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SOME NEW ESTIMATES OF HERMITE-HADAMARD INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we first establish an integral identity. Further, using this identity, some new estimates for Hermite-Hadamard inequalities for harmonically convex functions are established. Finally, some applications to special mean are showed.

1. Introduction

In this article, let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_{++} = (0, \infty)$.

Theory of convex functions and theory of inequalities are closely related to each other. Therefore, some literature on inequalities can be found for convex functions.

One of the most extensively research on inequalities is Hermite-Hadamard type inequalities.

Definition 1.1 ([1,2]) *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is convex function on I, if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \qquad \forall x, y \in I, t \in [0,1]. \tag{1.1}$$

f is concave function if $-f$ is convex function.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function. The following inequality is the well-known Hermite-Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad a, b \in I \text{ with } a < b. \tag{1.2}$$

Estimates for Hermite-Hadamard inequality for convex functions are studied in a rich literature [8–16].

Theorem 1.2 ([1]) *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0 and $a, b \in I^0$ with $a < b$. If $|f'(x)|^q$ is a convex function for $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \tag{1.3}$$

Recently, İscan [3] introduced the concept of harmonically convex functions and established Hermite-Hadamard type inequality for harmonically convex functions.

Definition 1.3 ([3,4]) *A function $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically convex function on I, if*

$$f\left(\frac{1}{tx^{-1} + (1-t)y^{-1}}\right) \leq tf(x) + (1-t)f(y), \qquad \forall x, y \in I, t \in [0,1]. \tag{1.4}$$

f is said to be harmonically concave function if $-f$ is harmonically convex function.

Theorem 1.4 ([3,4]) *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.5}$$

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Theorem 1.5 ([3,4]) *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function $a, b \in I$ with $a < b$ and $f' \in L[a, b]$. If $|f'(x)|^q$ is harmonically convex on $[a, b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mu_1|f'(a)|^q + \mu_2|f'(b)|^q\right)^{\frac{1}{q}}, \tag{1.6}$$

where

$$\mu_1 = \frac{\left[a^{2-2q} + b^{1-2q}[(b-a)(1-2q)-a]\right]}{2(b-a)^2(1-q)(1-2q)},$$
$$\mu_2 = \frac{\left[a^{2-2q} + b^{1-2q}[(b-a)(1-2q)-b]\right]}{2(b-a)^2(1-q)(1-2q)}.$$

For many recent results related to Hermite-Hadamard type inequalities for harmonically functions, see [3–7].

The aim of this paper is first to establish a integral identity. Then, using this identity, some new estimates for Hermite-Hadamard inequalities for harmonically convex functions are established by İ. İscan in [3] are derived.

2. Some lemmas

In order to prove our main results we need some lemmas.

Lemma 2.1 *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I^0 and $a, b \in I^0$ with $a < b$. If $f' \in L[0, 1]$, then for $\lambda \in [0, 1]$, one has*

$$\begin{aligned}(1-\lambda)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right) + \lambda f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathbf{d}x \\ = \frac{b-a}{ab}\left[\int_0^{1-\lambda} \frac{t}{[tb^{-1}+(1-t)a^{-1}]}f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t \right.\\ \left. - \int_{1-\lambda}^1 \frac{t-1}{[tb^{-1}+(1-t)a^{-1}]}f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t\right].\end{aligned} \tag{2.1}$$

Proof. Let $x = \frac{1}{tb^{-1}+(1-t)a^{-1}}$, $t \in [0, 1]$ and $x \in [a, b]$, then

$$\begin{aligned}&\left(\frac{1}{a} - \frac{1}{b}\right)\int_0^{1-\lambda} \frac{t}{(tb^{-1}+(1-t)a^{-1})^2}f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t \\ &= tf\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\Big|_0^{1-\lambda} - \int_a^u \frac{f(x)}{x^2}\frac{ab}{b-a}\mathbf{d}x \\ &= (1-\lambda)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right) - \frac{ab}{b-a}\int_a^u \frac{f(x)}{x^2}\mathbf{d}x.\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\left(\frac{1}{b} - \frac{1}{a}\right)\int_{1-\lambda}^1 \frac{t}{(tb^{-1}+(1-t)a^{-1})^2}f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t \\ &= tf\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\Big|_{1-\lambda}^1 - \int_u^b \frac{f(x)}{x^2}\frac{ab}{b-a}\mathbf{d}x \\ &= \lambda f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right) - \frac{ab}{b-a}\int_u^b \frac{f(x)}{x^2}\mathbf{d}x,\end{aligned} \tag{2.3}$$

where $u = \frac{1}{(1-\lambda)b^{-1}+\lambda a^{-1}}$. So (7) follows from (8) and (9).□ □

