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SOME COMMON FIXED POINT THEOREMS IN COMPLEX-VALUED METRIC SPACES USING IMPLICIT RELATION

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Abstract: The aim of this paper is to establish some new common fixed point theorems for generalized contractive maps in complex-valued metric space by using property (*E.A.*) and common property (*E.A.*).

1. INTRODUCTION

The Banach fixed point theorem for contraction mapping has been generalized and extended in many directions. Recently Azam *et al.*[3] introduced the complex valued metric space which is more general than classical metric space. Complex valued metric spaces is useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics; as well as in physics, including hydrodynamics, thermodynamics, mechanical engineering and electrical engineering. For more detail, one can read [4-5]. The concept of property $(E.A)$ in metric space has been introduced by Aamri *et al.* [1] which allows replacing the completeness requirement of the space with a more natural condition of closedness of the range.

The aim of this paper is to establish some new common fixed point theorems for generalized contractive maps in complex-valued metric space by using these new properties.

2. PRELIMINARIES

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Let C be the set of complex numbers throughout this paper and z_1, z_2 ∈ C, recall a natural partial order relation $\precsim$ on C as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$,

$z_1 \prec z_2$ if and only if $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

Definition 2.1. [3]. Let X be a nonempty set such that the map $d : X \times X \to$ C satisfies the following conditions:

(C1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(C3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex-valued metric on X, and (X, d) is called a complex-valued metric space.

Example 2.1[3]. Define complex-valued metric $d : X \times X \to$ C by $d(z_1, z_2) = e^{3i} | z_1 - z_2 |$. Then (X, d) is a complex-valued metric space

Definition 2.2[3]. Let (X, d) complex-valued metric space and $x \in X$. Then sequence $\{x_n\}$ in X is

(i) **convergent** if $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$, if for every $0 \prec c \in$ C, there is a natural number N such that $d(x_n, x) \prec c$, for all $n > N$. We denote it by $\lim_{n \to \infty} x_n = x$.

(ii) a **cauchy sequence**, if for every $c \in$ C, with $0 \prec$ c there is a natural number N such that $d(x_n, x_m) \prec$ c, for all $n, m > N$.

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Definition 2.3[2]. A pair of self-maps A and S of a complex-valued metric space (X, d) are weakly compatible if $ASx = SAx$ for all $x \in X$ at which $Ax = Sx$.

Example 2.2[5] Define complex-metric $d : X \times X \to$ C by: $d(z_1, z_2) = e^{ia} | z_1 - z_2 |$, where a is any real constant. Then (X, d) is a complex-valued metric space. Suppose self maps A and S be defined as:

$Az = 2e^{i\pi/4}$ if $\mathrm{Re}(z) \neq 0$,

$Az = 3e^{i\pi/3}$ if $\mathrm{Re}(z) = 0$,

and

$Sz = 2e^{i\pi/4}$ if $\mathrm{Re}(z) \neq 0$,
$Sz = 4e^{i\pi/6}$ if $\mathrm{Re}(z) = 0$.
Then maps A and S are weakly compatible at all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \neq 0$.

Definition 2.4[1]. A pair of Self maps A and S on a complex-valued metric space (X, d) satisfies the property $(E.A)$ if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Example 2.3. Let X = C and d be any complex-valued metric. Define self maps A and S by $Az = z^2$ and $Sz = z$, for all $z \in X$. Consider a sequence in X as $\{x_n\} = \left\{\frac{1}{n}\right\}$ where n = 1,2,3,... then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0$. Hence, the pair (A, S) satisfies property $(E.A)$ for the sequences $\{x_n\}$ in X.

Definition 2.5[2]. Two pairs of self maps (A, S) and (B, T) on a complex-valued metric space (X, d) satisfy Common property $(E.A)$ if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = p$ for some $p \in X$.

Definition 2.6[2]. Two finite families of self maps $\{A_i\}_{i=1}^{m}$ and $\{B_j\}_{j=1}^{n}$ on a set X are pairwise commuting if

(i) $A_i A_j = A_j A_i$, $i, j \in \{1, 2, 3, \ldots m\}$,
(ii) $B_i B_j = B_j B_i$, $i, j \in \{1, 2, 3, \ldots n\}$,
(iii) $A_i B_j = B_j A_i$, $i \in \{1, 2, 3, \ldots m\}$, $j \in \{1, 2, 3, \ldots n\}$.

