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ON CHEBYSHEV FUNCTIONAL AND OSTROWSKI-GRÜS TYPE INEQUALITIES FOR TWO COORDINATES

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Abstract. In this paper, we construct Chebyshev functional and Grüss inequality on two coordinates. Also we establish Ostrowski-Grüss type inequality on two coordinates. Related mean value theorems of Lagrange and Cauchy type are also given.

1. Introduction

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. We consider

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx - \frac{1}{b-a}\int_a^b f(x)g(x)dx. \tag{1}$$

If f and g are monotonic in same direction on $[a, b]$, then

$$T(f,g) \geq 0. \tag{2}$$

If f and g are monotonic in opposite directions on interval [a,b], then the reverse of the inequality (2) is valid.

The Chebyshev functional (1) has a long history and an extensive repertoire of applications in many fields including numerical quadrature, transform theory, probability and statistical problems and special functions. It is worthwhile noting that a number of identities relating to the Chebyshev functional already exist. In [11, Chapter IX and X], one can see lots of results related to the Chebyshev functional. One of them is famous as Korkine's identity (see [11, p. 243]) given by

$$T(f,g) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f(x)-f(y)\right)\left(g(x)-g(y)\right)dxdy. \tag{3}$$

This identity is often used to prove an inequality due to Grüss for functions bounded above and below (see in [6]). In literature it is known as Grüss inequality.

Theorem 1.1. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions such that $\phi \leq f(x) \leq \varphi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$, where ϕ, φ, γ and Γ are real constants. Then*

$$|T(f,g)| \leq \frac{1}{4}(\varphi - \phi)(\Gamma - \gamma). \tag{4}$$

An other celebrated inequality in this respect is by Ostrowski sated in the following theorem (see [12]).

Theorem 1.2. *Let $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o the interior of I and a, $b \in I^o$, $a < b$, If $\left|f^{'}(t)\right| \leq M$, for all $t \in [a, b]$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, \tag{5}$$

for all $x \in [a, b]$.

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Inequality in (5) is well known as Ostrowski inequality and has interesting consequences in numerical integration (see [3]). It has been improved by Dragomir and Wang in [4] using Grüss inequality in terms of the lower and upper bounds of the first derivative. That Ostrowski-Grüss type inequality is stated in the following theorem.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and its derivative satisfy the condition $\gamma \le f'(x) \le \Gamma$ for all $x \in [a, b]$, then we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(\frac{f(b) - f(a)}{b-a}\right)\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Gamma - \gamma), \tag{6}$$

for all $x \in [a, b]$.

In [2], Barnett et al. pointed out a similar result to the above for twice differentiable mappings in terms of the upper and lower bounds of the second derivative.

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and twice differentiable on (a, b) and assume that the second derivative $f'' : (a, b) \to \mathbb{R}$ satisfies the condition $\gamma \le f''(x) \le \Gamma$ for all $x \in [a, b]$. Then, for all $x \in [a, b]$, we have inequality*

$$\begin{aligned} &\left| f(x) - \left(x - \frac{a+b}{2}\right) f'(x) + \left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right]\left(\frac{f'(b) - f'(a)}{b-a}\right)\right. \\ &\qquad \left. - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{1}{8}(\Gamma - \gamma)\left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2}\right|\right]^2. \end{aligned} \tag{7}$$

Many authors considered different generalization of Chebyshev functional on two coordinates and found the bounds of these functional, for example see [1, 14] and references there in.

In this paper we give the Chebyshev functional and Grüss inequality on two coordinates and establish the Ostrowski-Grüss type inequality on two coordinates in terms of lower and upper bounds of first and second order partial derivatives. Also we give Lagrange and Cauchy type mean value theorems for the Chebyshev functional, as given in [5].

2. Main Results

Let $\Delta = [a, b] \times [c, d]$ be a bi-dimensional interval in $\mathbb{R}^2$ and $f : \Delta \to \mathbb{R}$ be a mapping. If $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, then we say $\mathbf{x} \le \mathbf{y}$ if $x_1 \le y_1$ and $x_2 \le y_2$. Also we say f is monotonically increasing on Δ if for all $\mathbf{x}, \mathbf{y} \in \Delta$

$$f(\mathbf{x}) \le f(\mathbf{y})$$

when $\mathbf{x} \le \mathbf{y}$.
If we take $F(x) = \int_a^b f(x, t)dt$, provided that the integral exists, then one can note that

$$F(x) = \int_c^d f(x, t)dt \le \int_c^d f(y, t)dt = F(y)$$

for $x < y$, that is F is monotonically increasing on $[a, b]$.

