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SLOW GROWTH AND OPTIMAL APPROXIMATION OF PSEUDOANALYTIC FUNCTIONS ON THE DISK

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Abstract. Pseudoanalytic functions (PAF) are constructed as complex combination of real-valued analytic solutions to the Stokes-Betrami System. These solutions include the generalized biaxisymmetric potentials. McCoy [10] considered the approximation of pseudoanalytic functions on the disk. Kumar et al. [9] studied the generalized order and generalized type of PAF in terms of the Fourier coefficients occurring in its local expansion and optimal approximation errors in Bernstein sense on the disk. The aim of this paper is to improve the results of McCoy [10] and Kumar et al. [9]. Our results apply satisfactorily for slow growth.

1. Introduction

Generalized biaxisymmetric potential (GBASP) that are harmonic at the origin may be expanded, in analogy with the Taylor's series for analytic functions of a single complex variable, in a convergent series of homogeneous harmonic polynomials on an open set.

Pseudoanalytic functions are constructed as complex combinations of real-valued function pair that are analytic solutions of Stokes-Beltrami System (SBS); a generalization of the Cauchy-Riemann equations that is linked to the GBASP equation by eliminating one of the dependent variables from the system. Pseudoanalytic functions provide sufficient basis for the transformation of Bernstein's ideas through transform and special function methods. The real part of pseudoanalytic function i.e., eliminating the harmonic conjugate gives the theory of GBASP. The GBASP equation frequently found in the summability theory of [2] Jacobi series as

$$r\frac{\partial}{\partial r}\left\{r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)r\frac{\partial u}{\partial r}\right\}+\frac{\partial}{\partial\theta}\left\{r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)\frac{\partial u}{\partial\theta}\right\}=0$$
$$\rho^{(\alpha,\beta)}(\theta)=(\sin\theta/2)^{2\alpha+1}(\cos\theta/2)^{2\beta+1},\alpha\geq\beta>-\frac{1}{2}$$

where (r,θ) are the plane polar coordinates. The domain of the potential u is a simply connected region with smooth boundary in the upper half plane $C^{+}=C\cap Re(z)\geq 0$. The existence of a harmonic conjugate ν of u is implied in the sense

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of the generalized Stokes-Beltrami System (SBS);

$$\begin{aligned} r\frac{\partial \nu}{\partial r} &= -r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)\frac{\partial u}{\partial \theta} \\ \frac{\partial \nu}{\partial \theta} &= r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)r\frac{\partial u}{\partial r} \end{aligned}$$

a system that reduces to the Cauchy-Riemann equations of analytic function theory in the limit $\alpha=\beta=-\frac{1}{2}$. Following along the lines of analytic function theory, a pseudoanalytic function [1,5,12] (PAF) is defined as the complex combination

$$F(re^{i\theta}) = u(r,\theta) + i\nu(r,\theta)$$

of a real valued analytic function pair formed from the potential u and the principal branch of its harmonic conjugate $\nu = \nu(r,\theta)$.

The disk $D_R(R>1)$ of maximum radius on which a PAF F exists, is designated by $F \in P(D_R)$. If F is an entire PAF, it has no singularities in the finite C^+-plane and writes $F \in P(C)$.

Kumar [8] studied the relationship between the pseudoanalytic functions and Bergman-Gilbert type integral operators for GBASP and polynomial approximation. Bergman [3] and Gilbert [5] generalize the operation of taking the real part. They obtained bounded linear operators which transform analytic functions to solution u, where integral operators are developed to provide the transformation from analytic functions to solutions of corresponding elliptic equation. Bers [4] and Vekua [11] have also extended function theory so that solutions u of elliptic equations can be obtained as $u = Re(f)$, where f is a pseudoanalytic function sharing many of the properties associated with classical analytic functions of a single complex variable. Also McCoy [10] considered the approximation of pseudoanalytic functions on the disk and obtained some coefficient and Bernstein type growth theorems. Kapoor and Nautiyal [6] characterized the order and type of GBASP's (not necessarily entire) in terms of rates of decay of approximation errors on both sup norm and $L^\delta-$norm, $1 \le \delta < \infty$. All these authors have not studied the generalized growth of pseudoanalytic functions on the disk. Our results and methods are different from all those authors mentioned above and apply satisfactorily for slow growth.