Remark 2.2 *From (7) we derive the following two identities.*

$$\begin{aligned}
& f\left(\frac{2ab}{a+b}\right)-\frac{2ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathbf{d}x \\
&\qquad =\frac{2(b-a)}{ab}\left[\int_0^{\frac{1}{2}}\frac{t}{[tb^{-1}+(1-t)a^{-1}]}f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t\right. \\
&\qquad\quad \left.-\int_{\frac{1}{2}}^1\frac{t-1}{[tb^{-1}+(1-t)a^{-1}]}f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t\right]; \qquad (2.4)\\
& \frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathbf{d}x \\
&\qquad =\frac{b-a}{2ab}\left[\int_0^1\frac{t}{[tb^{-1}+(1-t)a^{-1}]}f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t\right. \\
&\qquad\quad \left.-\int_0^1\frac{t-1}{[tb^{-1}+(1-t)a^{-1}]}f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t\right]. \qquad (2.5)
\end{aligned}$$

Proof. Take $\lambda=\frac{1}{2}$ in (7), we can derive (10).
We respectively take $\lambda=0$ and $\lambda=1$ in (7) and add two inequalities, then (11) is obtained. □

Lemma 2.3 *By integral calculation, then*

$$\begin{aligned}
C_1(a,b,\lambda) &= \int_0^{1-\lambda}t(1-t)[tb+(1-t)a]^2\mathbf{d}t \\
&= (1-\lambda)^3\left[\frac{1}{5}(b-a)^2(1-\lambda)^2+\frac{1}{2}a(b-a)(1-\lambda)+\frac{1}{3}\lambda^2\right]; \qquad (2.6)\\
C_2(a,b,\lambda) &= \int_0^{1-\lambda}t^2[tb+(1-t)a]^2\mathbf{d}t \\
&= (1-\lambda)^2\left[\left(\frac{1}{20}+\frac{1}{5}\lambda\right)(b-a)^2(1-\lambda)^2+\left(\frac{1}{6}+\frac{1}{2}\lambda\right)a(b-a)(1-\lambda)+\left(\frac{1}{6}+\frac{1}{3}\lambda\right)\lambda^2\right]; \qquad (2.7)
\end{aligned}$$

$$\begin{aligned}
C_3(a,b,\lambda) &= \int_{1-\lambda}^1 t(1-t)[tb+(1-t)a]^2\mathbf{d}t=\int_0^{\lambda}(1-u)u[(1-u)b+ua]^2\mathbf{d}u \\
&= -\frac{1}{5}(b-a)^2\lambda^5+\frac{1}{4}(a^2+3b^2-4ab)\lambda^4+\frac{1}{3}(2ab-3b^2)\lambda^3+\frac{1}{2}b^2\lambda^2; \qquad (2.8)\\
C_4(a,b,\lambda) &= \int_{1-\lambda}^1(1-t)^2[tb+(1-t)a]^2\mathbf{d}t=\int_0^{\lambda}u^2[(1-u)b+ua]^2\mathbf{d}u \\
&= \frac{1}{5}(b-a)^2\lambda^5+\frac{1}{2}b(a-b)\lambda^4+\frac{1}{3}b^2\lambda^3, \qquad (2.9)
\end{aligned}$$

where $u=1-t$.

3. Main results

Our main results are stated as follows.

Theorem 3.1 *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$. If $|f'(x)|^q$ is harmonically convex on $[a,b]$ with $0 \le a < b$ for $q \ge 1$, then*

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+\lambda f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathbf{d}x\right| \\
&\le \frac{b-a}{ab}\Big\{(C_1(a,b,\lambda))^{1-\frac{1}{q}}\left[|f'(b)|^qC_1(a,b,\lambda)+|f'(a)|^qC_2(a,b,\lambda)\right]^{\frac{1}{q}} \\
&+(C_3(a,b,\lambda))^{1-\frac{1}{q}}\left[|f'(b)|^qC_3(a,b,\lambda)+|f'(a)|^qC_4(a,b,\lambda)\right]^{\frac{1}{q}}\Big\}. \qquad (3.1)
\end{aligned}$$