In the following theorem, we introduce Chebyshev functional on two coordinates and generalize the Chebyshev inequality on a rectangle from the plane.

Theorem 2.1. *Let $f, g : \Delta \to \mathbb{R}$ be integrable functions. We consider*

$$\boldsymbol{A}(f; \Delta) = \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x, y)dydx, \tag{8}$$

and

$$\boldsymbol{T}(f, g; \Delta) = \boldsymbol{A}(f; \Delta)\boldsymbol{A}(g; \Delta) - \boldsymbol{A}(fg; \Delta) \tag{9}$$

If f and g are monotonic in same direction on Δ, then

$$\boldsymbol{T}(f, g; \Delta) \ge 0. \tag{10}$$

Proof. Considering the monotonicity of f and g on second coordinate and using (2), we get

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \leq \frac{1}{(d-c)^2}\int_c^d f(x,y)dy\int_c^d g(x,y)dy.$$

Integrating above inequality over $[a,b]$, we have

$$\frac{1}{d-c}\int_a^b\int_c^d f(x,y)g(x,y)dydx \leq \frac{1}{(d-c)^2}\int_a^b\left(\int_c^d f(x,y)dy\int_c^d g(x,y)dy\right)dx. \tag{11}$$

Now if we take $F(x)=\int_c^d f(x,y)dy$, then F is monotonic on $[a,b]$ by considering monotonicity of f on first coordinate. Similarly, we take $G(x)=\int_c^d g(x,y)dy$, then G is monotonic on $[a,b]$. If f and g are monotone in same direction so are F and G, then using the Chebyshev inequality, one has

$$\frac{1}{(b-a)}\int_a^b F(x)G(x)dx \leq \frac{1}{(b-a)^2}\int_a^b F(x)dx\int_a^b G(x)dx. \tag{12}$$

Using the above inequality in (11), we get

$$\boldsymbol{A}(fg;\Delta)\leq \boldsymbol{A}(f;\Delta)\boldsymbol{A}(g;\Delta),$$

which is equivalent to required result. □

It is easy to find that

$$\boldsymbol{T}(f,g;\Delta)= \frac{1}{2(b-a)^2(d-c)^2}\int_a^b\int_c^d\int_a^b\int_c^d (f(x,y)-f(u,v))(g(x,y)-g(u,v))dxdydudv.$$

This identity can be considered as Korkine's identity in two coordinates. Using this identity one can prove the following result similar to the proof of Theorem 1.1 (see also [11, p. 296])

Theorem 2.2. *Let $f,g:\Delta\to\mathbb{R}$ be integrable functions such that $\varphi\leq f(x,y)\leq\phi$ and $\gamma\leq g(x,y)\leq\Gamma$, for all $x,y\in\Delta$ where ϕ, φ, γ and Γ are constants. Then*

$$|\boldsymbol{T}(f,g:\Delta)|\leq\frac{1}{4}(\phi-\varphi)(\Gamma-\gamma). \tag{13}$$

In [9], an important result related to Grüss inequality is given, we can have a similar result related to Grüss inequality on two coordinates.

If $f,g:\Delta\to\mathbb{R}$ be integrable functions, then

$$\boldsymbol{T}(f,f;\Delta)\geq 0$$

and a following inequality holds:

$$\boldsymbol{T}^2(f,g;\Delta)\leq\boldsymbol{T}(f,f;\Delta)\boldsymbol{T}(g,g;\Delta). \tag{14}$$

By the combination of inequalities (13) and (14), we obtain the following result.