Let p and q are two positive functions defined on $(0,\infty)$, strictly increasing and infinitely differentiable such that

$$\begin{aligned} &\lim_{x\to\infty} p(x) = \lim_{x\to\infty} q(x) = \infty, \\ &\lim_{x\to\infty} \frac{p(cx)}{p(x)} = 1, \\ &\lim_{x\to\infty} \frac{q((1+w(x))x)}{q(x)} = 1, \lim_{x\to 0} w(x) = 0, \\ &p(x/q^{-1}(cp(x))) = (1+o(x))p(x), \ \ \texttt{for}\ x\to\infty, \\ &\lim_{x\to\infty}\left|\frac{d(q^{-1}(cp(x)))}{d(\log x)}\right| \le b, \end{aligned}$$

b is a non zero positive constant and $d(u)$ means the differential of u.

Kumar et al. [9] defined the $(p,q)-$order and $(p,q)-$type (or generalized order and generalized type) of pseudoanalytic functions $F \in P(D_R)$ with radial limits as:

$$\begin{aligned}\rho_0(p,q) &= \limsup_{r\to R} \frac{p(\log M_R(F))}{q(R/(R-r))},\\ \sigma_0(p,q) &= \limsup_{r\to R} \frac{p(\log M_R(F))}{[q(R/(R-r))]^{\rho_0(p,q)}}, 1<r<R,\end{aligned}$$

where

$$M_r(F) = \max\{|F(re^{i\theta})| : re^{i\theta} \in \overline{D}_R\}, r<R.$$

In [9] Kumar et al. studied the generalized order ($(p,q)-$order) and generalized type ($(p,q)-$type) of a PAF in terms of the Fourier coefficients occurring in its local expansion and optimal approximation errors in Bernstein sense on the disk. They obtained these results by using the following condition

$$\lim_{x\to\infty}\left|\frac{d(q^{-1}(cp(x)))}{d(\log x)}\right| < b,$$

this condition does not hold for $p=q$. Therefore the results fail to exist for $p=q$.

In this paper we shall improve the results of Kumar et al. [9] by using the concept of generalized order of slow growth introduced by Kapoor and Nautiyal [7] with the help of general function as:

Let L denote the class of functions h satisfying the following conditions:

(i) h is defined on $(0,\infty)$, strictly increasing to infinity differentiable such that

$$\lim_{x\to\infty} h(x) = \infty.$$

(ii) $\lim_{x\to\infty} \frac{h((1+w(x)).x)}{h(x)} = 1$, for every function w, such that $\lim_{x\to 0} w(x) = 0$.

Let Δ denote the class of function h satisfying condition (i) and

(iii) $\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$, for every $c>0$.

Let Ω be the class of functions h satisfying (i) and (iv) and $\overline{\Omega}$ be the class of functions satisfying (i) and (v) where

(iv) there exists a $\gamma \in \Omega$ and x_0, K_1 and K_2 such that, for all $x > x_0$.

$$0 < K_1 \le \lim_{x\to\infty} \frac{d(h(x))}{d(\gamma(\log x))} \le K_2 < \infty$$

(v) $\lim_{x\to\infty} \frac{d(h(x))}{d(\log x)} = K_3, 0 < K_3 < \infty.$

Now we define the $(p,p)-$order and $(p,p)-$type of $F \in P(D_R)$ (or generalized growth) by

$$\begin{aligned}\rho(p,p) &= \limsup_{r\to R} \frac{p(\log M_r(F))}{p(R/(R-r))},\\ \sigma(p,p) &= \limsup_{r\to R} \frac{p(\log M_r(F))}{[p(R/(R-r))]^{\rho(p,p)}}.\end{aligned}$$

Kapoor and Nautiyal [7] showed that classes Ω and $\overline{\Omega}$ are contained in Δ. Further, $\Omega \cup \overline{\Omega} = \phi$.

2. Generalized Order and Generalized Type with Fourier Coefficients of Pseudoanalytic Functions

The purpose of this section is to establish the relationship of the generalized growth ($p-$growth) of pseudoanalytic functions in a disk with Fourier coefficients occurring in its local expansion.

