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INTUITIONISTIC FUZZY TOPOLOGICAL POLYGROUPS

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Abstract. The notion of intuitionistic fuzzy set was introduced by Atanassov as a generalization of the notion of fuzzy set. Intuitionistic fuzzy topological spaces were introduced by Coker. This paper provides a new connection between algebraic hyperstructures and intuitionistic fuzzy sets. In this paper, we introduce and study the concept of intuitionistic fuzzy subpolygroup and intuitionistic fuzzy topological polygroup. We also investigate some interesting properties of an intuitionistic fuzzy subpolygroup and intuitionistic fuzzy normal subpolygroup.

1. Introduction

The hyperstructure theory was born in 1934 when Marty introduced the notion of hypergroup [24]. The concept of intuitionistic fuzzy sets was introduced by Atanassov [5]. Coker [7] has introduced the notions of intuitionistic fuzzy topological spaces. Biswas [6] introduced the concept of intuitionistic fuzzy subgroup and some other concepts. The concepts of quasi-coincidence for intuitionistic fuzzy point was introduced and developed by Gallego Lupianez [14]. On the other hand, in the last few decades, many connections between hyperstructures and intuitionistic fuzzy sets has been established and investigated. In [18], Heidari et, al introduced the notion of topological polygroups. Then in [1, 2, 3] Abbasizadeh et, al investigated to notion of fuzzy topological polygroups.

We recall some basic definitions and results to be used in the sequel.
Let H be a non-empty set. Then a mapping $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a *hyperoperation*, where $\mathcal{P}^*(H)$ is the family of non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A *hypergroupoid* $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z)=(x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, we have $x \circ H = H = H \circ x$. This condition is called the *reproduction axiom*. The couple $(H, \circ)$ is called a *hypergroup* if it is a *semihypergroup* and a *quasihypergroup* [9, 24].

For all $n > 1$, we define the relation β_n on a *semihypergroup* H, as follows:

$$a \ \beta_n \ b \Leftrightarrow \exists \ (x_1, \dots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

and $\beta = \bigcup \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. This relation was introduced by Koskas [21] and studied mainly by Corsini, Davvaz, Freni, Leoreanu, Vougiouklis and many others. Suppose that β^* is the smallest equivalence relation on a hypergroup (semihypergroup) H such that the qoutient H/β^* is a group (semigroup). If H is a hypergroup, then $\beta = \beta^*$ [13]. The relation β^* is called the *fundamental relation* on H and H/β^* is called the *fundamental groups*.

A special subclass of hypergroups is the class of polygroups. We recall the following definition from [8]. A *polygroup* is a system $P = \langle P, \circ, e, ^{-1} \rangle$, where $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, $e \in P$, $^{-1}$ is a unitary operation P and the following axioms hold for all $x, y, z \in P$:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$,
(2) $e \circ x = x = x \circ e$,

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(3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

The following elementary facts about polygroups follow easily from the axioms:
$e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x \circ y)^{-1}=y^{-1} \circ x^{-1}$.

A non-empty subset K of a polygroup P is a subpolygroup of P if and only if $a, b \in K$ implies $a \circ b \subseteq K$ and $a \in K$ implies $a^{-1} \in K$. The subpolygroup N of P is *normal* in P if and only if $a^{-1} \circ N \circ a \subseteq N$ for all $a \in P$. For a subpolygroup K of P and $x \in P$, denote the *right coset* of K by $K \circ x$ and let P/K be the set of all *right cosets* of K in P. If N is a normal subpolygroup of P, then $(P/N, \odot, N,^{-1})$ is a polygroup, where $N \circ x \odot N \circ y=\{N \circ z | z \in N \circ x \circ y\}$ and $(N \circ x)^{-1} = N \circ x^{-1}$. For more details about polygroups we refer to [11, 12, 15].

Let $P = \langle P, \circ, e,^{-1} \rangle$ be a polygroup and $(P, \mathcal{T})$ be a topological space. Then, the system $P = \langle P, \circ, e,^{-1}, \mathcal{T} \rangle$ is called a *topological polygroup* if the mapping $\circ : P \times P \longrightarrow \wp^*(P)$ and $^{-1} : P \longrightarrow P$ are continuous (see [18]).

Let $P = \langle P, \circ, e,^{-1} \rangle$ be a polygroup and $(P, \mathcal{T})$ be a fuzzy topological space. A triad $(P, \circ, \mathcal{T})$ is called a *fuzzy topological polygroup* or FTP for short, if (see [1, 2, 3]):

(i) For all $x, y \in P$ and any fuzzy open Q-neighborhood W of any fuzzy point z_λ of $x \circ y$, there are fuzzy open Q-neighborhood U of x_λ and V of y_λ such that:
$U \bullet V \leq W$.

(ii) For all $x \in P$ and any fuzzy open Q-neighborhood V of x_λ^{-1}, there exists a fuzzy open Q-neighborhood U of x_λ such that:
$U^{-1} \leq V$.

2. Preliminaries

For the sake of convenience and completeness of our study, in this section some basic definition and results of [4, 5, 7, 16, 17, 22, 23], which will be needed in the sequel are recalled here.

Let X be a non-empty set and I be the closed interval $[0, 1]$. A complex mapping $A = (\mu_A, \nu_A) : X \longrightarrow I \times I$ is called an *intuitionistic fuzzy set* (in short, IFS) on X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \longrightarrow I$ and $\nu_A : X \longrightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$, respectively. In particular, $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X) (see [5, 7]).

Let X be a non-empty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then (see [5]):

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$,
(2) $A = B$ iff $A \subset B$ and $B \subset A$,
(3) $A^c = (\nu_A, \mu_A)$,
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$,
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then (see [7]):

(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Let X and Y be non-empty sets and let $f : X \longrightarrow Y$ a mapping. Let $A = (\mu_A, \nu_A)$ be an IFS in X and $B = (\mu_B, \nu_B)$ be IFS on Y. Then (see [7]):

(1) The preimage of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by:

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B)(x) = \mu_B(f(x))$ and $f^{-1}(\nu_B)(x) = \nu_B(f(x))$.

(2) The image of A under f, denoted by $f(A)$, is the IFS in Y defined by:

$$f(A) = (f(\mu_A), f(\nu_A)),$$

where

$$f(\mu_A)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Throughout this paper, the symbol I will denote the unit interval $[0,1]$.

A *intuitionistic fuzzy topology* (in short, IFT) in Coker's sense on a non-empty set X is a family $\mathcal{T}$ on IFSs in X satisfying the following axioms:

(1) $0_\sim, 1_\sim \in \mathcal{T}$.
(2) For all $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$.
(3) For all $(A_j)_{j \in J}$, then $\bigcup_{j \in J} A_j \in \mathcal{T}$.

In this case, the pair (X,$\mathcal{T}$) is called an *intuitionistic fuzzy topological space* (in short, IFTS) in the sense of Coker, and each elements of $\mathcal{T}$ is called an intuitionistic fuzzy open set (in short, IFOS) in X. The complement A^c of an IFOS A in X is called an intuitionistic fuzzy closed set (in short, IFCS) in X.

We will denote the set of all the IFTs on a set X as IFT(X), and the set of all IFOSs and the set of all IFCSs in an IFTS(X) as IFO(X) and IFC(X), respectively (see [7]).

Example 1. (1) The family $\mathcal{T}$=$\{0_\sim$,$1_\sim\}$ is an intuitionistic fuzzy topology on X.
(2) The family of all intuitionistic fuzzy sets in X is an intuitionistic fuzzy topology on X.

Example 2. Let $X = \{a, b, c\}$ and $A =< x, (\frac{a}{0.4}, \frac{b}{0.3}, \frac{c}{0.2}), (\frac{a}{0.6}, \frac{b}{0.5}, \frac{c}{0.7}) >$. Then, the family $\mathcal{T} = \{0_\sim, A, 1_\sim\}$ of IFS's in X is an IFT on X.

Example 3. Let $X = [0,1]$ and consider the IFS $A = (\mu_A, \nu_A)$ as follows:

$$\mu_A(x) = \begin{cases} -\frac{3}{2}x + 1 & \text{if } 0 \leq x \leq \frac{2}{3} \\ 0 & \text{if } \frac{2}{3} \leq x \leq 1, \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \frac{x}{4} & \text{if } 0 \leq x \leq \frac{2}{3} \\ \frac{1}{6} & \text{if } \frac{2}{3} \leq x \leq 1. \end{cases}$$

Then, $\mathcal{T} = \{0_\sim, A^c, 1_\sim\}$ is an intuitionistic fuzzy topology on X.

Let X, Y be non-empty sets and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ IFSs of X and Y, respectively. Then $A \times B$ is an IFS of $X \times Y$ defined by (see [17]):

$$(A \times B)(x, y) = (\mu_A(x) \wedge \mu_B(y), \nu_A(x) \vee \nu_B(y)).$$

Let X be a non-empty set. An intuitionistic fuzzy point, (in short, IFP) in X denoted by $x_{r,s}$ is an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ such that

$$\mu_A(y) = \begin{cases} r & \text{if } y = x \\ 0 & \text{if } y \neq x, \end{cases}$$

and

$$\nu_A(y) = \begin{cases} s & \text{if } y = x \\ 1 & \text{if } y \neq x, \end{cases}$$

where $x \in X$ is a fixed point, the constants $r \in I_0,\ s \in I_1$ and $r + s \leq 1$.