3. Main Results:

Implicit relations play important role in establishing of common fixed point results.

Let M_6 be the set of all continuous functions satisfying the following conditions:

(A) $\phi(u, 0, u, 0, 0, u) \precsim 0 \Rightarrow u \precsim 0$
(B) $\phi(u, 0, 0, u, u, 0) \precsim 0 \Rightarrow u \precsim 0$
(C) $\phi(u, u, 0, 0, u, u) \precsim 0 \Rightarrow u \precsim 0$ for all $0 \precsim u$.

Example 3.1: Define $\phi : (C)^6 \to C$ as

$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \phi_1(\min\{t_2,t_3,t_4,t_5,t_6\})$,

where $\phi_1 : C \to C$ is increasing and continuous function such that $\phi_1(s) > s$ for all $s \in C$, Clearly, ϕ satisfies all conditions (*A*), (*B*) and (*C*). Therefore, $\phi \in M_6$.

Now, we begin with following observation:

Lemma 3.1: Let *A*, *B*, *S* and *T* be self mappings of a complex-valued metric space (X, d) satisfying the following:

(3.1) the pair (*A*, *S*) or (*B*, *T*) satisfies the property *E.A.*;

(3.2) for any $x, y \in X$, ϕ in M_6,

$$\phi\begin{pmatrix} d(Ax,By), d(Sx,Ty), d(Sx,Ax), \\ d(Ty,By), d(Sx,By), d(Ty,Ax) \end{pmatrix} \precsim 0$$

(3.3) $A(X) \subset T(X)$ or $B(X) \subset S(X)$.

Then the pairs (*A*, *S*) and (*B*, *T*) share the common (*E.A.*) property.

Proof: Suppose that the pair (*A*, *S*) satisfies property E.A., then there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$. Since $A(X) \subset T(X)$, hence for each $\{x_n\}$, there exist $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z$.

Now, we claim that $\lim_{n\to\infty} By_n = z$. Suppose that $\lim_{n\to\infty} By_n \neq z$, then applying inequality (3.2), we obtain

$$\phi\begin{pmatrix} d(Ax_n,By_n), d(Sx_n,Ty_n), d(Sx_n,Ax_n), \\ d(Ty_n,By_n), d(Sx_n,By_n), d(Ty_n,Ax_n) \end{pmatrix} \precsim 0$$

which on making $n \to \infty$ reduces to

$$\phi\begin{pmatrix} d(z,\lim_{n\to\infty} By_n), d(z,z,t), d(z,z), \\ d(z,\lim_{n\to\infty} By_n), d(z,\lim_{n\to\infty} By_n), d(z,z) \end{pmatrix} \precsim 0$$

$$\phi\begin{pmatrix} d(z,\lim_{n\to\infty} By_n), 0, 0, \\ d(z,\lim_{n\to\infty} By_n), d(z,\lim_{n\to\infty} By_n), 0 \end{pmatrix} \precsim 0$$

which is a contradiction to using (*B*), we get

$d(z, \lim_{n\to\infty} By_n) \precsim 0$ which gives, $\left|d(z, \lim_{n\to\infty} By_n)\right| \le 0$, a contradiction and therefore, $\lim_{n\to\infty} By_n = z$. Hence, the pairs (A, S) and (B, T) share the common $(E.A.)$ property.

Theorem 3.1: Let A, B, S and T be self mappings of a complex-valued metric space (X, d) satisfying the conditions (3.2) and

(3.4) the pair (A, S) and (B, T) share the common $(E.A.)$ property;

(3.5) $S(X)$ and $T(X)$ are closed subsets of X.

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof: In view of (3.4), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$ for some $z \in X$.