In a neighborhood of the origin, the pseudoanalytic function PAF $F \in P(D_R)$ has the local expansion

$$F(re^{i\theta}) = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta}), re^{i\theta} \in D_R,$$

$F_n(re^{i\theta}) = u_n(r,\theta) + i\nu_n(r,\theta), n = 0,1,2,\dots$ and a_n real-valued. Write

$$\limsup_{n\to\infty} \frac{p(n)}{p\left[\frac{n}{\log(n^{2\alpha+1}|a_n|R^n)}\right]} = \mu(p,p). \tag{2.1}$$

First we prove

Lemma 2.1. Let $p(x) \in \overline{\Omega}$ and $\mu > 0$. For every $r > 1$, the maximum of the function

$$x \to w(x,r) = x\log(r/R) + \frac{x}{p^{-1}(p(x)/\mu)}$$

is reached for $x = x_r$ solution of the equation

$$x = p^{-1}\left\{\mu p\left[\frac{1 - d\log(p^{-1}(p(x)/\mu))/d(\log x)}{\log(R/r)}\right]\right\}. \tag{2.2}$$

Proof. The proof follows on the lines of [9, Lemma 2.1] by simple calculation replacing q by p.

Lemma 2.2. Let $F(re^{i\theta}) = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta}), F \in P(D_R)$. For every $1 < r < R$ and $p(x) \in \overline{\Omega}$, we put

$$\begin{aligned} \overline{M}(r,F) &= \sup_n\{\|a_n w_n\| r^n, r > 0\}, \\ \|F\| &= \begin{cases} \|F\|_\delta = \left[\int\int_{\overline{D}_1} |F|^\delta r dr d\phi\right], & 1 \le \delta < \infty \\ \|F\|_\infty = M_1(F), & \delta = \infty \end{cases}, \end{aligned}$$

and

$$\overline{\rho}_0(p,p) = \limsup_{r\to R} \frac{p(\log \overline{M}(r,F))}{p(R/(R-r))}$$

then

$$\rho(p,p) \le \overline{\rho}_0(p,p).$$

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n \\ M(r,f) &\le \sum_{n=0}^{\infty} |a_n| r^n \le \sum_{n=0}^{\infty} |a_n| w_n r^n, \end{aligned}$$

substituting $r = r^{\xi}R^{1-\xi}(r/R)^{1-\xi}$ in above inequality we get

$$M(r,f) \leq \sum_{n=0}^{\infty} |a_n| w_n (r^{\xi}R^{1-\xi})^n (r/R)^{(1-\xi)^n}, (r/R) < 1$$

or

$$M(r,f) \leq \sum_{n=0}^{\infty} \sup(|a_n| w_n (r^{\xi}R^{1-\xi})^n (r/R)^{(1-\xi)^n})$$

or

$$\begin{aligned} M(r,f) &\leq \overline{M}(r^{'},F) \sum_{n=0}^{\infty} (r/R)^{(1-\xi)^n}, r^{'} = (r^{\xi}R^{1-\xi}), \\ &\leq \overline{M}(r^{'},F) \frac{1}{1-(r/R)^{1-\xi}} \end{aligned}$$

or

$$\log M(r,f) \leq \log \overline{M}(r^{'},F) - \log(1-(r/R)^{1-\xi}).$$

If the function $r \to \overline{M}(r^{'},F)$ is bounded, then $\rho(p,p) = \overline{\rho}_0(p,p) = 0$. So we can assume that $\overline{M}(r^{'},F) \to \infty$ as $r \to R$. Then, for every r sufficiently close to R

$$\frac{p(\log M(r,f))}{p(R/(R-r))} \leq \frac{p(\log \overline{M}(r^{'},F) - \log(1-(r/R)^{1-\xi}))}{p(R/(R-r^{\xi}R^{1-\xi}))} . \frac{p(R/(R-r^{\xi}R^{1-\xi}))}{p(R/(R-r))}.$$

Since

$$\frac{p(R/(R-r^{\xi}R^{1-\xi}))}{p(R/(R-r))} \to 1 \texttt{ as } r \to R,$$

we obtain by passing to limits on both sides

$$\rho(p,p) \leq \overline{\rho}_0(p,p).$$

Hence the proof is complete.