The intuitionistic fuzzy point $x_{r,s}$ is said to be contained in an intuitionistic fuzzy set A, denoted by $x_{r,s} \in A$, if and only if $\mu_A(x) \geq r$ and $\nu_A(x) \leq s$. In particular

$$x_{r,s} \subseteq y_{m,n} \Leftrightarrow x = y \quad \text{and} \quad r \leq m, \quad s \geq n.$$

The intuitionistic fuzzy characteristic mapping of a subset A of a set X is denoted by χ_A is defined as

$$\chi_A(x) = \begin{cases} (1,0) & \text{if } x \in A \\ (0,1) & \text{otherwise.} \end{cases}$$

Obviously an intuitionistic characteristic function χ_A is also an intuitionistic fuzzy set on X and for any non-empty subsets A and B of a set X, we have $A \subseteq B$ if and only if $\chi_A \subseteq \chi_B$ (see [22]).

Let $x_{r,s}$ be an IFP in X and let $A = (\mu_A, \nu_A)$ be an IFS in X. We say that $x_{r,s}$ is *quasi-coincident* with A, written $x_{r,s}\ q\ A$, if $\mu_A(x) + r > 1$ and $\nu_A(x) + s < 1$ (see [16]).

Let $(X, \mathcal{T})$ be an IFTS, and let p be an IFP of X. Say that an IFS N of X is a Q-neighbourhood of p if there exists an IFOS A of $(X, \mathcal{T})$ such that $p\ q\ A$ and $A \subseteq N$ (see [23]).

Let X, Y be two non-empty sets, let $f : X \longrightarrow Y$ be a map, let $\mathcal{T}$ be an IFT in X and let σ be an IFT in Y. Then, $f : (X, \mathcal{T}) \longrightarrow (Y, \sigma)$ is continuous if and only if, for each IFP p of X and for each Q-neighbourhood V of $f(p)$, there exists a Q-neighbourhood U of p such that $f(U) \subseteq V$ (see [23]).

3. Intuitionistic fuzzy subpolygroups

Definition 3.1. Let P be a polygroup and $A \in IFS(P)$. Then A is called *intuitionistic fuzzy subpolygroup* (in short, IFSP) of P if it satisfies the following conditions:

(1) $\mu_A(z) \geq \mu_A(x) \wedge \mu_A(y)$ and
$\nu_A(z) \leq \nu_A(x) \vee \nu_A(y)$ for each $z \in x \circ y$ and $x, y \in P$.

(2) $\mu_A(x^{-1}) \geq \mu_A(x)$ and
$\nu_A(x^{-1}) \leq \nu_A(x)$ for each $x \in P$.

We will denote the set of all IFSPs of P as IFSP(P).

Example 4. Let $P = \{e, a, b\}$. Then, P together with the following hyperoperation

$\circ$	e	a	b
e	e	a	b
a	a	e	b
b	b	b	$\{e, a\}$

is a polygroup. Let $A =< x, (\frac{e}{0.7}, \frac{a}{0.5}, \frac{b}{0.3}), (\frac{e}{0.1}, \frac{a}{0.3}, \frac{b}{0.5}) >$. Then, A is an IFSP of P.

Definition 3.2. [28] Let P be a polygroup. A fuzzy subset μ of P is called a *fuzzy subpolygroup* if

(1) $\min\{\mu(x), \mu(y)\} \leq \mu(z)$, for all $x, y \in P$ and for all $z \in x \circ y$,

(2) $\mu(x) \leq \mu(x^{-1})$, for all $x \in P$.

The following elementary facts about fuzzy subpolygroups follow easily from the axioms: $\mu(x) = \mu(x^{-1})$ and $\mu(x) \leq \mu(e)$, for all $x \in P$.

Proposition 3.3. *Let A be an IFSP of a polygroup P. Then $A(x^{-1}) = A(x)$, that is, $\mu_A(x) = \mu_A(x^{-1})$, $\nu_A(x) = \nu_A(x^{-1})$ and $\mu_A(x) \leq \mu_A(e)$, $\nu_A(x) \geq \nu_A(e)$ for each $x \in P$, where e is the identity element of P.*

Proof. By Definition 3.2, we have $\mu_A(x) = \mu_A(x^{-1})$ and $\mu_A(x) \leq \mu_A(e)$ for each $x \in P$. Thus it is enough to show that $\nu_A(x) = \nu_A(x^{-1})$ and $\nu_A(x) \geq \nu_A(e)$ for each $x \in P$.
Let $x \in P$. Then,

$$\nu_A(x) = \nu_A((x^{-1})^{-1}) \leq \nu_A(x^{-1}) \leq \nu_A(x).$$

On the other hand, for each $z \in x \circ x^{-1}$, we have $\nu_A(z) \leq \nu_A(x) \vee \nu_A(x^{-1})$. Since $e \in x \circ x^{-1} \cap x^{-1} \circ x$, so $\nu_A(e) \leq \nu_A(x) \vee \nu_A(x^{-1}) = \nu_A(x)$. Thus $\nu_A(e) \leq \nu_A(x)$ for each $x \in P$. This complete the proof. □

Proposition 3.4. *Let P be a polygroup.*

(1) *If μ_A is a fuzzy subpolygroup of P, then $A = (\mu_A, \mu_{A^c}) \in IFSP(P)$.*

(2) *If $A \in IFSP(P)$, then μ_A and ν_{A^c} are fuzzy subpolygroups of P.*

(3) *$A = (\chi_T, \chi_{T^c}) \in IFSP(P)$ if and only if T is a subpolygroup of P.*

Proof. It is straightforward. □

Proposition 3.5. *Let $\{A_\alpha\}_{\alpha \in J} \subset IFSP(P)$. Then $\bigcap_{\alpha \in J} A_\alpha \in IFSP(P)$.*

Proof. It is straightforward. □

Proposition 3.6. *If A be an IFSP of a polygroup P then,*

$$P_A = \{x \in P : A(x) = A(e),\ \text{that is,}\ \mu_A(x) = \mu_A(e)\ \text{ and }\ \nu_A(x) = \nu_A(e)\}$$

is a subpolygroup of P.

Proof. We have to show that:

(1) $x \circ y \subseteq P_A$ for each $x, y \in P_A$.

(2) If $x \in P_A$ then, $x^{-1} \in P_A$.

Let $x, y \in P_A$ and $z \in x \circ y$. Since $x, y \in P_A$ then, $\mu_A(x) = \mu_A(e)$, $\nu_A(x) = \nu_A(e)$ and $\mu_A(y) = \mu_A(e)$, $\nu_A(y) = \nu_A(e)$. Since $A \in IFSP(P)$ and $z \in x \circ y$,

$$\begin{aligned} \mu_A(z) &\geq \mu_A(x) \wedge \mu_A(y) \\ &= \mu_A(e) \wedge \mu_A(e) = \mu_A(e), \end{aligned}$$

and

$$\begin{aligned} \nu_A(z) &\leq \nu_A(x) \vee \nu_A(y) \\ &= \nu_A(e) \vee \nu_A(e) = \nu_A(e). \end{aligned}$$

So $\mu_A(z) \geq \mu_A(e)$ and $\nu_A(z) \leq \nu_A(e)$. Then $\mu_A(z) = \mu_A(e)$, $\nu_A(z) = \nu_A(e)$ and $z \in P_A$, that is, $x \circ y \subseteq P_A$.

Now, if $x \in P_A$ then, $\mu_A(x) = \mu_A(e)$ and $\nu_A(x) = \nu_A(e)$. Since $\mu_A(x^{-1}) = \mu_A(x) = \mu_A(e)$ and $\nu_A(x^{-1}) = \nu_A(x) = \nu_A(e)$, that is, $x^{-1} \in P_A$. Hence P_A is a subpolygroup of P. □

Definition 3.7. [26] Let A be an IFS in a set X and let $\alpha, \beta \in I$ with $\alpha + \beta \leq 1$. Then the set

$$C_{\alpha,\beta}(A) = \{x \in X \ : \ \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$$

is called a (α, β)- cut set of A.

Proposition 3.8. *Let A be an IFSP of a polygroup P. Then for each $(\alpha, \beta) \in I \times I$ with $(\alpha, \beta) \leq A(e)$, that is, $\alpha \leq \mu_A(e)$, $\beta \geq \nu_A(e)$, $C_{\alpha,\beta}(A)$ is a subpolygroup of P, where e is the identity of P.*

Proof. Clearly, $C_{\alpha,\beta}(A) \neq \varnothing$. Let $x, y \in C_{\alpha,\beta}(A)$. We show that $x \circ y \subseteq C_{\alpha,\beta}(A)$.

Let $z \in x \circ y$. Since $x, y \in C_{\alpha,\beta}(A)$, then $A(x) \geq (\alpha, \beta)$ and $A(y) \geq (\alpha, \beta)$, that is, $\mu_A(x) \geq \alpha$, $\nu_A(x) \leq \beta$ and $\mu_A(y) \geq \alpha$, $\nu_A(y) \leq \beta$. Since $A \in IFSP(P)$, $\mu_A(z) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha$ and $\nu_A(z) \leq \nu_A(x) \vee \nu_A(y) \leq \beta$. Thus $A(z) \geq (\alpha, \beta)$. So $z \in C_{\alpha,\beta}(A)$ and $x \circ y \subseteq C_{\alpha,\beta}(A)$.