Proof. From Lemma 2.1 and using Hölder inequality, further, since $|f'(x)|^q$ is harmonically convex on $[a,b]$, we have

$$\begin{aligned}
I &= \left|(1-\lambda)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+\lambda f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathbf{d}x\right| \\
&\le \frac{b-a}{ab}\left[\int_0^{1-\lambda}\frac{t}{[tb^{-1}+(1-t)a^{-1}]}\left|f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t\right|\right. \\
&\left.+\int_{1-\lambda}^1\frac{1-t}{[tb^{-1}+(1-t)a^{-1}]}\left|f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\mathbf{d}t\right|\right] \\
&\le \frac{b-a}{ab}\left[\left(\int_0^{1-\lambda}\frac{t}{[tb^{-1}+(1-t)a^{-1}]^2}\mathbf{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{1-\lambda}\frac{t}{[tb^{-1}+(1-t)a^{-1}]^2}\left|f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\right|^q\mathbf{d}t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{1-\lambda}^1\frac{1-t}{[tb^{-1}+(1-t)a^{-1}]^2}\mathbf{d}t\right)^{1-\frac{1}{q}}\int_{1-\lambda}^1\frac{1-t}{[tb^{-1}+(1-t)a^{-1}]^2}\left|f'\left(\frac{1}{tb^{-1}+(1-t)a^{-1}}\right)\right|^q\mathbf{d}t\right] \\
&\le \frac{b-a}{ab}\left\{\left(\int_0^{1-\lambda}\frac{t}{[tb^{-1}+(1-t)a^{-1}]^2}\mathbf{d}t\right)^{1-\frac{1}{q}}\left[\int_0^{1-\lambda}\frac{t\left[t|f'(b)|^q+(1-t)|f'(a)|^q\right]}{[tb^{-1}+(1-t)a^{-1}]^2}dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{1-\lambda}^1\frac{1-t}{[tb^{-1}+(1-t)a^{-1}]^2}\mathbf{d}t\right)^{1-\frac{1}{q}}\left[\int_{1-\lambda}^1\frac{(1-t)\left[t|f'(b)|^q+(1-t)|f'(a)|^q\right]}{[tb^{-1}+(1-t)a^{-1}]^2}dt\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Noticing that $[tb^{-1}+(1-t)a^{-1}]^{-1} \le ta+(1-t)b$, and using (12-15), then, from above inequality we have

$$\begin{aligned}
I &\le \frac{b-a}{ab}\left\{\left(\int_0^{1-\lambda}t[tb+(1-t)a]^2\mathbf{d}t\right)^{1-\frac{1}{q}}\left[\int_0^{1-\lambda}t[tb+(1-t)a]^2\left[t|f'(b)|^q+(1-t)|f'(a)|^q\mathbf{d}t\right]\right]^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{1-\lambda}^1(1-t)[tb+(1-t)a]^2\mathbf{d}t\right)^{1-\frac{1}{q}}\left[\int_{1-\lambda}^1(1-t)[tb+(1-t)a]^2\left[t|f'(b)|^q+(1-t)|f'(a)|^q\right]\mathbf{d}t\right]^{\frac{1}{q}}\right\} \\
&= \frac{b-a}{ab}\left\{(C_1(a,b,\lambda))^{1-\frac{1}{q}}\left[\int_0^{1-\lambda}\left[|f'(b)|^qt^2[tb+(1-t)a]^2+|f'(a)|^qt(1-t)[tb+(1-t)a]^2\right]\mathbf{d}t\right]^{\frac{1}{q}}\right. \\
&\left.+(C_3(a,b,\lambda))^{1-\frac{1}{q}}\left[\int_{1-\lambda}^1\left[|f'(b)|^qt(1-t)[tb+(1-t)a]^2+|f'(a)|^q(1-t)^2[tb+(1-t)a]^2\right]\mathbf{d}t\right]^{\frac{1}{q}}\right\} \\
&= \frac{b-a}{ab}\Big\{(C_1(a,b,\lambda))^{1-\frac{1}{q}}\left[|f'(b)|^qC_1(a,b,\lambda)+|f'(a)|^qC_2(a,b,\lambda)\right]^{\frac{1}{q}} \\
&+(C_3(a,b,\lambda))^{1-\frac{1}{q}}\left[|f'(b)|^qC_3(a,b,\lambda)+|f'(a)|^qC_4(a,b,\lambda)\right]^{\frac{1}{q}}\Big\}.
\end{aligned}$$

So the proof is complete. □

Corollary 3.2 *Assume that all the assumptions of Theorem 3.1 are satisfied. If we take* $q = 1$ *and* $M = \max\{|f'(a)|, |f'(b)|\}$*, then*

$$\left|(1-\lambda)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+\lambda f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathbf{d}x\right|$$
$$\leq \frac{M(b-a)}{ab}\Big[C_1(a,b,\lambda)+C_2(a,b,\lambda)+C_3(a,b,\lambda)+C_4(a,b,\lambda)\Big]. \tag{3.2}$$

Corollary 3.3 *Assume that all the assumptions of Theorem 3.1 are satisfied. If we take* $\lambda = q = 1$ *and* $M = \max\{|f'(a)|, |f'(b)|\}$*, then*

$$\left|f(a)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathbf{d}x\right|\leq\frac{M(b-a)}{ab}\left[\frac{1}{4}(b-a)^2+\frac{1}{6}b(b-4a)\right]. \tag{3.3}$$