Theorem 2.3. *Let $f,g:\Delta\to\mathbb{R}$ be two integrable functions. If $\varphi\leq f(x,y)\leq\phi$, for all $x\in[a,b]$ and $y\in[c,d]$, where ϕ and φ are some constants, then*

$$|\boldsymbol{T}(f,g;\Delta)|\leq\frac{1}{2}(\phi-\varphi)\sqrt{\boldsymbol{T}(g,g;\Delta)}. \tag{15}$$

Proof. Setting $g=f$ in (13), we get

$$\boldsymbol{T}(f,f;\Delta)=|\boldsymbol{T}(f,f;\Delta)|\leq\frac{1}{4}(\phi-\varphi)^2. \tag{16}$$

Combining (16) with (14) we get

$$\boldsymbol{T}^2(f,g;\Delta)\leq\frac{1}{4}(\phi-\varphi)^2\boldsymbol{T}(g,g;\Delta),$$

this is equivalent to required result (15). □

In the following result we construct Ostrowski-Grüss type inequality on two coordinates in terms of the lower and upper bounds of the first order partial derivatives.

Theorem 2.4. *Let $f : \Delta \to \mathbb{R}$ be continuous on Δ and its partial derivative satisfy the condition $\gamma_1 \leq \frac{\partial f}{\partial x} \leq \Gamma_1$ and $\gamma_2 \leq \frac{\partial f}{\partial y} \leq \Gamma_2$ on Δ. Then we have*

$$\left| \int_a^b \frac{f(x,c)+f(x,d)}{2} dx + \int_c^d \frac{f(a,y)+f(b,y)}{2} dy - \left(\frac{1}{b-a} + \frac{1}{d-c} \right) \right.$$

$$\left. \int_a^b \int_c^d f(x,y) dx dy \right| \leq \frac{(b-a)(d-c)}{4} \left[(\Gamma_2 - \gamma_2) + (\Gamma_1 - \gamma_1) \right]. \tag{17}$$

Proof. For all $(x,y) \in \Delta$, consider two mappings $f_y : [a,b] \to \mathbb{R}$ and $f_x : [c,d] \to \mathbb{R}$ defined by $f_y(t) = f(t,y)$ and $f_x(t) = f(x,t)$ respectively.

Applying (6) for mapping f_y at $x = b$, we have

$$\left| f(b,y) - \frac{1}{b-a} \int_a^b f(t,y) dt - \frac{f(b,y) - f(a,y)}{2} \right| \leq \frac{1}{4}(b-a)(\Gamma_1 - \gamma_1).$$

On integrating over $[c,d]$, we have

$$\left| \int_c^d f(b,y) dy - \frac{1}{b-a} \int_a^b \int_c^d f(x,y) dy dx - \frac{(d-c)(f(b,y) - f(a,y))}{2} \right|$$

$$\leq \frac{1}{4}(b-a)(d-c)(\Gamma_1 - \gamma_1). \tag{18}$$

Applying (6) for mapping f_y at $x = a$ and then integrating over $[c,d]$, we get

$$\left| \int_c^d f(a,y) dy - \frac{1}{b-a} \int_a^b \int_c^d f(x,y) dy dx + \frac{(d-c)(f(b,y) - f(a,y))}{2} \right|$$

$$\leq \frac{1}{4}(b-a)(d-c)(\Gamma_1 - \gamma_1). \tag{19}$$

Addition of (18) and (19) lead us to

$$\left| \int_c^d \frac{f(a,y)+f(b,y)}{2} dy - \frac{1}{b-a} \int_a^b \int_c^d f(x,y) dy dx \right|$$

$$\leq \frac{1}{4}(b-a)(d-c)(\Gamma_1 - \gamma_1). \tag{20}$$

Similarly using inequalities getting after applying (6) for mapping f_x first at $y = c$ then at $y = d$ and integrating over $[a,b]$, we can have

$$\left| \int_a^b \frac{f(x,c)+f(x,d)}{2} dx - \frac{1}{d-c} \int_a^b \int_c^d f(x,y) dy dx \right| \leq \frac{1}{4}(b-a)(d-c)(\Gamma_2 - \gamma_2). \tag{21}$$

Using (20) and (21), we have (17). □

In the following we establish the similar result to the Theorem 2.4 for twice differentiable mappings in terms of the lower and upper bounds of the second order partial derivative.