Since $S(X)$ is a closed subset of X, therefore, there exists a point $u \in X$ such that $z = Su$. We claim that $Au = z$. If $Au \neq z$, then by (3.2), take $x = u$, $y = y_n$,

$$\phi\begin{pmatrix} d(Au, By_n), d(Su, Ty_n), d(Su, Au), \\ d(Ty_n, By_n), d(Su, By_n), d(Ty_n, Au) \end{pmatrix} \precsim 0$$

taking the limit as $n \to \infty$, we get

$$\phi\begin{pmatrix} d(Au, z), d(z, z), d(z, Au), \\ d(z, z), d(z, z), d(z, Au) \end{pmatrix} \precsim 0$$

$$\phi(d(Au, z), 0, d(Au, z), 0, 0, d(Au, z)) \precsim 0.$$

Using (A), we get $d(Au, z) \precsim 0$.

This gives, $|d(Au, z)| \le 0$, a contradiction. Therefore, $Au = z = Su$ which shows that u is a coincidence point of the pair (A, S).

Since $T(X)$ is also a closed subset of X, therefore $\lim_{n\to\infty} Ty_n = z$ in $T(X)$ and hence there exists $v \in X$ such that $Tv = z = Au = Su$. Now, we show that $Bv = z$.

If $Bv \neq z$, then by using inequality (3.2), take $x = u$, $y = v$, we have

$$\phi\begin{pmatrix} d(Au,Bv), d(Su,Tv), d(Su,Au), \\ d(Tv,Bv), d(Su,Bv), d(Tv,Au) \end{pmatrix} \precsim 0$$

$$\phi(d(z,Bv),0,0,d(z,Bv),d(z,Bv),0) \precsim 0.$$

Using (B), we get $d(z,Bv) \precsim 0$ which gives, $|d(z,Bv)| \leq 0$, a contradiction, hence, $Bv = z = Tv$ which shows that v is a coincidence point of the pair (B, T).

Since the pairs (A, S) and (B, T) are weakly compatible and $Au = Su$, $Bv = Tv$, therefore, $Az = ASu = SAu = Sz$, $Bz = BTv = TBv = Tz$.

If $Az \neq z$, then by using inequality (3.2), we have

$$\phi\begin{pmatrix} d(Az,Bv), d(Sz,Tv), d(Sz,Az), \\ d(Tv,Bv), d(Sz,Bv), d(Tv,Az) \end{pmatrix} \precsim 0$$

$$\phi\begin{pmatrix} d(Az,z), d(Az,z), d(Az,Az), \\ d(Bv,Bv), d(Az,z), d(z,Az) \end{pmatrix} \precsim 0$$

$$\phi(d(Az,z),d(Az,z),0,0,d(Az,z),d(Az,z)) \precsim 0$$

Using (C), we get $d(Az,z) \precsim 0$ which gives, $|d(Az,z)| \leq 0$, a contradiction. Hence $Az = z = Sz$.

Similarly, one can prove that $Bz = Tz = z$. Hence, $Az = Bz = Sz = Tz$, and z is common fixed point of A, B, S and T.

Uniqueness: Let z and w be two common fixed points of A, B, S and T. If $z \neq w$, then by using inequality (3.2), we have

$$\phi\begin{pmatrix} d(Az,Bw), d(Sz,Tw), d(Sz,Az), \\ d(Tw,Bw), d(Sz,Bw), d(Tw,Az) \end{pmatrix} \precsim 0$$

$$\phi\begin{pmatrix} d(z,w), d(z,w), d(z,z), \\ d(w,w), d(z,w), d(w,z) \end{pmatrix} \precsim 0$$

$$\phi\begin{pmatrix} d(z,w), d(z,w), 0, \\ 0, d(z,w), d(z,w) \end{pmatrix} \precsim 0$$

Using (C), we have $d(z,w) \precsim 0$ which gives, $|d(z,w)| \leq 0$, a contradiction. Therefore, $z = w$. □

By choosing A, B, S and T suitably, one can derive corollaries involving two or more mappings. As a sample, we deduce the following natural result for a pair of self mappings by setting $B = A$ and $T = S$ in above theorem:

Corollary 3.1: Let A and S be self mappings of complex-valued metric space (X, d) satisfying the following:

(3.6) the pair (A, S) satisfies the property $(E.A.)$;

(3.7) for any $x, y \in X$, ϕ in M_6,

$$\phi\begin{pmatrix} d(Ax, Ay), d(Sx, Sy), d(Sx, Ax), \\ d(Sy, Ay), d(Sx, Ay), d(Sy, Ax) \end{pmatrix} \precsim 0;$$

(3.8) $S(X)$ is a closed subset of X.