On the other hand, $\mu_A(x^{-1}) \geq \mu_A(x) \geq \alpha$ and $\nu_A(x^{-1}) \leq \nu_A(x) \leq \beta$. Thus $A(x^{-1}) \geq (\alpha, \beta)$. So $x^{-1} \in C_{\alpha,\beta}(A)$. Hence $C_{\alpha,\beta}(A)$ is a subpolygroup of P. □

Proposition 3.9. *Let A be an IFS in a polygroup P such that $C_{\alpha,\beta}(A)$ is a subpolygroup of P for each $(\alpha, \beta) \in I \times I$ with $(\alpha, \beta) \leq A(e)$. Then A is an IFSP of P.*

Proof. For any $x, y \in P$, let $A(x) = (t_1, s_1)$ and $A(y) = (t_2, s_2)$. Then clearly $x \in C_{t_1,s_1}(A)$ and $y \in C_{t_2,s_2}(A)$. Suppose $t_1 < t_2$ and $s_1 > s_2$, then $C_{t_1,s_1}(A) \subset C_{t_2,s_2}(A)$. Thus $y \in C_{t_1,s_1}(A)$. Since $C_{t_1,s_1}(A)$ is a subpolygroup of P, $x \circ y \subseteq C_{t_1,s_1}(A)$. Then for each $z \in x \circ y$, $A(z) = (t_1, s_1)$, that is, $\mu_A(z) \geq t_1$ and $\nu_A(z) \leq s_1$. So $\mu_A(z) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(z) \leq \nu_A(x) \vee \nu_A(y)$. On the other hand, for each $x \in P$, let $A(x) = (\alpha, \beta)$. Then $x \in C_{\alpha,\beta}(A)$. Since $C_{\alpha,\beta}(A)$ is a subpolygroup of P, $x^{-1} \in C_{\alpha,\beta}(A)$. So $A(x^{-1}) \geq (\alpha, \beta)$, that is, $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$. Hence A is an IFSP of P. □

Theorem 3.10. *Let A and B be two IFSP's of a polygroup P. Then $A \cap B$ is IFSP of polygroup P.*

Proof. By Theorems 3.8 and 3.9, $A \cap B$ is IFSP of polygroup P if and only if $C_{\alpha,\beta}(A \cap B)$ is a subpolygroup of P. Clearly, $C_{\alpha,\beta}(A \cap B) = C_{\alpha,\beta}(A) \cap C_{\alpha,\beta}(B)$ and both $C_{\alpha,\beta}(A)$ and $C_{\alpha,\beta}(B)$ are subpolygroups of P and intersection of two subpolygroups of a polygroup is a subpolygroup of P implies that $C_{\alpha,\beta}(A \cap B)$ is a subpolygroup of P and hence $A \cap B$ is IFSP of polygroup P. □

Theorem 3.11. *Let A and B IFSP of polygroups P_1 and P_2 respectively. Then $A \times B$ is also IFSP of polygroup $P_1 \times P_2$.*

Proof. Let A and B be IFSP of polygroups P_1 and P_2 respectively, then, $C_{\alpha,\beta}(A)$ and $C_{\alpha,\beta}(B)$ are subpolygroups of polygroups P_1 and P_2 respectively, for all $\alpha, \beta \in I$ with $\alpha + \beta \leq 1$. So $C_{\alpha,\beta}(A) \times C_{\alpha,\beta}(B)$ is subpolygroup of polygroup $P_1 \times P_2$. Hence $C_{\alpha,\beta}(A \times B)$ is subpolygroup of polygroup $P_1 \times P_2$. Therefore $A \times B$ is an IFSP of polygroup $P_1 \times P_2$. □

Proposition 3.12. *Let A and B IFS of the polygroups P_1 and P_2 respectively such that $\mu_A(x) \leq \mu_B(e_2)$ and $\nu_A(x) \geq \nu_B(e_2)$ hold for $x \in P_1$, e_2 being the identity element of P_2. If $A \times B$ is an IFSP of $P_1 \times P_2$, then, A is IFSP of polygroup P_1.*

Proof. Let $x, y \in P_1$ and $z \in x \circ y$, we have:

$$\begin{aligned} \mu_A(z) &= \mu_A(z) \wedge \mu_B(e_2) = \mu_{A\times B}(z, e_2) \\ &\geq \mu_{A\times B}(x, e_2) \wedge \mu_{A\times B}(y, e_2) \\ &= [\mu_A(x) \wedge \mu_B(e_2)] \wedge [\mu_A(y) \wedge \mu_B(e_2)] \\ &= \mu_A(x) \wedge \mu_A(y). \end{aligned}$$

Then, $\mu_A(z) \geq \mu_A(x) \wedge \mu_A(y)$. Also,

$$\begin{aligned} \nu_A(z) &= \nu_A(z) \vee \nu_B(e_2) = \nu_{A\times B}(z, e_2) \\ &\leq \nu_{A\times B}(x, e_2) \vee \nu_{A\times B}(y, e_2) \\ &= [\nu_A(x) \vee \nu_B(e_2)] \vee [\nu_A(y) \vee \nu_B(e_2)] \\ &= \nu_A(x) \vee \nu_A(y). \end{aligned}$$

Hence, $\nu_A(z) \leq \nu_A(x) \vee \nu_A(y)$. Therefore A is an IFSP of P_1. □

Proposition 3.13. *Let A and B IFS of the polygroups P_1 and P_2 respectively such that $\mu_B(y) \leq \mu_A(e_1)$ and $\nu_B(y) \geq \nu_A(e_1)$ hold for $y \in P_2$, e_1 being the identity element of P_1. If $A \times B$ is an IFSP of $P_1 \times P_2$, then, B is IFSP of polygroup P_2.*

Proof. The proof is similar to the proof of Proposition 3.12. □

Corollary 3.14. Let A and B IFS of the polygroups P_1 and P_2 respectively. If $A \times B$ is an IFSP of $P_1 \times P_2$, then, either A is IFSP of P_1 or B is IFSP of polygroup P_2.

Definition 3.15. [12] Let $\langle P, \cdot, e, ^{-1} \rangle$ and $\langle P', *, e', ^{-I} \rangle$ be two polygroups. Let f be a mapping from P to P' such that $f(e) = e'$. Then, f is called a *strong homomorphism* or a *good homomorphism* if $f(x \cdot y) = f(x) * f(y)$, for all $x, y \in P$.

Definition 3.16. [19] Let $A \in IFS(P)$. Then A is said to have the *sup property* if for any $T \in \mathcal{P}^*(P)$, there exists a $t_0 \in T$ such that $A(t_0) = \bigcup\limits_{t\in T} A(t)$, that is, $\mu_A(t_0) = \bigvee\limits_{t\in T} \mu_A(t)$ and $\nu_A(t_0) = \bigwedge\limits_{t\in T} \nu_A(t)$.

Proposition 3.17. *Let $f : P \longrightarrow P'$ be a strong polygroup homomorphism and $A \in IFSP(P)$, $B \in IFSP(P')$. Then, the following hold:*

(1) *If A has the sup property then, $f(A) \in IFSP(P')$.*
(2) *$f^{-1}(B) \in IFSP(P)$.*

Proof. (1) First, we suppose that A is an IFSP of P. In order to prove that $f(A)$ is an IFSP of P', by proposition 3.8, it is sufficient to show that each non-empty (α, β)-cut set $C_{\alpha,\beta}(f(A))$ is an subpolygroup of P'. So, suppose that $C_{\alpha,\beta}(f(A))$ is non-empty set for some $(\alpha, \beta) \in I \times I$ with $(\alpha, \beta) \leq A(e)$. Let $y_1, y_2 \in C_{\alpha,\beta}(f(A))$. We show that $y_1 * y_2 \subseteq C_{\alpha,\beta}(f(A))$. We have

$$\mu_{f(A)}(y_1) \geq \alpha,\ \nu_{f(A)}(y_1) \leq \beta \quad \text{and} \quad \mu_{f(A)}(y_2) \geq \alpha,\ \nu_{f(A)}(y_2) \leq \beta,$$

which implies that

$$\bigvee_{x\in f^{-1}(y_1)} \mu_A(x) \geq \alpha, \bigwedge_{x\in f^{-1}(y_1)} \nu_A(x) \leq \beta \quad \text{and} \bigvee_{x\in f^{-1}(y_2)} \mu_A(x) \geq \alpha, \bigwedge_{x\in f^{-1}(y_2)} \nu_A(x) \leq \beta.$$

Since A has the sup property, it follows that there exist $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ such that

$$\mu_A(x_1) \geq \alpha,\ \nu_A(x_1) \leq \beta \quad \text{and} \quad \mu_A(x_2) \geq \alpha,\ \nu_A(x_2) \leq \beta.$$

Since f is strong homomorphism, it follows that $y_1 * y_2 = f(x_1) * f(x_2) = f(x_1.x_2)$. Let $z \in y_1 * y_2$. Then, there exists $x' \in x_1.x_2$ such that $f(x') = z$. Thus, $f(x) = f(x') = z$. Since A is an IFSP of P, we have

$$\mu_A(x') \geq \mu_A(x_1) \wedge \mu_A(x_2) \geq \alpha \quad \text{and} \quad \nu_A(x') \leq \nu_A(x_1) \vee \nu_A(x_2) \leq \beta.$$

Therefore, we obtain

$$\mu_{f(A)}(z) = \bigvee_{x\in f^{-1}(z)} \mu_A(x) \geq \alpha \ \text{ and } \ \nu_{f(A)}(z) = \bigwedge_{x\in f^{-1}(z)} \nu_A(x) \leq \beta.$$

Hence $z \in C_{\alpha,\beta}(f(A))$. Thus, $y_1 * y_2 \subseteq C_{\alpha,\beta}(f(A))$.
Next, for $y \in C_{\alpha,\beta}(f(A))$, we have $\mu_{f(A)}(y) \geq \alpha$ and $\nu_{f(A)}(y) \leq \beta$. Thus,

$$\bigvee_{x\in f^{-1}(y)} \mu_A(x) \geq \alpha \ \text{ and } \ \bigwedge_{x\in f^{-1}(y)} \nu_A(x) \leq \beta.$$

Hence, we have

$$\bigvee_{x^{-1}\in f^{-1}(y^{-1})} \mu_A(x^{-1}) \geq \alpha \quad \text{and} \quad \bigwedge_{x^{-1}\in f^{-1}(y^{-1})} \nu_A(x^{-1}) \leq \beta.$$

Therefore, $\mu_{f(A)}(y^{-1}) \geq \alpha$ and $\nu_{f(A)}(y^{-1}) \leq \beta$ implies that $y^{-1} \in C_{\alpha,\beta}(f(A))$.