Theorem 2.1. Let Let $p(x) \in \overline{\Omega}$ and $F(re^{i\theta}) = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta}, F \in P(D_R), re^{i\theta} \in D_R$ such that $\mu(p,p)$ defined by (2.1) is finite. Then F is the restriction of a pseudo analytic function in $P(D_R)(R>1)$ and its $(p,p)-$order $\rho(p,p) = \mu(p,p)$.

Proof. It can be seen [9] that for every $1 < r < R$ the series $\sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta})$ is convergent in D_R. Now first we show that $\rho(p,p) \leq \mu(p,p)$. By (2.1) we have for every $\varepsilon > 0$, there exists $n(\varepsilon)$ such that for every $n > n(\varepsilon)$,

$$p(n) \leq (\mu(p,p)+\varepsilon) p\left(\frac{n}{\log(|a_n| n^{2\alpha+1} R^n)}\right), \tag{2.3}$$

since

$$\log(|a_n| n^{2\alpha+1} r^n) = n \log(r/R) + \log(|a_n| n^{2\alpha+1} R^n) \tag{2.4}$$

using (2.3) in (2.4) we get

$$\log(|a_n| n^{2\alpha+1} r^n) \leq n \left[\log(r/R) + \frac{1}{p^{-1}\left(\frac{p(n)}{\mu}\right)}\right]. \tag{2.5}$$

Choose

$$n = x_r = p^{-1}\left\{\overline{\mu} p\left[\frac{1-d\left(p^{-1}\left(\frac{p(x)}{\overline{\mu}}\right)\right)/d(\log x)}{\log(R/r)}\right]\right\}.$$

Using the properties of the function $p\left(\frac{d\log\left(p^{-1}\left(\frac{p(x)}{\overline{\mu}}\right)\right)}{d(\log x)} = 0(1)\right)$, and the function $t \to \log t, (\log(1+x) = (1+o(1)).x, x \to 0)$, we have

$$n = x_r = (1+o(1))p^{-1}\left[\overline{\mu}p(R/(R-r))\right],$$

and replacing in (2.5), we have

$$\log(|a_n|n^{2\alpha+1}r^n) \le (1+o(1))p^{-1}(\overline{\mu}p(R/(R-r)))\left(\log(r/R) + \frac{1}{R/(R-r)}\right)$$

since $\frac{R}{R-r} > 1$, it gives

$$\log(|a_n|n^{2\alpha+1}r^n) \le C_0 p^{-1}(\overline{\mu}p(R/(R-r)).$$

By the properties of function p and proceeding the limit supremum as r sufficiently close to R we get

$$\limsup_{r\to R}\frac{p(\log \overline{M}(r,F))}{p(R/(R-r))} \le \overline{\mu} = \mu + \varepsilon,$$

or

$$\overline{\rho}(p,p) \le \mu(p,p).$$

Using Lemma 2.2 we obtain

$$\rho(p,p) \le \mu(p,p). \tag{2.6}$$

Now we show that $\rho(p,p) \ge \mu(p,p)$. By the definition of $\rho(p,p)$, for every $\varepsilon > 0$, there exists $r_\varepsilon \in]1,R[$ such that for every $r \ge r_\varepsilon$ we have

$$\log M_r(F) \le p^{-1}[(\rho(p,p)+\varepsilon)p(R/(R-r))]. \tag{2.7}$$

Since for $r \in]1,R[$,

$$\log(|a_n|n^{2\alpha+1}R^n) = -n\log(r/R) + \log(|a_n|n^{2\alpha+1}r^n). \tag{2.8}$$

Thus

$$\frac{p(n)}{p\left(\frac{C_1 n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)} \le \rho(p,p) + \varepsilon.$$

Now proceeding to limits supremum as $n \to \infty$, we get

$$\mu(p,p) \le \rho(p,p). \tag{2.9}$$

Combining (2.6) and (2.9) we complete the proof.