Theorem 2.5. *Let $f : \Delta^2 \to \mathbb{R}$ be continuous on Δ^2 and differentiable for all $x \in (a,b)$ and $y \in (c,d)$ and assume that the second order partial derivative satisfies the condition $\gamma_2 \leq \frac{\partial^2 f}{\partial x^2} \leq \Gamma_2$ for all*

$x \in [a,b]$ and $\gamma_1 \leq \frac{\partial^2 f}{\partial y^2} \leq \Gamma_1$ for all $y \in [c,d]$, then we have

$$\begin{aligned}
(22)\quad & \left| \frac{1}{2}\left[\int_a^b (f(x,c)+f(x,d))\,dx + \int_c^d (f(a,y)+f(b,y))\,dy\right] + \frac{1}{12} \right. \\
& \left[(b-a)\int_c^d \left(\frac{\partial f(a,y)}{\partial x} - \frac{\partial f(b,y)}{\partial x}\right) dy + (d-c)\int_a^b \left(\frac{\partial f(x,c)}{\partial y} - \frac{\partial f(x,d)}{\partial y}\right) dx\right] - \\
& \left.\left(\frac{1}{b-a}+\frac{1}{d-c}\right)\int_a^b\int_c^d f(x,y)dydx\right| \leq \frac{1}{8}(d-c)(b-a)\left((\Gamma_1-\gamma_1)(b-a)\right. \\
& \left. +(\Gamma_2-\gamma_2)(d-c)\right).
\end{aligned}$$

Proof. For all $(x,y) \in \Delta$, consider two mappings $f_y : [a,b] \to \mathbb{R}$ and $f_x : [c,d] \to \mathbb{R}$ defined by $f_y(t) = f(t,y)$ and $f_x(t) = f(x,t)$ respectively.

Applying (7) for mapping f_y at $x = b$, we have

$$\left| f(b,y) - \frac{(b-a)}{6}\left(\frac{\partial f(a,y)}{\partial x} + 2\frac{\partial f(b,y)}{\partial x}\right) - \frac{1}{b-a}\int_a^b f(t,y)dt \right| \leq \frac{1}{8}(\Gamma_1-\gamma_1)(b-a)^2.$$

Integrating over $[c,d]$, we have

$$\begin{aligned}
& \left| \int_c^d f(b,y)dy - \frac{(b-a)}{6}\int_c^d \left(\frac{\partial f(a,y)}{\partial x} + 2\frac{\partial f(b,y)}{\partial x}\right) dy \right. \\
(23)\quad & \left. -\frac{1}{b-a}\int_a^b\int_c^d f(x,y)dydx \right| \leq \frac{1}{8}(d-c)(\Gamma_1-\gamma_1)(b-a)^2.
\end{aligned}$$

Applying (7) for mapping f_y at $x = a$ and integrating over $[c,d]$, we get

$$\begin{aligned}
& \left| \int_c^d f(a,y)dy + \frac{(b-a)}{6}\int_c^d \left(\frac{\partial f(b,y)}{\partial x} + 2\frac{\partial f(a,y)}{\partial x}\right) dy \right. \\
(24)\quad & \left. -\frac{1}{b-a}\int_a^b\int_c^d f(x,y)dydx \right| \leq \frac{1}{8}(d-c)(\Gamma_1-\gamma_1)(b-a)^2.
\end{aligned}$$

Using (23) and (24), we get

$$\begin{aligned}
(25)\quad & \left| \frac{1}{2}\int_c^d (f(a,y)dy + f(b,y))dy + \frac{(b-a)}{12}\int_c^d \left(\frac{\partial f(a,y)}{\partial x} - \frac{\partial f(b,y)}{\partial x}\right) dy \right. \\
& \left. -\frac{1}{b-a}\int_a^b\int_c^d f(x,y)dydx \right| \leq \frac{1}{8}(d-c)(\Gamma_1-\gamma_1)(b-a)^2.
\end{aligned}$$

Similarly using inequalities getting after applying (7) for mapping f_x first at $y = c$ then at $y = d$ and integrating over $[a,b]$, we have

$$\begin{aligned}
(26)\quad & \left| \frac{1}{2}\int_a^b (f(x,c)dy + f(x,d))dx + \frac{(d-c)}{12}\int_a^b \left(\frac{\partial f(x,c)}{\partial y} - \frac{\partial f(x,d)}{\partial y}\right) dx \right. \\
& \left. -\frac{1}{d-c}\int_a^b\int_c^d f(x,y)dydx \right| \leq \frac{1}{8}(b-a)(\Gamma_2-\gamma_2)(d-c)^2.
\end{aligned}$$

Using (25) and (26), we get (22). $\square$

3. Mean Value Theorems

In this section, we give mean value theorems of Lagrange and Cauchy type for Chebyshev functional on two coordinates. Before presenting our main results, one can note: if a function $f : \Delta \to \mathbb{R}$ has non-negative first order partial derivatives on Δ, then it is increasing on Δ.