(2) Let $x, y \in P$ and $z \in x.y$. Then,

$$\begin{aligned} \mu_{f^{-1}(B)}(z) = \mu_B(f(z)) \quad &\geq \mu_B(f(x)) \wedge \mu_B(f(y)) \\ &= \mu_{f^{-1}(B)}(x) \wedge \mu_{f^{-1}(B)}(y), \end{aligned}$$

and

$$\begin{aligned} \nu_{f^{-1}(B)}(z) = \nu_B(f(z)) \quad &\leq \nu_B(f(x)) \vee \nu_B(f(y)) \\ &= \nu_{f^{-1}(B)}(x) \vee \nu_{f^{-1}(B)}(y). \end{aligned}$$

Also, for all $x \in P$

$$\mu_{f^{-1}(B)}(x^{-1}) = \mu_B(f(x^{-1})) = \mu_B(f(x)^{-1}) \geq \mu_B(f(x)) = \mu_{f^{-1}(B)}(x),$$

and

$$\nu_{f^{-1}(B)}(x^{-1}) = \nu_B(f(x^{-1})) = \nu_B(f(x)^{-1}) \leq \nu_B(f(x)) = \nu_{f^{-1}(B)}(x).$$

Hence $f^{-1}(B) \in IFSP(P)$. □

Definition 3.18. Let A be an IFSP in a polygroup P. Then A is called an intuitionistic fuzzy normal subpolygroup (in short, IFNSP) of P if for all $x, y \in P$

$$A(z) = A(z') \ i.e\ \mu_A(z) = \mu_A(z') \text{ and } \nu_A(z) = \nu_A(z'),$$

for all $z \in x \circ y$ and $z' \in y \circ x$.

It is obvious that if A is an intuitionistic fuzzy normal subpolygroup of P, then for all $x, y \in P$,

$$A(z) = A(z') \quad \text{for all} \quad z, z' \in x \circ y.$$

Theorem 3.19. *Let A be an intuitionistic fuzzy subpolygroup of P. Then A is an intuitionistic fuzzy normal subpolygroup if and only if*

(1) $\mu_A(z) = \mu_A(y)$ *and*
(2) $\nu_A(z) = \nu_A(y)$, *for all $x, y \in P$ and for all $z \in x \circ y \circ x^{-1}$.*

Proof. It is straightforward. □

Theorem 3.20. *Let A be an intuitionistic fuzzy normal subpolygroup of polygroup P. Then $C_{\alpha,\beta}(A)$ is normal subpolygroup of polygroup P, where $\mu_A(e) \geq \alpha$, $\nu_A(e) \leq \beta$ and e is the identity element of P.*

Proof. Let $y \in C_{\alpha,\beta}(A)$ and $x \in P$ be any element. Then $\mu_A(y) \geq \alpha$, $\nu_A(y) \leq \beta$. Since A be intuitionistic fuzzy normal subpolygroup of polygroup P, so $\mu_A(z) = \mu_A(y)$ and $\nu_A(z) = \nu_A(y)$ for all $x, y \in P$ and for all $z \in x \circ y \circ x^{-1}$. Therefore $\mu_A(z) = \mu_A(y) \geq \alpha$ and $\nu_A(z) = \nu_A(y) \leq \beta$ implies that $\mu_A(z) \geq \alpha$, $\nu_A(z) \leq \beta$ and so $z \in C_{\alpha,\beta}(A)$, $x \circ y \circ x^{-1} \subseteq C_{\alpha,\beta}(A)$. Hence $C_{\alpha,\beta}(A)$ is normal subpolygroup of P. □

Proposition 3.21. *If A is an IFNSP of P then, P_A is a normal subpolygroup of P.*

Proof. It is straightforward. □

Theorem 3.22. *Let A and B IFNSP of polygroups P_1 and P_2 respectively. Then $A \times B$ is also IFNSP of polygroup $P_1 \times P_2$.*

Proof. The proof is similar to the proof of Theorem 3.11. □

4. t-Intuitionistic fuzzy subpolygroups and t-intuitionistic fuzzy quotient polygroups

In this section, the notion of t-intuitionistic fuzzy (normal) subpolygroup, t-intuitionistic fuzzy cosets of an intuitionistic fuzzy normal subpolygroup and t-intuitionistic fuzzy quotient polygroup are defined and discussed.

Definition 4.1. [27] Let A be an IFS of a polygroup P. Then the IFS A^t of P is called the *t-intuitionistic fuzzy subset* of P and is defined as $A^t = (\mu_{A^t}, \nu_{A^t})$, where

$$\mu_{A^t}(x) = \mu_A(x) \wedge t \quad \text{and} \quad \nu_{A^t}(x) = \nu_A(x) \vee (1-t),$$

for all $x \in P$ and $t \in [0,1]$.

Definition 4.2. Let A be an IFS of a polygroup P. Then A is called the *t-intuitionistic fuzzy subpolygroup* (in short t-IFSP) of P if A^t is IFSP of P, i.e, the following conditions hold:

(1) $\mu_{A^t}(z) \geq \mu_{A^t}(x) \wedge \mu_{A^t}(y)$ and
$\nu_{A^t}(z) \leq \nu_{A^t}(x) \vee \nu_{A^t}(y)$ for each $z \in x \circ y$ and $x, y \in P$.

(2) $\mu_{A^t}(x^{-1}) \geq \mu_{A^t}(x)$ and
$\nu_{A^t}(x^{-1}) \leq \nu_{A^t}(x)$ for each $x \in P$.

Proposition 4.3. *If A is IFSP of a polygroup P, then A is also t-IFSP of P.*

Proof. Let $x, y \in P$ and $z \in x \circ y$. Then, we have:

$$\begin{aligned} \mu_{A^t}(z) &= \mu_A(z) \wedge t \\ &\geq (\mu_A(x) \wedge \mu_A(y)) \wedge t \\ &= (\mu_A(x) \wedge t) \wedge (\mu_A(y) \wedge t) \\ &= \mu_{A^t}(x) \wedge \mu_{A^t}(y). \end{aligned}$$

Thus $\mu_{A^t}(z) \geq \mu_{A^t}(x) \wedge \mu_{A^t}(y)$. Similarly we can show that $\nu_{A^t}(z) \leq \nu_{A^t}(x) \vee \nu_{A^t}(y)$. Also,

$$\mu_{A^t}(x^{-1}) = \mu_A(x^{-1}) \wedge t = \mu_A(x) \wedge t = \mu_{A^t}(x).$$

Similarly, we can show that $\nu_{A^t}(x^{-1}) = \nu_{A^t}(x)$. Hence A is t-IFSP of P. □

Definition 4.4. Let P be a polygroup, A be an intuitionistic fuzzy subpolygroup of P and $t \in [0,1]$. Then, the intuitionistic fuzzy subset A_a^t of P which is defined by

$$A_a^t(x) = (\mu_{A_a^t}(x), \nu_{A_a^t}(x)),$$

where

$$\mu_{A_a^t}(x) = (\bigwedge_{z \in x \circ a} \mu_A(z)) \wedge t \text{ and } \nu_{A_a^t}(x) = (\bigwedge_{z \in x \circ a} \mu_A(z)) \vee (1-t),$$

is called the *t- intuitionistic fuzzy right coset* of A. The *t- intuitionistic fuzzy left coset* ${}_aA^t$ of A is defined similarly.

Proposition 4.5. *Let A be an intuitionistic fuzzy normal subpolygroup of P and a an arbitrary element of P. Then, the t- intuitionistic fuzzy right coset A_a^t is same as the t- intuitionistic fuzzy left coset ${}_aA^t$.*

Proof. Let A be an intuitionistic fuzzy normal subpolygroup of P, $a \in P$ and $t \in [0,1]$. Then, for any $x \in P$ and $z \in x \circ a$, $z' \in a \circ x$, $\mu_A(z) = \mu_A(z')$, $\nu_A(z) = \nu_A(z')$. So,

$$\begin{aligned} \mu_{A_a^t}(x) &= (\bigwedge_{z \in x \circ a} \mu_A(z)) \wedge t \\ &= (\bigwedge_{z' \in a \circ x} \mu_A(z')) \wedge t \\ &= \mu_{{}_aA^t}(x), \end{aligned}$$

and

$$\begin{aligned} \nu_{A_a^t}(x) &= (\bigwedge_{z \in x \circ a} \nu_A(z)) \vee (1-t) \\ &= (\bigwedge_{z' \in a \circ x} \nu_A(z')) \vee (1-t) \\ &= \nu_{{}_aA^t}(x). \end{aligned}$$

Hence $A_a^t =_a A^t$. □

Definition 4.6. Let A be t-IFSP of a polygroup P. Then A is called *t-intuitionistic fuzzy normal subpolygroup* (in short t-IFNSP) of P if and only if $A^t_a =_a A^t$ for all $a \in P$.

Lemma 4.7. *Let A be t-IFNSP of a polygroup P. Then*

$$A^t_a = A^t_b \Leftrightarrow Na = Nb$$

for all $a, b \in P$, where $N = C_{t,1-t}(A)$.

Proof. It is straightforward. □

Consider the set $P/A^t = \{A^t_a \mid a \in P\}$ of all t- intuitionistic fuzzy right coset of A.

Theorem 4.8. *Let A be an intuitionistic fuzzy normal subpolygroup of a polygroup P and $N = C_{t,1-t}(A)$. Then, there is a bijection between P/A^t and P/N.*

Proof. The proof is similar to the proof of Theorem 2.3.8 in [10]. □

Corollary 4.9. Let P be a polygroup, A be an intuitionistic fuzzy normal subpolygroup of P and $a \in P$. Then, $A^t_z = A^t_a$ for all $z \in Na$, where $N = C_{t,1-t}(A)$ and $t \in [0, 1]$.