Let $F = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta})$ be pseudo analytic function of $(p,p)-$order $\rho = \rho(p,p)$ and write

$$T(p,p) = \limsup_{n\to\infty}\frac{p(n/\rho)}{\left\{p\left(\frac{\rho}{(\rho-1)}\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho-1}}.$$

Now we prove

Lemma 2.3. Let $p(x) \in \overline{\Omega}$ and $F = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta})$. For every $r \in]1, R[$,

$$\overline{\sigma}_1(p,p) = \limsup_{r\to R} \frac{p(\log \overline{M}(r,F))}{(p(R/(R-r)))^{\rho}},$$

then

$$\sigma(p,p) \leq \overline{\sigma}_1(p,p).$$

Proof. The proof can be obtain by using the same reasoning as in the proof of Lemma 2.2 as

$$\frac{p(\log M(r,f))}{[p(R/(R-r))]^{\rho}} \leq \frac{p(\log \overline{M}(R^{\xi}R^{1-\xi}, F) - \log(1-(r/R)^{1-\xi}))}{p(R/(R-r^{\xi}R^{1-\xi}))} . \frac{p(R/(R-r^{\xi}R^{1-\xi}))}{p(R/(R-r))}.$$

Proceeding the limit, we get

$$\sigma(p,p) \leq \overline{\sigma}_1(p,p).$$

In view of (2.7) and [10, eq. 2.8], (2.8) gives that

$$\log(|a_n|n^{2\alpha+1}R^n) \leq -n\log(r/R) + \log(n+2)n^{2\alpha+1}A) + p^{-1}[(\rho(p,p)+\varepsilon)p(R/(R-r))].$$

or

$$\frac{\log(|a_n|n^{2\alpha+1}R^n)}{n} \leq \varphi(r,n)$$

where

$$\varphi(r,n) = \log(R/r) + \frac{1}{n}\log((n+2)n^{2\alpha+1}A) + \frac{1}{n}p^{-1}[(\rho(p,p)+\varepsilon).p(R/(R-r))]$$

and $A = \|\rho^{(\alpha,\beta)}\|_{\delta'}, \frac{1}{\delta} + \frac{1}{\delta'} = 1$.

For r sufficiently close to R and for sufficiently large $n, \varphi(r,n)$ is equivalent to $\log(R/r)$ for $n \to \infty$ and $\log(R/r)$ is equivalent to $\frac{R-r}{r} = \frac{R}{r} - 1$ for $r \to R$. Then

$$\varphi(r,n) = (1+o(1))\log(R/r) \;\; \texttt{as} \;\; n \to \infty,$$

and

$$\log(R/r) = (1+o(1))\frac{(R-r)}{r}(r \to R).$$

Therefore for r sufficiently close to R and n sufficiently large

$$\frac{\log(|a_n|n^{2\alpha+1}R^n)}{n} \leq (1+o(1))(R/r-1).$$

Substituting

$$r = \frac{R}{\left[1 + p^{-1}\left(\frac{p(n)}{(\rho(p,p)+\varepsilon)}\right)\right]},$$

and applying the properties of function p, we obtain

$$\log(|a_n|n^{2\alpha+1}R^n) \leq C_1 \frac{n}{\left[p^{-1}\left(\frac{p(n)}{(\rho(p,p)+\varepsilon)}\right)\right]},$$

Theorem 2.2. Let $p(x) \in \overline{\Omega}$ and $F = \sum_{n=0}^{\infty} a_n w_n F_n(re^{i\theta})$ be a pseudoanalytic function on the closed unit disk. If F is of finite generalized $(p,p)-$order $\rho(p,p)$, and

$$T(p,p) = \limsup_{n\to\infty} \frac{p(n/\rho)}{\left\{p\left(\frac{\rho}{\rho-1}\cdot\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho-1}} < \infty. \tag{2.10}$$

Then F is the restriction of a pseudoanalytic function in $P(D_R)(R>1)$ and its $(p,p)-$type $\sigma(p,p) = T(p,p)$.

Proof. The function F is the restriction of a pseudoanalytic function in $P(D_R)$ by the definition of $T(p,p)$ and arguments used in Theorem 2.1. Put $T = T(p,p), \rho = \rho(p,p); \sigma = \sigma(p,p)$.