Lemma 3.1. *Let $f : \Delta \to \mathbb{R}$ be an integrable function and also monotonically increasing on coordinates, such that $m_1 \leq \frac{\partial f(x,y)}{\partial x} \leq M_1$ and $m_2 \leq \frac{\partial f(x,y)}{\partial y} \leq M_2$ for all interior points (x, y) in Δ. Consider the functions $h, k : \Delta \to \mathbb{R}$ defined as*

$$h(x, y) = \max\{M_1, M_2\}(x + y) - f(x, y)$$

and

$$k(x, y) = f(x, y) - \min\{m_1, m_2\}(x + y).$$

Then h and k are monotonically increasing on Δ.

Proof. Since

$$\frac{\partial h(x, y)}{\partial x} = \max\{M_1, M_2\} - \frac{\partial f(x, y)}{\partial x} \geq 0 \tag{27}$$

and

$$\frac{\partial h(x, y)}{\partial y} = \frac{\partial f(x, y)}{\partial y} - \min\{m_1, m_2\} \geq 0 \tag{28}$$

for all interior points (x, y) in Δ, h is monotonically increasing on coordinates.

Similarly it can also be proved that k is monotonically increasing on coordinates on Δ. □

Theorem 3.2. *Let $f, g : \Delta \to \mathbb{R}$ be functions such that f has continuous partial derivatives of first order in Δ and g is increasing on Δ. Then there exists (ξ_1, η_1) and (ξ_2, η_2) in the interior of Δ such that*

$$\boldsymbol{T}(f, g; \Delta) = \frac{\partial f(\xi_1, \eta_1)}{\partial x} \boldsymbol{T}(r, g; \Delta) \tag{29}$$

and

$$\boldsymbol{T}(f, g; \Delta) = \frac{\partial f(\xi_2, \eta_2)}{\partial y} \boldsymbol{T}(r, g; \Delta), \tag{30}$$

where $r(x, y) = x + y$ and $\boldsymbol{T}(r, g; \Delta) \neq 0$.

Proof. Since f has continuous partial derivatives of first order in Δ, there exist real numbers m_1, m_2, M_1 and M_2, such that $m_1 \leq \frac{\partial f(x,y)}{\partial x} \leq M_1$ and $m_2 \leq \frac{\partial f(x,y)}{\partial y} \leq M_2$ for all $(x, y) \in \Delta$.

Now consider function h defined in Lemma 3.1. As h is increasing on coordinates in Δ, therefore

$$\boldsymbol{T}(h, g; \Delta) \geq 0,$$

that is

$$\boldsymbol{T}(\max\{M_1, M_2\}r - f, g; \Delta) \geq 0.$$

This gives us

$$\boldsymbol{T}(f, g; \Delta) \leq \max\{M_1, M_2\} \boldsymbol{T}(r, g; \Delta). \tag{31}$$

On the other hand for the function k defined in Lemma 3.1, one has

$$\min\{m_1, m_2\} \boldsymbol{T}(r, g; \Delta) \leq \boldsymbol{T}(f, g; \Delta). \tag{32}$$

As $\boldsymbol{T}(r, g; \Delta) \neq 0$, combining above inequalities (31) and (32), we get

$$\min\{m_1, m_2\} \leq \frac{\boldsymbol{T}(f, g; \Delta)}{\boldsymbol{T}(r, g; \Delta)} \leq \max\{M_1, M_2\}.$$

Then there exist (ξ_1, η_1) and (ξ_2, η_2) in the interior of Δ, such that

$$\frac{\boldsymbol{T}(f, g; \Delta)}{\boldsymbol{T}(r, g; \Delta)} = \frac{\partial f(\xi_1, \eta_1)}{\partial x}$$

and

$$\frac{\boldsymbol{T}(f,g;\Delta)}{\boldsymbol{T}(r,g;\Delta)} = \frac{\partial f(\xi_2,\eta_2)}{\partial y}.$$

Hence the required results are proved. □

Theorem 3.3. *Let $f, g : \Delta \to \mathbb{R}$ be functions having partial derivatives in Δ and g is increasing on Δ. Then there exists (ξ_i, η_i), $i = 1, 2, 3, 4$ in the interior of Δ such that*

$$\boldsymbol{T}(f,g;\Delta) = \frac{\partial f(\xi_1,\eta_1)}{\partial x}\frac{\partial g(\xi_3,\eta_3)}{\partial x}\,\boldsymbol{T}(r,r;\Delta) \tag{33}$$

and

$$\boldsymbol{T}(f,g;\Delta) = \frac{\partial f(\xi_2,\eta_2)}{\partial y}\frac{\partial g(\xi_4,\eta_4)}{\partial y}\,\boldsymbol{T}(r,r;\Delta), \tag{34}$$

where $r(x,y) = x + y$.