Proposition 4.10. *Let P be a polygroup, A be an intuitionistic fuzzy normal subpolygroup of a polygroup P. Then, $(P/A^t, \otimes)$ is a polygroup (called the polygroup of t- intuitionistic fuzzy coset induced by A and t), where the hyperoperation $\otimes$ is defined as follows:*

$$\otimes : P/A^t \times P/A^t \longrightarrow \mathcal{P}^*(P/A^t)$$

$$(A^t_a\ ,\ A^t_b) \mapsto \{A^t_c \mid c \in N.a.b\}$$

and $^{-1}$ on P/A^t is defined by $(A^t_a)^{-1} = A^t_{a^{-1}}$.

Proof. The proof is similar to the proof of Theorem 2.3.10 in [10]. □

Definition 4.11. Let P be a polygroup, A be an intuitionistic fuzzy subset of P and β^* the fundamental relation on P. Define the intuitionistic fuzzy subset A_{β^*} on P/β^* as follows:

$$\begin{aligned} &A_{\beta^*} : P/\beta^* \longrightarrow I \times I \\ &A_{\beta^*}(\beta^*(x)) = (\mu_{A_{\beta^*}}(\beta^*(x)), \nu_{A_{\beta^*}}(\beta^*(x))) \\ &\qquad\qquad\quad = (\bigvee_{a\in\beta^*(x)} \mu_A(a), \bigwedge_{a\in\beta^*(x)} \nu_A(a)). \end{aligned}$$

Theorem 4.12. *Let P be a polygroup, A be an intuitionistic fuzzy subpolygroup of P. Then A_{β^*} is an intuitionistic fuzzy subgroup of the group P/β^*.*

Proof. We have

$$\begin{aligned} \nu_{A_{\beta^*}}(\beta^*(x)) \vee \nu_{A_{\beta^*}}(\beta^*(y)) &= (\bigwedge_{a\in\beta^*(x)} \nu_A(a)) \vee (\bigwedge_{b\in\beta^*(y)} \nu_A(b)) \\ &= \bigwedge_{\substack{a\in\beta^*(x)\\ b\in\beta^*(y)}} [\nu_A(a) \vee \nu_A(b)] \\ &\geq \bigwedge_{\substack{a\in\beta^*(x)\\ b\in\beta^*(y)}} (\bigvee_{z\in a\circ b} \nu_A(z)) \\ &\geq \bigwedge_{\substack{a\in\beta^*(x)\\ b\in\beta^*(y)}} (\bigwedge_{z\in a\circ b} \nu_A(z)) \\ &\geq \bigwedge_{\substack{a\in\beta^*(x)\\ b\in\beta^*(y)}} (\bigwedge_{z\in\beta^*(a.b)} \nu_A(z)) \\ &= \bigwedge_{\substack{a\in\beta^*(x)\\ b\in\beta^*(y)}} (\nu_{A_{\beta^*}}(\beta^*(a.b))) \\ &= \bigwedge_{\substack{a\in\beta^*(x)\\ b\in\beta^*(y)}} (\nu_{A_{\beta^*}}(\beta^*(a) \odot \beta^*(a))) \\ &= \nu_{A_{\beta^*}}(\beta^*(x) \odot \beta^*(y)). \end{aligned}$$

Similarly we can show that

$$\mu_{A_{\beta^*}}(\beta^*(x)) \wedge \mu_{A_{\beta^*}}(\beta^*(y)) \leq \mu_{A_{\beta^*}}(\beta^*(x) \odot \beta^*(y)).$$

Now, suppose that $\beta^*(x)$ is an arbitrary element of P/β^*. Then,

$$\begin{aligned}\nu_{A_{\beta^*}}(\beta^*(x)^{-1}) &= \nu_{A_{\beta^*}}(\beta^*(x^{-1}))\\ &= \bigwedge_{a\in\beta^*(x^{-1})} \nu_A(a)\\ &= \bigwedge_{a\in\beta^*(x^{-1})} \nu_A(a^{-1})\\ &= \bigwedge_{b\in\beta^*(x)} \nu_A(b)\\ &= \nu_{A_{\beta^*}}(\beta^*(x)).\end{aligned}$$

Similarly, we can show that $\mu_{A_{\beta^*}}(\beta^*(x)^{-1}) = \mu_{A_{\beta^*}}(\beta^*(x))$. Thus, the proof is complete. □

5. Intuitionistic fuzzy topological polygroups

In this section, we define and study the concept of intuitionistic fuzzy topological polygroups, and we prove some properties in this respect.

Definition 5.1. Let $(P,\mathcal{T})$ be a polygroup and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ are two intuitionistic fuzzy sets in P. We define AB and A^{-1} by the respective formula:

(1)

$$\mu_{AB}(x) = \begin{cases} \bigvee_{(x_1,x_2)\in X\times X} [\mu_A(x_1)\wedge\mu_B(x_2)] & \text{if } x\in x_1\circ x_2.\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\nu_{AB}(x) = \begin{cases} \bigwedge_{(x_1,x_2)\in X\times X} [\nu_A(x_1)\vee\nu_B(x_2)] & \text{if } x\in x_1\circ x_2.\\ 1 & \text{otherwise.}\end{cases}$$

(2)

$$\mu_{A^{-1}}(x) = \mu_A(x^{-1}) \quad \text{and} \quad \nu_{A^{-1}}(x) = \nu_A(x^{-1}).$$

Definition 5.2. Let $(P,\circ)$ be a polygroup and $(P,\mathcal{T})$ be an intuitionistic fuzzy topological space. Let $U = (\mu_U, \nu_U)$, $V = (\mu_V, \nu_V)$ and $W = (\mu_W, \nu_W)$ be an intuitionistic fuzzy sets in P. $(P,\circ,\mathcal{T})$ is called an *intuitionistic fuzzy topological polygroup* or IFTP for short, if and only if:

(1) For all $x, y \in P$ and any fuzzy open Q-neighborhood W of any intuitionistic fuzzy point $z_{r,s}$ of $x\circ y$, there are fuzzy open Q-neighborhoods U of $x_{r,s}$ and V of $y_{r,s}$ such that: $UV \subseteq W$.

(2) For all $x \in P$ and any fuzzy open Q-neighborhood V of an intuitionistic fuzzy point $(x^{-1})_{r,s}$, there exists a fuzzy open Q-neighborhood U of $x_{r,s}$ such that: $U^{-1} \subseteq V$.

Evidently, every intuitionistic fuzzy topological group is an intuitionistic fuzzy topological polygroup. We give some other examples.

Example 5. Let P be a polygroup and $\mathcal{T}$ is the collection of all constant intuitionistic fuzzy sets in P. Then $(P,\mathcal{T})$ is an intuitionistic fuzzy topological polygroup.

Example 6. Let $P = \{e, a, b\}$. Then, P together with the following hyperoperation

$\circ$	e	a	b
e	e	a	b
a	a	e	b
b	b	b	$\{e,a\}$

is a polygroup. It is clear that $a^{-1} = a$, $b^{-1} = b$. Consider on P the fuzzy topology $\mathcal{T} = \{0_\sim, A, 1_\sim\}$, where $A =< x, (\frac{e}{0.7}, \frac{a}{0.5}, \frac{b}{0.3}), (\frac{e}{0.3}, \frac{a}{0.5}, \frac{b}{0.7}) >$. Then, $(P,\mathcal{T})$ is an intuitionistic fuzzy topological polygroup.

Definition 5.3. [25] Let $(X,\mathcal{T})$ be an intuitionistic fuzzy topological space. Let $\alpha, \beta \in [0,1]$. An intuitionistic fuzzy set $(\alpha\beta)^* = (\mu_{(\alpha\beta)^*}, \nu_{(\alpha\beta)^*})$, where $\mu_{(\alpha\beta)^*}(x) = \alpha$ and $\nu_{(\alpha\beta)^*}(x) = \beta$, for every $x \in X$ such that $\mu_{(\alpha\beta)^*}(x) + \nu_{(\alpha\beta)^*}(x) = 1$. Then $(X,\mathcal{T})$ is called a fully stratified space if for every $\alpha, \beta \in [0,1]$, $(\alpha\beta)^* \in \mathcal{T}$.

