$$

The Kronecker product of matrices is associative, noncommutative and verifies, among other properties, the equality:

$$(21) \quad (X_1 X_2) \otimes (Y_1 Y_2) = (X_1 \otimes Y_1)(X_2 \otimes Y_2).$$

where $X_i, Y_j, i = 1, 2; j = 1, 2$ are matrices such that the product X_1X_2 and Y_1Y_2 exist; see [8] and [12].

Definition 3.5. Let $\mathfrak{T}_{\sigma_1}, \mathfrak{T}_{\sigma_2}$ be in $\Delta(\widehat{G})$. The Kronecker product of \mathfrak{T}_{σ_1} by \mathfrak{T}_{σ_2} , denoted by $\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}$, is the mapping from $M^1(G, \mathcal{A})$ into $\mathcal{M}_{d_{\sigma_1}d_{\sigma_2}}(\mathcal{A})$ such that

$$(22) \quad \forall \mu \in M^1(G, \mathcal{A}), [\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}](\mu) = \mathcal{M}(\widehat{\mu}(\sigma_1)) \otimes \mathcal{M}(\widehat{\mu}(\sigma_2)).$$

Remark. Since the Kronecker product of matrices is noncommutative, so is the extended Kronecker product \boxtimes .

Many properties of the Kronecker product of matrices are easily extended to the Kronecker product of the mappings \mathfrak{T}_{σ} . As an example we prove the associativity in the following theorem.

Theorem 3.6. Let $\sigma_i, i = 1, 2, 3$, be in \widehat{G} and μ, ν be in $M^1(G, \mathcal{A})$. We have the following associative relation:

$$(23) \quad (\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}) \boxtimes \mathfrak{T}_{\sigma_3} = \mathfrak{T}_{\sigma_1} \boxtimes (\mathfrak{T}_{\sigma_2} \boxtimes \mathfrak{T}_{\sigma_3}).$$

And we have the following effect on the convolution of vector measures:

$$(24) \quad [\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}](\mu * \nu) = [\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}](\nu)[\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}](\mu).$$

Proof.

$$1. [(\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}) \boxtimes \mathfrak{T}_{\sigma_3}](\mu) = [\mathcal{M}(\widehat{\mu}(\sigma_1)) \otimes \mathcal{M}(\widehat{\mu}(\sigma_2))] \otimes \mathcal{M}(\widehat{\mu}(\sigma_3)).$$

The proof can be completed by using the fact that the Kronecker product of matrices is associative.

$$\begin{aligned} 2. [\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}](\mu * \nu) &= \mathcal{M}(\widehat{\mu * \nu}(\sigma_1)) \otimes \mathcal{M}(\widehat{\mu * \nu}(\sigma_2)) \\ &= [\mathcal{M}(\widehat{\nu}(\sigma_1))\mathcal{M}(\widehat{\mu}(\sigma_1))] \otimes [\mathcal{M}(\widehat{\nu}(\sigma_2))\mathcal{M}(\widehat{\mu}(\sigma_2))] \\ &= [\mathcal{M}(\widehat{\nu}(\sigma_1)) \otimes \mathcal{M}(\widehat{\nu}(\sigma_2))][\mathcal{M}(\widehat{\mu}(\sigma_1)) \otimes \mathcal{M}(\widehat{\mu}(\sigma_2))] \\ &= [\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}](\nu)[\mathfrak{T}_{\sigma_1} \boxtimes \mathfrak{T}_{\sigma_2}](\mu), \end{aligned}$$

using meanwhile the property (21) of Kronecker product of matrices. \square

4. CONCLUSION

We interpreted the Fourier-Stieltjes transform of \mathcal{A} (Banach algebra)-valued measures on a compact group G as a collection of operators from a tensor product of Hilbert spaces into the Banach algebra \mathcal{A} . Therefore it was possible to associate a matrix with each of such operators. Then we extended the Kronecker product of matrices to some matrix valued mappings acting on vector measures. We also computed the effect of these mappings and that of their Kronecker product on the convolution of vector measures. Many other properties could be found if this discussion is deepened. These are the aims of some forthcoming papers. What can be the possible applications to Physics (for example) for joining Kronecker product with Fourier-Stieltjes transform of vector measures?

REFERENCES

- [1] Assiamoua, V.S.K. and Olubummo, A., Fourier-Stieltjes transforms of vector-valued measures on compact groups, *Acta Sci. Math.(Szeged)*, 53 (1989), 301-307.
- [2] Davio, M., Kronecker product and shuffle algebra, *IEEE trans. Comput.*, C-30 (1981), 116-125.
- [3] Diestel, J. and Uhl, J. J., *Vector measures*, Math. Surveys 15, Amer. Math. Soc., Providence, 1977.
- [4] Folland, G. B., *A Course in Abstract Harmonic Analysis*, CRC Press , 1995.
- [5] Hewitt, E. and Ross, K.A., *Abstract Harmonic Analysis*, Vol II, Springer-Verlag, New York-Berlin-Heidelberg, 1970.
- [6] Kawabe, J., Compactnes criteria for the weak convergence of vector measures in locally convex spaces, *Publ. Math. Debrecen.*, 60 (2002),115-130.
- [7] Van Loan, C. F., The ubiquitous Kronecker product, *Journal Compt. Appl. Math.*, 123 (2000) 85-100.
- [8] Marcus, M. and Minc, H., *A survey of matrix theory and matrix inequalities*, Dover publications, Inc., New York, 1992.
- [9] Rao, K. N. S., *Linear algebra and group theory for Physicists*, Hindustan Book Agency, India, 2006.
- [10] Regalia, P. A. and Mitra, S., Kronecker products, unitary matrices and signal processing applications, *SIAM Rev.*, 31 (1989), 586-613.
- [11] Steeb, W.-H., *Matrix calculus and Kronecker product with applications and C++ programs*, World Scientific Publishing, Singapore, 1997.
- [12] Steeb, W.-H. and Hardy, Y., *Matrix Calculus and Kronecker Product: A Practical Approach to Linear and Multilinear Algebra*, World Scientific Publishing, Singapore (2011).
- [13] Tinkham, M., *Group theory and quantum mechanics*, NY McGraw-Hill, 1964.
- [14] Tolimieri, R. , Lu, C. and An, M., *Algorithms for discrete Fourier transform and convolution*, Springer, New York, 1989.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOMÉ, TOGO, B. P. 1515, LOMÉ, TOGO

AND

INTERNATIONAL CHAIR IN MATHEMATICAL PHYSICS AND APPLICATIONS (ICMPA UNESCO-CHAIR),UNIVERSITY OF ABOMEY-CALAVI, BENIN