If $T < \infty$ by (2.10), for every $\varepsilon > 0$, there exists $n_0 \le n$ such that

$$p(n/\rho) \le (T+\varepsilon)\left\{p\left(\frac{\rho}{\rho-1}\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho-1}$$

or

$$\log(|a_n|n^{2\alpha+1}R^n) \le \frac{\rho}{(\rho-1)}\frac{n}{p^{-1}\left(\left(\frac{1}{\overline{T}}p(n/\rho)\right)^{1/(\rho-1)}\right)}, \overline{T} = T+\varepsilon. \tag{2.11}$$

Since

$$\log(|a_n|n^{2\alpha+1}r^n) \le n\log(r/R) + \log(|a_n|n^{2\alpha+1}R^n).$$

Using (2.11), we get

$$\log(|a_n|n^{2\alpha+1}r^n) \le n\log(r/R) + \frac{\rho}{\rho-1}\frac{n}{p^{-1}\left(\left(\frac{1}{\overline{T}}p(n/\rho)\right)^{1/(\rho-1)}\right)}.$$

For every $r \in]1,R[$, and r sufficiently close to R, we put

$$\phi(x,r) = x\log(r/R) + \frac{\rho}{\rho-1}\frac{x}{p^{-1}\left(\left(\frac{p(x/\rho)}{\overline{T}}\right)^{1/(\rho-1)}\right)}.$$

then

$$\frac{\partial\phi(x,r)}{\partial x} = \log(r/R) + \frac{\rho}{\rho-1}\frac{d}{dx}\left\{\frac{x}{p^{-1}\left(\left(\frac{p(x/\rho)}{\overline{T}}\right)^{1/(\rho-1)}\right)}\right\}.$$

Then the maximum of the function $x \to \phi(x,r)$ is reached for $x = x_r$ where x_r is the unique solution of the equation

$$\frac{\partial\phi}{\partial x}(x,r) = 0.$$

If we put $S = S(x,\overline{T},\frac{1}{\rho-1}) = p^{-1}\left(\left(\frac{1}{\overline{T}}p(x/\rho)\right)^{1/(\rho-1)}\right)$, then

$$\phi(x,r) = x\log(r/R) + \frac{x}{S}$$

we have

$$\frac{\partial\phi}{\partial x}(x,r)=0 \Leftrightarrow \log(r/R)+\frac{\rho}{\rho-1}\left(\frac{S-x\frac{dS}{dx}}{S}\right)=0,$$

or

$$\log(R/r)=\frac{\rho}{\rho-1}\left(\frac{1-\frac{d(\log S)}{d(\log x)}}{S}\right),$$

as $\frac{dS}{dx}=\frac{dS}{d(\log x)}\frac{d(\log x)}{dx}=\frac{1}{x}\frac{dS}{d(\log x)}$.

Since

$$\log(R/r)=\log\left(\frac{r-R}{R}+1\right)\sim\left(\frac{r-R}{R}\right)\left(\texttt{ as } \frac{r-R}{R}\to 0\right)$$

and

$$\left|\frac{d\left[\log\left(p^{-1}((p(x/\rho))^{1/(\rho-1)})\right)\right]}{d\log x}\right|\leq b,$$

where b is a constant positive. Then by the properties of function $p\in\overline{\Omega}$, we have

$$S=(1+o(1))\frac{\rho}{\rho-1}\left(\frac{R}{R-r}\right),$$

thus

$$p^{-1}\left(\left(\frac{p(x/\rho)}{\overline{T}}\right)^{1/\rho-1}\right)=(1+o(1))\frac{\rho}{\rho-1}\left(\frac{R}{R-r}\right),$$

therefore

$$x_r=(1+o(1))\rho p^{-1}(\overline{T}(p(R/(R-r)))^{1/(\rho-1)}).$$

Substituting in the relation (2.11), we have

$$\log(|a_n|n^{2\alpha+1}r^n)\leq\sup\phi(x,r)=\phi(x_r,r).$$

Replacing x by x_r in this last relation we get

$$\log(|a_n|n^{2\alpha+1}r^n)\leq\frac{(1+o(1))\frac{\rho-1}{\rho}p^{-1}(\overline{T}(p(R/(R-r)))^{\rho-1})}{R/(R-r)}.$$