Then, A and S have a point of coincidence each. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point.

The following example illustrates Theorem 3.1.

As an application of Theorem 3.1, we prove a common fixed point theorem for four finite families of maps on metric spaces. While proving our result, we utilize Definition 2.6 which is a natural extension of commutativity condition to two finite families.

Theorem 3.2: Let $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_n\}$, $\{S_1, S_2, \ldots, S_p\}$ and $\{T_1, T_2, \ldots, T_q\}$ be four finite families of self maps of a complex-valued metric space (X, d) such that $A = A_1.A_2.....A_m$, $B = B_1.B_2.....B_n$, $S = S_1.S_2.....S_p$ and $T = T_1.T_2.....T_q$ satisfy the condition (3.2) and

(i) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$)

(ii) the pair (A,S) (or (B,T)) satisfy property $(E.A)$.

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover finite families of self maps A_i, S_k, B_r and T_t have a unique common fixed point provided that the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_t\})$ commute pairwise for all $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, p$, $r = 1, 2, \ldots, n$, $t = 1, 2, \ldots, q$.

Proof: Since self maps A, B, S, T satisfy all the conditions of theorem 3.1, the pairs (A, S) and (B, T) have a point of coincidence. Also the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_t\})$ commute pairwise, we first show that $AS = SA$ as

$$AS = (A_1A_2 \ldots A_m)(S_1S_2 \ldots S_p) = (A_1A_2 \ldots A_{m-1})(A_m S_1S_2 \ldots S_p)$$
$$= (A_1A_2 \ldots A_{m-1})(S_1S_2 \ldots S_p A_m) = (A_1A_2 \ldots A_{m-2})(A_{m-1} S_1S_2 \ldots S_p A_m)$$
$$= (A_1A_2 \ldots A_{m-2})(S_1S_2 \ldots S_p A_{m-1}A_m) = \ldots = A_1(S_1S_2 \ldots S_p A_2 \ldots A_m)$$
$$= (S_1S_2 \ldots S_p)(A_1A_2 \ldots A_m) = SA.$$

Similarly one can prove that $BT = TB$. And hence, obviously the pair (A, S) and (B, T) are weakly compatible. Now using Theorem 3.1, we conclude that A, S, B and T have a unique common fixed point in X, say z.

Now, one needs to prove that z remains the fixed point of all the component maps.

For this consider

$$A(A_i z) = ((A_1A_2 \ldots A_m)A_i)z = (A_1A_2 \ldots A_{m-1})(A_m A_i)z$$
$$= (A_1A_2 \ldots A_{m-1})(A_i A_m)z = (A_1A_2 \ldots A_{m-2})(A_{m-1} A_i A_m)z$$
$$= (A_1A_2 \ldots A_{m-2})(A_i A_{m-1} A_m)z = \ldots = A_1(A_iA_2 \ldots A_m)z$$
$$= (A_1A_i)(A_2 \ldots A_m)z$$
$$= (A_i A_1)(A_2 \ldots A_m)z = A_i (A_1A_2 \ldots A_m)z = Ai\, Az = A_iz.$$

Similarly, one can prove that

$$A(S_k z) = S_k (Az) = S_kz,\ S(S_kz) = S_k(Sz) = S_kz,$$
$$S(A_i z)= A_i(Sz) = A_iz,\ B(B_rz) = B_r(Bz) = B_rz,$$
$$B(T_t z) = T_t(Bz) = T_t z,\ T(T_tz) = T_t(Tz) = T_tz$$

and

$$T(B_rz) = B_r(Tz) = B_rz,$$

which shows that (for all i, r, k and t) A_iz and S_kz are other fixed point of the pair (A, S) whereas B_rz and T_tz are other fixed points of the pair (B, T). As A, B, S and T have a unique common fixed point, so, we get

$$z = A_i z = S_k z = B_r z = T_t z, \quad \text{for all } i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, p,$$
$$r = 1, 2, \ldots, n, \quad t = 1, 2, \ldots, q.$$

which shows that z is a unique common fixed point of $\{A_i\}_{i=1}^{m}, \{S_k\}_{k=1}^{p}$, $\{B_r\}_{r=1}^{n}$ and $\{T_t\}_{t=1}^{q}$.

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