Proposition 5.4. *Suppose $(P,\mathcal{T})$ is a fully stratified space. Let $(P,\circ,\mathcal{T})$ be an intuitionistic fuzzy topological polygroup. Then the mapping $f: x \longrightarrow x^{-1}$ is intuitionistic fuzzy homeomorphic function of $(P,\mathcal{T})$ onto itself.*

Proof. It is seen that f is invertible. Hence the only thing which needs to be proved that f is intuitionistic fuzzy continuous. Let $(P,\circ,\mathcal{T})$ be an intuitionistic fuzzy topological polygroup and V be a fuzzy open Q-neighbourhood of intuitionistic fuzzy point $(x^{-1})_{r,s}$. Then, there exists a fuzzy open Q-neighbourhood U of $x_{r,s}$ such that $U^{-1} \subseteq V$. Since

$$\begin{aligned} \mu_{U^{-1}}(x) + r &= \mu_U(x^{-1}) + r \\ &> \mu_U(x) + r > 1, \end{aligned}$$

and

$$\begin{aligned} \nu_{U^{-1}}(x) + s &= \nu_U(x^{-1}) + s \\ &< \nu_U(x) + s < 1. \end{aligned}$$

This implies that $x_{r,s} q\ U^{-1}$. Hence U^{-1} is a fuzzy open Q-neighbourhood of $x_{r,s}$. Thus $f(U) = U^{-1} \subseteq V$. Then f is an intuitionistic fuzzy continuous function at the intuitionistic fuzzy point $x_{r,s}$. Therefore, f is an intuitionistic fuzzy continuous function. □

Proposition 5.5. *Let $(P,\circ,\mathcal{T})$ be an intuitionistic fuzzy topological polygroup.*

(1) *If U is an intuitionistic fuzzy compact subset of P then, U^{-1} is an intuitionistic fuzzy compact.*
(2) *If U is an intuitionistic fuzzy open set in $\mathcal{T}$ then, U^{-1} is an intuitionistic fuzzy open set in $\mathcal{T}$.*

Proof. It is straightforward. □

Definition 5.6. [7] Let $(X,\mathcal{T})$ be an IFTS and A an IFS in X. Then the fuzzy closure is defined by

$$cl(A) = \cap\{F \mid A \subseteq F,\ F^c \in \mathcal{T}\},$$

and the fuzzy interior is defined by

$$int(A) = \cup\{G \mid A \supseteq G,\ G \in \mathcal{T}\}.$$

Definition 5.7. Let P be a polygroup and A be IFSP of polygroup P. Let $a \in P$ be a fixed element. Then the set $aA = <\mu_{aA}, \nu_{aA}>$ where

$$\mu_{aA}(x) = \bigvee_{z \in a^{-1}\circ x} \mu_A(z) \qquad \text{for all } x \in P,$$

and

$$\nu_{aA}(x) = \bigwedge_{z \in a^{-1}\circ x} \nu_A(z) \qquad \text{for all } x \in P,$$

is called *intuitionistic fuzzy left coset of P determined by A and a.*
Similarly, the set $Aa = <\mu_{Aa}, \nu_{Aa}>$ where

$$\mu_{Aa}(x) = \bigvee_{z \in x\circ a^{-1}} \mu_A(z) \qquad \text{for all } x \in P,$$

and

$$\nu_{Aa}(x) = \bigwedge_{z \in x\circ a^{-1}} \nu_A(z) \qquad \text{for all } x \in P,$$

is called *intuitionistic fuzzy right coset of P determined by A and a.*

If A is an intuitionistic fuzzy normal subpolygroup of P and a an arbitrary element of P, then the intuitionistic fuzzy right coset Aa is same as the intuitionistic fuzzy left coset aA. Consider the set $P/A = \{Aa \mid a \in P\}$ of all intuitionistic fuzzy right cosets of A. Now we give a structure on P/A by defining the operation $\otimes$ between two intuitionistic fuzzy right cosets as

$$Aa \otimes Ab = \{Ac \mid c \in a \circ b\}.$$

If A is an intuitionistic fuzzy normal subpolygroup of a polygroup P, then the operation $\otimes$ defined on P/A is well defined. Then, $(P/A,\otimes)$ becomes a polygroup and is called the *intuitionistic fuzzy quotient polygroup* relative to the intuitionistic fuzzy normal subpolygroup A.

Proposition 5.8. *Let $(P,\mathcal{T})$ be an intuitionistic fuzzy topological polygroup. Then, the family $\mathcal{B} = \{\tilde{A} \in IFS(\mathcal{P}^*(P)) \mid A \in \mathcal{T}\}$, where $\mu_{\tilde{A}}(X) = \bigvee_{x\in X} \mu_A(x)$ and $\nu_{\tilde{A}}(X) = \bigwedge_{x\in X} \nu_A(x)$ is a base for an intuitionistic fuzzy topology $\mathcal{T}^*$ on $\mathcal{P}^*(P)$.*

Proof. $\mathcal{B}$ is a base for an intuitionistic fuzzy topology on $\mathcal{P}^*(P)$ because:

(1) For any $\tilde{A}_1, \tilde{A}_2 \in \mathcal{B}$, with $A_1, A_2 \in \mathcal{T}$, it follows that $\tilde{A}_1 \cap \tilde{A}_2 \in \mathcal{B}$, because $\tilde{A}_1 \cap \tilde{A}_2 = \widetilde{A_1 \cap A_2}$ and $A_1 \cap A_2 \in \mathcal{T}$.

Indeed, for any $X \in \mathcal{P}^*(P)$, we have

$$\begin{aligned}\mu_{\widetilde{A_1\cap A_2}}(X) &= \bigvee_{x\in X} \mu_{(A_1\cap A_2)}(x) = \bigvee_{x\in X} (\mu_{A_1}(x) \wedge \mu_{A_2}(x))\\ &= (\bigvee_{x\in X} \mu_{A_1}(x)) \wedge (\bigvee_{x\in X} \mu_{A_2}(x))\\ &= \mu_{\tilde{A}_1}(X) \wedge \mu_{\tilde{A}_2}(X)\\ &= \mu_{(\tilde{A}_1\cap\tilde{A}_2)}(X),\end{aligned}$$

and

$$\begin{aligned}\nu_{\widetilde{A_1\cap A_2}}(X) &= \bigwedge_{x\in X} \nu_{(A_1\cap A_2)}(x) = \bigwedge_{x\in X} (\nu_{A_1}(x) \vee \nu_{A_2}(x))\\ &= (\bigwedge_{x\in X} \nu_{A_1}(x)) \vee (\bigwedge_{x\in X} \nu_{A_2}(x))\\ &= \nu_{\tilde{A}_1}(X) \vee \nu_{\tilde{A}_2}(X)\\ &= \nu_{(\tilde{A}_1\cap\tilde{A}_2)}(X),\end{aligned}$$

(2) Since $1_\sim \in \mathcal{T}$ it follows that $\tilde{1_\sim}(X) = 1$ for any $X \in \mathcal{P}^*(P)$ and thus $\bigcup_{\tilde{A}\in\mathcal{B}} = 1$. □

Lemma 5.9. *Let U be an intuitionistic fuzzy open subset of an intuitionistic fuzzy topological polygroup P. Then, aU and Ua are intuitionistic fuzzy open subsets of P for every $a \in P$.*

Proof. Suppose that U be an intuitionistic fuzzy open subset of P. Then,

$$\begin{aligned}\mu_{({}_{a^{-1}}\phi^{-1}(\tilde{U}))}(z) &= \mu_{\tilde{U}}({}_{a^{-1}}\phi(z)) = \mu_{\tilde{U}}(a^{-1}\circ z)\\ &= \bigvee_{t\in a^{-1}\circ z} \mu_U(t) = \mu_{aU}(z),\end{aligned}$$

and

$$\begin{aligned}\nu_{({}_{a^{-1}}\phi^{-1}(\tilde{U}))}(z) &= \nu_{\tilde{U}}({}_{a^{-1}}\phi(z)) = \nu_{\tilde{U}}(a^{-1}\circ z)\\ &= \bigwedge_{t\in a^{-1}\circ z} \nu_U(t) = \nu_{aU}(z).\end{aligned}$$

Since the mapping ${}_{a^{-1}}\phi^{-1} : P \longrightarrow \mathcal{P}^*(P), x \mapsto a^{-1}\circ x$, is intuitionistic fuzzy continuous, thus aU is intuitionistic fuzzy open. Similarly, we can prove that Ua is intuitionistic fuzzy open. □

Proposition 5.10. *Let $(P,\mathcal{T})$ be a fully stratified space. Let $(P,\circ,\mathcal{T})$ be an intuitionistic fuzzy topological polygroup and $U = (\mu_U, \nu_U)$ be an intuitionistic fuzzy set of P. If $IFcl(U)$ is an intuitionistic fuzzy closed set , then $aIFcl(U)$, $IFcl(U)a$ are intuitionistic fuzzy closed sets, where $a \in P$ is a definite point.*

Proof. It is straightforward. □

Proposition 5.11. *Let A be an IFSP of polygroup P. Then for each $(r,s) \in I\times I$ with $(r,s) \geq A(e)$, $x_{r,s}A = xA$, where $x \in P$ and e is the identity of P.*

Proof. We have $(x_{r,s}A)(t) = (\mu_{x_{r,s}A}(t), \nu_{x_{r,s}A}(t)$, where

$$\begin{aligned}\mu_{x_{r,s}A}(t) &= \bigvee_{t\in t_1\circ t_2} [\mu_{x_{r,s}}(t_1) \wedge \mu_A(t_2)]\\ &= \begin{cases}\bigvee_{t\in t_1\circ t_2} [r\wedge \mu_A(t_2)] & \text{if } t_1 = x\\ 0 & \text{if } t_1 \neq x\end{cases}\\ &= \bigvee_{t_2\in x^{-1}\circ t} \mu_A(t_2)\\ &= \mu_{xA}(t),\end{aligned}$$

and

$$\begin{aligned}\nu_{x_{r,s}A}(t) &= \bigwedge_{t\in t_1\circ t_2}[\nu_{x_{r,s}}(t_1)\vee\nu_A(t_2)]\\ &= \begin{cases}\bigwedge_{t\in t_1\circ t_2}[s\vee\nu_A(t_2)] & \text{if } t_1=x\\ 1 & \text{if } t_1\neq x\end{cases}\\ &= \bigwedge_{t_2\in x^{-1}\circ t}\nu_A(t_2)\\ &= \nu_{xA}(t).\end{aligned}$$

Hence $x_{r,s}A = xA$. □

Proposition 5.12. *Let $(P,\mathcal{T})$ be a fully stratified space. Let $(P,\circ,\mathcal{T})$ be an intuitionistic fuzzy topological polygroup and $U=(\mu_U,\nu_U)$ be an intuitionistic fuzzy set of P. If $IFcl(U)$ is an intuitionistic fuzzy closed set , then $a_{r,s}IFcl(U)$, $IFcl(U)a_{r,s}$ and $IFcl(U)^{-1}$ are intuitionistic fuzzy closed sets.*

Theorem 5.13. *In an intuitionistic fuzzy topological polygroup P, V is a Q-neighbourhood of $e_{r,s}$ if and only if V^{-1} is a Q-neighbourhood of $e_{r,s}$.*