Since $\frac{R}{R-r}>1$ and $\frac{\rho-1}{\rho}<1$, then

$$\log(|a_n|n^{2\alpha+1}r^n)\leq Cp^{-1}(\overline{T}(p(R/(R-r)))^{\rho-1}).$$

Then

$$\log(M(r,f))\leq Cp^{-1}(\overline{T}(p(R/(R-r)))^{\rho-1}).$$

Thus

$$\frac{p(\log M(r,f))}{p(R/(R-r))}\leq\overline{T}.$$

Proceeding to the limit supremum as $r\to R$, we get

$$\sigma(p,p)\leq T(p,p).$$

The result is obviously holds for $T=\infty$.

To complete the proof it remains to show that $\sigma(p,p) \geq T(p,p)$. Put $\overline{\sigma} = \sigma(p,p) + \varepsilon, \rho = \rho(p,p)$.

Suppose that $\sigma < \infty$. By definition of $\sigma(p,p)$, we have for every $\varepsilon > 0$, there exist $r_0 \in]1, R[$, such that for every $r > r_0 (R > r > n_0 > 1)$

$$\log M_r(F) \leq p^{-1}[\overline{\sigma}(p(R/(R-r)))^{\rho}]. \tag{2.12}$$

Since for every $r \in]1, R[$

$$\log(|a_n| n^{2\alpha+1} R^n) = -n \log(r/R) + \log(|a_n| n^{2\alpha+1} r^n)$$

then in view of (2.12) and [10,eq. 2.8], we get

$$\log(|a_n| n^{2\alpha+1} R^n) \leq -n \log(r/R) + \log((n+2) n^{2\alpha+1} A) + p^{-1}[\overline{\sigma}(p(R/(R-r)))^{\rho}].$$

Let

$$\frac{\log(|a_n| n^{2\alpha+1} R^n)}{n} \leq w(r,n)$$

where

$$w(r,n) = -\log(r/R) + \frac{1}{n} \log((n+2) n^{2\alpha+1} A) + \frac{1}{n} p^{-1}[\overline{\sigma}(p(R/(R-r)))^{\rho}].$$

For r sufficiently close to R we have

$$\lim_{n\to\infty} w(r,n) = -\log(r/R) = \log(R/r).$$

Then for n sufficiently large and r sufficiently close to R, we have

$$w(r,n) = (1 + 0(1)) \log(R/r), n \to \infty,$$

then

$$\frac{1}{n} \log(|a_n| n^{2\alpha+1} R^n) \leq (1 + o(1)) \log(R/r). \tag{2.13}$$

Assume

$$r = \frac{R(\rho-1)p^{-1}\left(\frac{1}{\overline{\sigma}} p(n/\rho)\right)^{1/\rho-1}}{\rho + (\rho-1)p^{-1}\left(\frac{p(n/\rho)}{\overline{\sigma}}\right)^{1/(\rho-1)}}. \tag{2.14}$$

Now using (2.13) and the properties of the function $p \in \overline{\Omega}$ and $t \to \log t$, for r sufficiently close to R, we get

$$\frac{\log(|a_n| n^{2\alpha+1} R^n)}{n} \leq (1 + o(1))((R/r) - 1).$$

From (2.14) we have

$$\begin{aligned} \frac{R}{r} - 1 &= \frac{\rho + (\rho-1)p^{-1}\left(\frac{1}{\overline{\sigma}} p(n/\rho)\right)^{1/(\rho-1)}}{(\rho-1)p^{-1}\left(\frac{p(n/\rho)}{\overline{\sigma}}\right)^{1/(\rho-1)}} - 1 \\ &= \frac{\rho}{(\rho-1)p^{-1}\left(\frac{p(n/\rho)}{\overline{\sigma}}\right)^{1/(\rho-1)}}. \end{aligned}$$