Proof. Since $T(r,g;\Delta) = T(g,r;\Delta)$ and $r(x,y) = x+y$ is increasing on Δ, by Theorem 3.2 there exists (ξ_3,η_3) in the interior of Δ such that

$$\boldsymbol{T}(r,g;\Delta) = \frac{\partial g(\xi_3,\eta_3)}{\partial x}\,\boldsymbol{T}(r,r;\Delta) \tag{35}$$

Using above expression in (29) gives us (33).

In a similar way, one can deduce (34). □

In [15], Pečarić gave many interesting result related to Chebyshev functional. A similar result is also valid for Chebyshev functional on two coordinates. Namely, the following corollary.

Corollary 3.4. *Let $f, g : \Delta \to \mathbb{R}$ be functions, such that g is increasing on Δ and f has partial derivatives of first order in Δ with $\left|\frac{\partial f}{\partial x}\right| \le M_1$, $\left|\frac{\partial f}{\partial y}\right| \le M_2$, $\left|\frac{\partial g}{\partial x}\right| \le N_1$ and $\left|\frac{\partial f}{\partial x}\right| \le N_2$. Then one has*

$$\boldsymbol{T}(f,g;\Delta) \le M_i N_i\,\boldsymbol{T}(r,r;\Delta), \quad i = 1, 2, \tag{36}$$

where $r(x,y) = x + y$.

Theorem 3.5. *Let $f_1, f_2, g : \Delta \to \mathbb{R}$ be functions, such that f has partial derivatives of first order in Δ and g is increasing on Δ. Then there exists (ξ_1, η_1) and (ξ_2, η_2) in the interior of Δ such that*

$$\frac{\boldsymbol{T}(f_1,g;\Delta)}{\boldsymbol{T}(f_2,g;\Delta)} = \frac{\frac{\partial f_1(\xi_1,\eta_1)}{\partial x}}{\frac{\partial f_2(\xi_1,\eta_1)}{\partial x}}$$

and

$$\frac{\boldsymbol{T}(f_1,g;\Delta)}{\boldsymbol{T}(f_2,g;\Delta)} = \frac{\frac{\partial f_1(\xi_1,\eta_1)}{\partial y}}{\frac{\partial f_2(\xi_1,\eta_1)}{\partial y}}.$$

Proof. We define the function $h : \Delta \to \mathbb{R}$, such that

$$h = c_1 f_1 - c_2 f_2,$$

where $c_1 = \boldsymbol{T}(f_2,g;\Delta)$ and $c_2 = \boldsymbol{T}(f_1,g;\Delta)$.

Now, using Theorem 3.2 with $f = h$, we have

$$0 = \left(c_1\frac{\partial f_1(\xi_1,\eta_1)}{\partial x} - c_2\frac{\partial f_2(\xi_1,\eta_1)}{\partial x}\right)\boldsymbol{T}(r,g;\Delta)$$

and

$$0 = \left(c_1\frac{\partial f_1(\xi_2,\eta_2)}{\partial y} - c_2\frac{\partial f_2(\xi_2,\eta_2)}{\partial y}\right)\boldsymbol{T}(r,g;\Delta).$$

Since $\boldsymbol{T}(r,g;\Delta) \neq 0$, we have

$$\frac{c_2}{c_1} = \frac{\frac{\partial f_1(\xi_1,\eta_1)}{\partial x}}{\frac{\partial f_2(\xi_1,\eta_1)}{\partial x}}$$

and

$$\frac{c_2}{c_1} = \frac{\frac{\partial f_1(\xi_1,\eta_1)}{\partial y}}{\frac{\partial f_2(\xi_1,\eta_1)}{\partial y}},$$

this complete the proof. □

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