Proof. Let V be a Q-neighbourhood of $e_{r,s}$. Then there exists intuitionistic fuzzy open set A such that $e_{r,s}qA\subseteq V$ that is,

$$\mu_A(e)+r>1,\ A\subseteq V,$$
$$\nu_A(e)+s<1,\ A\subseteq V.$$

For all $x\in P$, $\mu_A(x^{-1})\le\mu_V(x^{-1})$ and $\nu_A(x^{-1})\ge\nu_V(x^{-1})$, so $\mu_{A^{-1}}(x)\le\mu_{V^{-1}}(x)$ and $\nu_{A^{-1}}(x)\ge\nu_{V^{-1}}(x)$ then, $A^{-1}\subseteq V^{-1}$.
Now,

$$\mu_{A^{-1}}(e)+\mu_{e_{r,s}}(e)=\mu_{A^{-1}}(e)+r>1,$$
$$\nu_{A^{-1}}(e)+\nu_{e_{r,s}}(e)=\nu_{A^{-1}}(e)+s<1.$$

Hence, $e_{r,s}qA^{-1}$ and $A^{-1}\subseteq V^{-1}$. Therefore, V^{-1} is a Q-neighbourhood of $e_{r,s}$.
Conversely, let V^{-1} be a Q-neighbourhood of $e_{r,s}$. Then there exists intuitionistic fuzzy open set A such that $e_{r,s}qA\subseteq V^{-1}$. As above, $A^{-1}\subseteq V$ and $e_{r,s}qA^{-1}$. That is, V is a Q-neighbourhood of $e_{r,s}$. □

Proposition 5.14. *Let $(P,\mathcal{T})$ be a fully stratified space. Let $(P,\circ,\mathcal{T})$ be an intuitionistic fuzzy topological polygroup and $U=(\mu_U,\nu_U)$ be an intuitionistic fuzzy set of P. If U is a Q-neighbourhood of $e_{r,s}$, then $x_{1,0}U$ is a Q-neighbourhood of $x_{r,s}$.*

Proof. Since U is a Q-neighbourhood of $e_{r,s}$, there exists an intuitionistic fuzzy open set A such that $r+\mu_A(e)>1$ and $s+\nu_A(e)<1$, $A\subseteq U$. So,

$$\begin{aligned}\mu_{x_{1,0}A}(x) &= \bigvee_{x\in xy}[\mu_{x_{1,0}}(x)\wedge\mu_A(y)]\\ &\ge 1\wedge\mu_A(e)\\ &=\mu_A(e),\end{aligned}$$

and

$$r+\mu_{x_{1,0}A}(x)\ge r+\mu_A(e)>1.$$

Also

$$\begin{aligned}\nu_{x_{1,0}A}(x) &= \bigwedge_{x\in xy}[\nu_{x_{1,0}}(x)\vee\nu_A(y)]\\ &\ge 0\vee\nu_A(e)\\ &=\nu_A(e),\end{aligned}$$

and

$$s+\nu_{x_{1,0}A}(x)\le s+\nu_A(e)<1.$$

Thus, for all $z\in P$,

$$\begin{aligned}x_{1,0}U(z)=xU(z) &= (\bigvee_{t\in x^{-1}\circ z}\mu_U(t),\ \bigwedge_{t\in x^{-1}\circ z}\nu_U(t))\\ &\supseteq (\bigvee_{t\in x^{-1}\circ z}\mu_A(t),\ \bigwedge_{t\in x^{-1}\circ z}\nu_A(t))\\ &= x_{1,0}A(z).\end{aligned}$$

Hence $x_{r,s} q\ x_{1,0}A \subseteq x_{1,0}U$ and since $x_{1,0}A$ is an intuitionistic fuzzy open set, Therefore $x_{1,0}U$ is a Q-neighbourhood of $x_{r,s}$. □

Proposition 5.15. [25] *An intuitionistic fuzzy point $x_{r,s} \subseteq IFcl(A)$ if and only if each Q-neighbourhood of $x_{r,s}$ is quasi-coincident with A.*

Proposition 5.16. *Let $(P, \mathcal{T})$ be a fully stratified space. Let $(P, \circ, \mathcal{T})$ be an intuitionistic fuzzy topological polygroup and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of P. If $x_{r,s} \subseteq IFclA$, then $(\bigcap_{C\in\{C\}} AC)(x) = (\bigcap_{C\in\{C\}} CA)(x) \supset 0$, where $\{C\}$ is the system of all Q- neighbourhood of $e_{a,b}$ in P with $a \leq r$ and $b \geq s$.*

Proof. Since $x_{r,s} \subseteq IFclA$ then, each Q- neighbourhood of $x_{r,s}$ is quasi-coincident with A. For any $C \in \{C\}$, there exists IFOS B such that $e_{a,b}\ q\ B \subseteq C$, that is,

$$\mu_B(e) + a > 1,\ B \subseteq C,$$

$$\nu_B(e) + b < 1,\ B \subseteq C.$$

Hence, $x_{1,0}B^{-1}$ is an IFOS. Moreover, we have

$$\begin{aligned}\mu_{x_{1,0}B^{-1}}(x) &= \bigvee_{x\in xy} [\mu_{x_{1,0}}(x) \wedge \mu_{B^{-1}}(y)] \\ &\geq 1 \wedge \mu_{B^{-1}}(e) \\ &= 1 \wedge \mu_B(e^{-1}) \\ &= \mu_B(e),\end{aligned}$$

and

$$\mu_{x_{1,0}B^{-1}}(x) + r \geq \mu_B(e) + r \geq \mu_B(e) + a > 1.$$

Also, we have

$$\begin{aligned}\nu_{x_{1,0}B^{-1}}(x) &= \bigwedge_{x\in xy} [\nu_{x_{1,0}}(x) \vee \nu_{B^{-1}}(y)] \\ &\geq 0 \vee \nu_{B^{-1}}(e) \\ &= 0 \vee \nu_B(e^{-1}) \\ &= \nu_B(e),\end{aligned}$$

and

$$\nu_{x_{1,0}B^{-1}}(x) + s \leq \nu_B(e) + s \leq \nu_B(e) + b < 1.$$

Hence, we conclude that

$$\begin{aligned}x_{1,0}C^{-1}(z) = xC^{-1}(z) &= (\bigvee_{t\in x^{-1}\circ z} \mu_{C^{-1}}(t), \bigwedge_{t\in x^{-1}\circ z} \nu_{C^{-1}}(t)) \\ &\supseteq (\bigvee_{t\in x^{-1}\circ z} \mu_{B^{-1}}(t), \bigwedge_{t\in x^{-1}\circ z} \nu_{B^{-1}}(t)) \\ &= xB^{-1}(z) \\ &= x_{1,0}B^{-1}(z).\end{aligned}$$

This implies that $x_{r,s}\ q\ x_{1,0}B^{-1} \subseteq x_{1,0}C^{-1}$. Since $x_{1,0}C^{-1}$ and A are quasi-coincident, there exists $y \in P$ such that

$$\mu_{x_{1,0}C^{-1}}(y) + \mu_A(y) > 1 \text{ and } \nu_{x_{1,0}C^{-1}}(y) + \nu_A(y) < 1.$$

Also

$$\begin{aligned}\mu_{x_{1,0}C^{-1}}(y) &= \bigvee_{y\in xz} [\mu_{x_{1,0}}(x) \wedge \mu_{C^{-1}}(z)] \\ &= \bigvee_{y\in xz} [1 \wedge \mu_{C^{-1}}(z)] \\ &= \bigvee_{z\in x^{-1}y} \mu_{C^{-1}}(z) \\ &= \mu_{xC^{-1}}(y),\end{aligned}$$

and

$$\begin{aligned}\nu_{x_{1,0}C^{-1}}(y) &= \bigwedge_{y\in xz}[\nu_{x_{1,0}}(x)\vee\nu_{C^{-1}}(z)]\\ &= \bigwedge_{y\in xz}[0\vee\nu_{C^{-1}}(z)]\\ &= \bigwedge_{z\in x^{-1}y}\nu_{C^{-1}}(z)\\ &= \nu_{xC^{-1}}(y).\end{aligned}$$

Thus

$$\begin{aligned}\mu_{AC}(x) &= \bigvee_{x\in t_1t_2}[\mu_A(t_1)\wedge\mu_C(t_2)]\\ &\geq \mu_A(y)\wedge(\bigvee_{z\in y^{-1}x}\mu_C(z))\\ &= \mu_A(y)\wedge(\bigvee_{z^{-1}\in x^{-1}y}\mu_{C^{-1}}(z))\\ &= \mu_A(y)\wedge\mu_{x_{1,0}C^{-1}}(y)\\ &> 0,\end{aligned}$$

and

$$\begin{aligned}\nu_{AC}(x) &= \bigwedge_{x\in t_1t_2}[\nu_A(t_1)\vee\nu_C(t_2)]\\ &\leq \nu_A(y)\vee(\bigwedge_{z\in y^{-1}x}\nu_C(z))\\ &= \nu_A(y)\vee(\bigwedge_{z^{-1}\in x^{-1}y}\nu_{C^{-1}}(z))\\ &= \nu_A(y)\vee\nu_{x_{1,0}C^{-1}}(y)\\ &< 1.\end{aligned}$$

That is, $AC(x)\supset 0$ for every $C\in\{C\}$. Hence $(\cap AC)(x)=\bigwedge_{C\in\{C\}}AC(x)\supset 0$. It is easy to prove $\cap AC=\cap CA$. □