Then for r sufficiently close to R and n sufficiently large we obtain

$$\frac{\log(|a_n| n^{2\alpha+1} R^n)}{n} \leq \frac{\rho}{(\rho-1)p^{-1}\left(\frac{p(n/\rho)}{\overline{\sigma}}\right)^{1/(\rho-1)}},$$

or

$$\frac{(\rho-1)}{\rho p^{-1}\left(\frac{p(n/\rho)}{\overline{\sigma}}\right)^{1/(\rho-1)}} \leq \frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}$$

or

$$\left(\frac{1}{\overline{\sigma}}p(n/\rho)\right)^{1/(\rho-1)} \leq p\left(\frac{\rho}{(\rho-1)}\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right).$$

Therefore

$$\frac{p(n/\rho)}{\overline{\sigma}} \leq \left\{p\left(\frac{\rho}{(\rho-1)}\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho-1}$$

or

$$\frac{p(n/\rho)}{\left\{p\left(\frac{\rho}{(\rho-1)}\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho-1}} \leq \overline{\sigma} = \sigma+\varepsilon.$$

Proceeding to the limit supremum as $n \to \infty$ we get

$$\sigma(p,p) \geq T(p,p).$$

The result is obviously holds for $\sigma(p,p)=\infty$.

3. Generalized Growth and Optimal Polynomial Approximation of Pseudoanalytic Functions

The purpose of this section is to give the relationship between the generalized order $(\rho(p,p))$ and generalized type $(T(p,p))$ of a pseudoanalytic function PAF and optimal rate of convergence to 0 in the norm defined in Lemma 2.2.

The approximating pseudoanalytic polynomials of (fixed) degree n are taken from the sets

$$\pi_n = \left[P : P(re^{i\theta}) = \sum_{k=0}^{n} c_k w_k F_k(re^{i\theta}), c_k \ \texttt{real}\right], n = 0,1,2,\ldots.$$

The optimal approximates minimize the error $\|F-P\|$ for $P \in \pi_n$ in Bernstein sense as

$$E_n(F) = \inf\{\|F-P\| : P \in \pi_n\}\, n = 0,1,2,\ldots. \tag{3.1}$$

Lemma 3.1. Let $p(x) \in \overline{\Omega}$ and $E_n(F)$ is defined as (3.1) then

$$\limsup_{n\to\infty}\frac{p(n)}{p\left[\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right]} = \limsup_{n\to\infty}\frac{p(n)}{p\left[\frac{n}{\log(E_n(F)n^{2\alpha+1}R^n)}\right]}, \tag{3.2}$$

and

(3.3)

$$\limsup_{n\to\infty}\frac{p(n/\rho)}{\left\{p\left(\frac{\rho}{\rho-1}\frac{n}{\log(|a_n|n^{2\alpha+1}R^n)}\right)\right\}^{\rho-1}} = \limsup_{n\to\infty}\frac{p(n/\rho)}{\left\{p\left(\frac{\rho}{\rho-1}\frac{n}{\log(E_n(F)n^{2\alpha+1}R^n)}\right)\right\}^{\rho-1}}.$$

Proof. The proof can be obtain by following on the basis of proofs of Theorem 2.1 and 2.2 and using the identity [10, eq. 12]

$$\limsup_{n\to\infty}|a_n|^{1/n} = \limsup_{n\to\infty}[E_n(F)]^{1/n}.$$

In the consequence of (3.2) and (3.3) we can prove the following theorem.

Theorem 3.1. Let $p(x) \in \overline{\Omega}$ and F be a pseudoanalytic function on $P(D_R)(R > 1)$. Then

(i) Then the generalized $(p,p)-$order of F is

$$\rho(p,p) = \limsup_{n\to\infty} \frac{p(n)}{p\left[\frac{n}{\log(E_n(F)n^{2\alpha+1}R^n)}\right]}$$

(ii) The generalized $(p,p)-$type of F is $T(p,p)$, if and only if,

$$T(p,p) = \limsup_{n\to\infty} \frac{p(n/\rho)}{\left\{p\left(\frac{\rho}{\rho-1}\frac{n}{\log(E_n(F)n^{2\alpha+1}R^n)}\right)\right\}^{\rho-1}}$$

when $0 < \rho(p,p) < \infty$.

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