Corollary 3.4 *Assume that all the assumptions of Theorem 3.1 are satisfied. If we take* $\lambda = 0$*,* $q = 1$ *and* $M = \max\{|f'(a)|, |f'(b)|\}$*, then*

$$\left|f(b)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathbf{d}x\right|\leq\frac{M(b-a)}{ab}\left[\frac{1}{4}(b-a)^2+\frac{2}{3}a(b-a)\right]. \tag{3.4}$$

Corollary 3.5 *Assume that all the assumptions of Theorem 3.1 are satisfied. If we take* $\lambda = \frac{1}{2}$*,* $q = 1$ *and* $M = \max\{|f'(a)|, |f'(b)|\}$*, then*

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathbf{d}x\right|\leq\frac{M(b-a)}{ab}$$
$$\times\left\{C_1(a,b,\frac{1}{2})\big[C_1(a,b,\frac{1}{2})+C_2(a,b,\frac{1}{2})\big]+C_3(a,b,\frac{1}{2})\big[C_3(a,b,\frac{1}{2})+C_4(a,b,\frac{1}{2})\big]\right\}. \tag{3.5}$$

Remark 3.6 *By (18) and (19) we obtain the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathbf{d}x\right|\leq\frac{M(b-a)}{ab}\left[\frac{1}{12}(4b^2-6ab-a^2)\right]. \tag{3.6}$$

4. Some Applications for Special Means

Let a, b are two nonnegative number with $a < b$. Let us recall the following special means of a and b.

(1) The arithmetic mean $A = A(a,b) := \frac{a+b}{2}$; (2) The geometric mean $G = G(a,b) := \sqrt{ab}$;
(3) The harmonic mean $H = H(a,b) := \frac{2ab}{a+b}$; (4) The logarithmic mean $L = L(a,b) := \frac{b-a}{\ln b - \ln a}$;
(5) The -logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\ a\neq b, p\in\mathbf{R}, p\neq 0,-1.$$

These means are often applied to numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq A. \tag{4.1}$$

It is also known that L_p is monotonically increasing respecting to $p \in \mathbf{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 1. *Let $0 < a < b$. Then we obtain the following inequalities:*

$$\left|H - \frac{G^2}{L}\right| \le \frac{b-a}{ab}$$
$$\times \left\{C_1(a,b,\frac{1}{2})\left[C_1(a,b,\frac{1}{2}) + C_2(a,b,\frac{1}{2})\right] + C_3(a,b,\frac{1}{2})\left[C_3(a,b,\frac{1}{2}) + |C_4(a,b,\frac{1}{2})\right]\right\};$$
$$\left|A - \frac{G^2}{L}\right| \le \frac{b-a}{ab}\left[\frac{1}{12}(4b^2 - 6ab - a^2)\right].$$

Proof. The assertion follows from inequalities (20) and (21), respectively, for $f(x) = x$, $x \in \mathbf{R}_{++}$. □

Proposition 2. *Let $0 < a < b$. Then we derive the following inequalities:*

$$\left|H^2 - G^2\right| \le \frac{2(b-a)}{a}$$
$$\times \left\{C_1(a,b,\frac{1}{2})\left[C_1(a,b,\frac{1}{2}) + C_2(a,b,\frac{1}{2})\right] + C_3(a,b,\frac{1}{2})\left[C_3(a,b,\frac{1}{2}) + |C_4(a,b,\frac{1}{2})\right]\right\};$$
$$\left|A(a^2,b^2) - \frac{G^2}{L}\right| \le \frac{2(b-a)}{a}\left[\frac{1}{12}(4b^2 - 6ab - a^2)\right].$$

Proof. The assertion follows from inequalities (20) and (21), respectively, for $f(x) = x^2$, $x \in \mathbf{R}_{++}$. □

Proposition 3. *Let $0 < a < b$. Then we have the following inequalities:*

$$\left|H^{p+2} - L_p^p G^2\right| \le \frac{2b^{p+2}(b-a)}{ab}$$
$$\times \left\{C_1(a,b,\frac{1}{2})\left[C_1(a,b,\frac{1}{2}) + C_2(a,b,\frac{1}{2})\right] + C_3(a,b,\frac{1}{2})\left[C_3(a,b,\frac{1}{2}) + |C_4(a,b,\frac{1}{2})\right]\right\};$$
$$\left|A(a^{p+2},b^{p+2}) - L_p^p G^2\right| \le \frac{2b^{n+2}(b-a)}{ab}\left[\frac{1}{12}(4b^2 - 6ab - a^2)\right].$$

Proof. The assertion follows from inequalities (20) and (21), respectively, for $f(x) = x^{p+2}$, $x \in \mathbf{R}_{++}$ and $p \in (-1,\infty)\backslash\{0\}$. □

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