Proposition 5.17. *Let $(P,\mathcal{T})$ be a fully stratified space. Let $(P,\circ,\mathcal{T})$ be an intuitionistic fuzzy topological polygroup and $A=(\mu_A,\nu_A)$ be an intuitionistic fuzzy subset of P. If $x_{r,s}\subseteq\bigcap_{C\in\{C\}}AC=\bigcap_{C\in\{C\}}CA$ and $r>0.5,\ s<0.5$, then $x_{r,s}\subseteq IFcl(A)$, where $\{C\}$ is the system of all Q- neighbourhood of $e_{a,b}$ in P with $a\leq r$ and $b\geq s$.*

Proof. Let $x_{r,s}\subseteq AC$ for each $C\in\{C\}$. Then $\mu_{AC}(x)\geq r$ and $\nu_{AC}(x)\leq s$. Let D be an arbitrary Q-neighbourhood of $x_{r,s}$. Then there exists an IFOS B such that $x_{r,s}\ q\ B\subseteq D$. That is,

$$\mu_B(x)+r>1,\ B\subseteq D,$$

$$\nu_B(x)+s<1,\ B\subseteq D.$$

Since $\mu_B(x)+r>1,\ \ r>0.5$ and $\nu_B(x)+s<1,\ \ s<0.5$, thus $D(x)\supseteq B(x)\supseteq 0$. Hence, $B^{-1}x_{1,0}$ is an IFOS. Moreover we have

$$\begin{aligned}\mu_{B^{-1}x_{1,0}}(e) &= \bigvee_{e\in yx}[\mu_{B^{-1}}(y)\wedge\mu_{x_{1,0}}(x)]\\ &\geq \mu_{B^{-1}}(x^{-1})\wedge 1\\ &= \mu_{B^{-1}}(x^{-1})\\ &= \mu_B(x).\end{aligned}$$

and

$$\mu_{B^{-1}x_{1,0}}(e)+a\geq\mu_B(x)+a\geq\mu_B(x)+r>1.$$

Similarly, we have

$$\begin{aligned}\nu_{B^{-1}x_{1,0}}(e) &= \bigwedge_{e\in yx}[\nu_{B^{-1}}(y)\vee\nu_{x_{1,0}}(x)]\\ &\leq \nu_{B^{-1}}(x^{-1})\vee 0\\ &= \nu_{B^{-1}}(x^{-1})\\ &= \nu_B(x).\end{aligned}$$

and

$$\nu_{B^{-1}x_{1,0}}(e)+b\leq\nu_B(x)+b\leq\nu_B(x)+s<1.$$

Hence

$$\begin{aligned} B^{-1}x_{1,0}(z) = B^{-1}x(z) &= (\bigvee_{t\in z\circ x^{-1}} \mu_{B^{-1}}(t), \bigwedge_{t\in z\circ x^{-1}} \nu_{B^{-1}}(t)) \\ &\subseteq (\bigvee_{t\in z\circ x^{-1}} \mu_{D^{-1}}(t), \bigwedge_{t\in z\circ x^{-1}} \nu_{D^{-1}}(t)) \\ &= D^{-1}x_{1,0}(z). \end{aligned}$$

This implies that $e_{a,b}\ q\ B^{-1}x_{1,0} \subseteq D^{-1}x_{1,0}$. So $D^{-1}x_{1,0} \in \{C\}$. Thus $\mu_{AD^{-1}x_{1,0}}(x) \geq r$ and $\nu_{AD^{-1}x_{1,0}}(x) \leq s$. Moreover, we have

$$\begin{aligned} \mu_{AD^{-1}x_{1,0}}(x) &= \bigvee_{x\in yx} [\mu_{AD^{-1}}(y) \wedge \mu_{x_{1,0}}(x)] \\ &\geq \mu_{AD^{-1}}(e) \wedge 1 \\ &= \mu_{AD^{-1}}(e), \end{aligned}$$

and

$$\begin{aligned} \nu_{AD^{-1}x_{1,0}}(x) &= \bigwedge_{x\in yx} [\nu_{AD^{-1}}(y) \vee \nu_{x_{1,0}}(x)] \\ &\leq \nu_{AD^{-1}}(e) \vee 0 \\ &= \nu_{AD^{-1}}(e). \end{aligned}$$

$$\begin{aligned} \mu_{AD^{-1}}(e) &= \bigvee_{e\in t_1t_2} [\mu_A(t_1) \wedge \mu_{D^{-1}}(t_2)] \\ &\geq \mu_A(k) \wedge \mu_{D^{-1}}(k^{-1}) \\ &= \mu_A(k) \wedge \mu_D(k), \end{aligned}$$

and

$$\begin{aligned} \nu_{AD^{-1}}(e) &= \bigwedge_{e\in t_1t_2} [\nu_A(t_1) \vee \nu_{D^{-1}}(t_2)] \\ &\leq \nu_A(k) \vee \nu_{D^{-1}}(k^{-1}) \\ &= \nu_A(k) \vee \nu_D(k). \end{aligned}$$

Thus there exists $z \in P$ such that

$$\mu_{AD^{-1}}(e) \geq \mu_A(z) \wedge \mu_D(z) \text{ and } \nu_{AD^{-1}}(e) \leq \nu_A(z) \vee \nu_D(z).$$

Since $\mu_{AD^{-1}}(e) \geq r$ and $\nu_{AD^{-1}}(e) \leq s$ then, $\mu_A(z) \geq r$, $\nu_A(x) \leq s$ and $\mu_D(z) \geq r, \nu_D(x) \leq s$. Since $r > 0.5$ and $s < 0.5$,

$$\mu_A(z) + \mu_D(z) \geq r + r = 2r > 1 \text{ and } \nu_A(z) + \nu_D(z) \leq s + s = 2s < 1.$$

That is, D is quasi-coincident with A. Therefore $x_{r,s} \subseteq IFcl(A)$. □

References

[1] N. Abbasizadeh and B. Davvaz, Topological polygroups in the framework on fuzzy sets, J. Intell. Fuzzy Syst, 30 (2016) 2811-2820.

[2] N. Abbasizadeh, B. Davvaz and V. Leoreanu-Fotea, Studies on fuzzy topological polygroups, J. Intell. Fuzzy Syst, in press.

[3] N. Abbasizadeh and B. Davvaz, Category of fuzzy topological polygroups, International Journal of Analysis and Applications 11 (2016), 124-136.

[4] T. C. Ahn, K. Hur and K. W. Jang, intuitionistic fuzzy subgroups and level subgroups, International Journal of Fuzzy Logic and Intelligent System 6(3) (2006), 240-246.

[5] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems, 20 (1986), 87-96.

[6] R. Biswas Intuitionistic fuzzy subgroups, Mathematical Forum 10 (1996), 39-44.

[7] D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88 (1997), 81-89.

[8] S. D. Comer, Polygroups derived from cogroups, J. Algebra 89 (1984), 397-405.

[9] P. Corsini, Prolegomena of Hypergroup Theory, Aviani Editore, 1993.

[10] B. Davvaz and I. Cristea, Fuzzy Algebraic Hyperstructures, Springer, 2015.

[11] B. Davvaz, Isomorphism theorems of polygroups, Bull. Math. Sci. Soc., 33(3) (2010), 11-19.

[12] B. Davvaz, Polygroup Theory and Related System, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[13] D. Freni, Une note sur le coeur dun hypergroupe et sur la cloture transitive β^* de β, Riv. di Mat. Pura Appl. 8 (1991), 153-156.

[14] F. Gallego, Hausdorffness in intuitionistic fuzzy topological space, Mathware and Soft Computing 10 (2003), 17-22.

[15] M. Jafarpour, H. Aghabozorgi and B. Davvaz, On nilpotent and solvable polygroups, Bulletin of Iranian Mathematical Society 39 (2013), 487-499.

[16] Y. B. Jun, On (ϕ, ψ)- intuitionistic fuzzy mappings, Int. J. Pure Appl. Math. 12(2) (2004), 133-139.

[17] I. M. Hanafy, Completely continuous functions in intuitionistic fuzzy topological spaces, Czechoslovak Math. J. 53 (2003), 793-803.
[18] D. Heidari, B. Davvaz and S. M. S. Modarres, Topological hypergroups in thr sense of Marty, Comm. Algebra 42 (2014), 4712-4721.
[19] K. Hur, S. Y. Jang and H. W. Kang, intuitionistic fuzzy subgroupoids, International Journal of Fuzzy Logic and Intelligent System 3 (2003), 72-77.
[20] K. Hur, S. Y. Jang and H. W. Kang, intuitionistic fuzzy subgroups and cosets, Honam Mathematical J. 26 (2004), 17-41.
[21] M. Koskas, Groupoides, demi-hypergroupes et hypergroupes, J. Math. Pure Appl. 49(9) (1970), 155-192.
[22] S. J. Lee and P. E. Lee, The category of intuitionistic fuzzy topological spaces, Bull. Korean. Math. Soc. 37 (2000), 63-76.
[23] F. G. Lupianez, Quasicoincident for intuitionistic fuzzy points, Int. J. Math. Sci. 10 (2005), 1539-1542.
[24] F. Marty, Sur une generalization de la notion de groupe, 8^{iem} Congress Math. Scandinaves, Stockholm, (1934), 45-90.
[25] S. Padmapriya, M. K. Uma and E. Roja, A study on intuitionistic fuzzy topological groups, Ann. Fuzzy Math. Inform. 76 (2014), 991-1004.
[26] P. K. Sharma, (α, β)-cut of Intuitionistic fuzzy groups, International Mathematics Forum 6 (2011), 2605-2614.
[27] P. K. Sharma, t- Intuitionistic fuzzy subgroups, International Journal of Fuzzy Mathematics and Systems 3 (2012), 233-243.
[28] M.M. Zahedi, M. Bolurian and A. Hasankhani, On polygroups and fuzzy subpolygroups, J. Fuzzy Math. 3 (1995), 1-15.

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