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ON COMPARISON THEOREMS FOR CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper the more general comparison theorems for conformable fractional differential equations is proposed and tested. Thus we prove some inequalities for conformable integrals by using the generalization of Sturm's separation and Sturm's comparison theorems. The results presented here would provide generalizations of those given in earlier works. The numerical example is also presented to verify the proposed theorem.

1. Introduction

An important point is that the fractional derivative at a point x is a local property only when a is an integer; in non-integer cases we cannot say that the fractional derivative at x of a function f depends only on values of f very near x, in the way that integer-power derivatives certainly do. Therefore it is expected that the theory involves some sort of boundary conditions, involving information on the function further out. To use a metaphor, the fractional derivative requires some peripheral vision. As far as the existence of such a theory is concerned, the foundations of the subject were laid by Liouville in a paper from 1832. The fractional derivative of a function to order a is often now defined by means of the Fourier or Mellin integral transforms. Various types of fractional derivatives were introduced: Riemann-Liouville, Caputo, Hadamard, Erdelyi-Kober, Grunwald-Letnikov, Marchaud and Riesz are just a few to name [9]-[14]. Recently a new local, limit-based definition of a so-called conformable derivative has been formulated in [1], [6] as follows

$$D_{\alpha}\left(f\right)\left(t\right)=\lim_{\varepsilon\to0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f\left(t\right)}{\varepsilon}$$

provied the limits exits. Note that if f is fully differentiable at t, then the derivative is $D_{\alpha}\left(f\right)\left(t\right)=t^{1-\alpha}f'(t)$. The reader interested on the subject of conformable calculus is referred in [1]-[8].

The aim of this paper is to establish the comparison theorems for conformable fractional differential equations which are based on conditions of mixed type, point-wise and integral inequalities. Thus we provide the generalization for comparison of the solution of linear second order conformable differential equations investigated in [13].

The remaining part of the paper proceeds as follows: the second section of this paper will review the basic tools of conformable fractional calculus. The third section begins by laying out the main results and looks at how can be applied it to the conformable fractional differential equations. The fourth section concludes this study with some remarks.

2. Preliminary

Here we recall basic notions, and provide results helpful for the main section. The basic definition is from [1]-[3].

Definition 1. ***(Conformable fractional derivative)*** *Given a function $f:[0,\infty)\to\mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_{\alpha}\left(f\right)\left(t\right)=\lim_{\epsilon\to0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f\left(t\right)}{\epsilon} \tag{2.1}$$

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for all $t>0,\ \alpha\in(0,1)$. *If* f *is* $\alpha-$*differentiable in some* $(0,a)$, $\alpha>0$, $\lim_{t\to 0^+} f^{(\alpha)}(t)$ *exist, then define*

$$f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t). \tag{2.2}$$

We can write $f^{(\alpha)}(t)$ *for* $D_\alpha(f)(t)$ *to denote the conformable fractional derivatives of* f *of order* α. *In addition, if the conformable fractional derivative of* f *of order* α *exists, then we simply say* f *is* $\alpha-$*differentiable. For* $2\le n\in N$, *we denote* $D_\alpha^n(f)(t)=D_\alpha D_\alpha^{n-1}(f)(t)(t)$

Theorem 1. *Let* $\alpha\in(0,1]$ *and* f,g *be* $\alpha-$*differentiable at a point* $t>0$. *Then*

i. $D_\alpha(af+bg)=aD_\alpha(f)+bD_\alpha(g)$, for all $a,b\in\mathbb{R}$,

ii. $D_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$,

iii. $D_\alpha(fg)=fD_\alpha(g)+gD_\alpha(f)$,

iv. $D_\alpha\left(\frac{f}{g}\right)=\frac{fD_\alpha(g)-gD_\alpha(f)}{g^2}$.

If f is differentiable, then

$$D_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \tag{2.3}$$

Theorem 2 (Mean value theorem for conformable fractional differentiable functions)**.** *Let* $\alpha\in(0,1]$ *and* $f:[a,b]\to\mathbb{R}$ *be a continuous on* $[a,b]$ *and an* α*-fractional differentiable mapping on* (a,b) *with* $0\le a<b$. *Then, there exists* $c\in(a,b)$, *such that*

$$D_\alpha(f)(c)=\frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}.$$

Definition 2 (Conformable fractional integral)**.** *Let* $\alpha\in(0,1]$ *and* $0\le a<b$. *A function* $f:[a,b]\to\mathbb{R}$ *is* α*-fractional integrable on* $[a,b]$ *if the integral*

$$\int_a^b f(x)\,d_\alpha x:=\int_a^b f(x)\,x^{\alpha-1}dx \tag{2.4}$$

exists and is finite.

Remark 1.

$$I_\alpha^a(f)(t)=I_1^a\left(t^{\alpha-1}f\right)=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

Theorem 3. *Let* $f:(a,b)\to\mathbb{R}$ *be differentiable and* $0<\alpha\le 1$. *Then, for all* $t>a$ *we have*

$$I_\alpha^a D_\alpha^a f(t)=f(t)-f(a). \tag{2.5}$$

Theorem 4. ***(Integration by parts)*** *Let* $f,g:[a,b]\to\mathbb{R}$ *be two functions such that* fg *is differentiable. Then*

$$\int_a^b f(x)\,D_\alpha^a(g)(x)\,d_\alpha x=\left.fg\right|_a^b-\int_a^b g(x)\,D_\alpha^a(f)(x)\,d_\alpha x. \tag{2.6}$$

Theorem 5. *Assume that* $f:[a,\infty)\to\mathbb{R}$ *such that* $f^{(n)}(t)$ *is continuous and* $\alpha\in(n,n+1]$. *Then, for all* $t>a$ *we have*

$$D_\alpha^a f(t)\,I_\alpha^a=f(t).$$

Definition 3. *For two functions* y_1 *and* y_2 *satisfying the* α*-conformable fractional equation and* $\alpha\in(0,1]$, *we set*

$$W_\alpha(y_1,y_2)=\begin{vmatrix} y_1 & y_2\\ D_\alpha y_1 & D_\alpha y_2\end{vmatrix}.$$

In this paper, we establish the following comparison theorems for conformable fractional differential equations are based in conditions of a mixed type, point-wise and integral inequalities, and generalizes the results in [12] and [13]. The results presented here would provide generalizations of those given in earlier works.

3. Main results

In [13], Pospisil and Skripkova give the Sturm's comparison theorems for conformable fractional differential equations as follows:

Theorem 6 (Sturm Separation Theorem)**.** *Let $x(t)$ and $y(t)$ be linearly independent solutions of*

$$D_\alpha^2 x(t) + p(t) D_\alpha x(t) + q(t) x(t) = 0 \tag{3.1}$$

where $p(t)$ and $q(t)$ are continuous functions, on an open interval (a,b) and $0 < \alpha \le 1$. Then $x(t)$ has a zero between any two successive zeros of $y(t)$. Thus the zeros of x and y occur alternately.

Theorem 7 (Sturm Comparison Theorem)**.** *Let $x(t)$ and $y(t)$ be non-trivial solutions of*

$$D_\alpha^2 x(t) + r(t) x(t) = 0 \tag{3.2}$$

$$D_\alpha^2 y(t) + r_1(t) y(t) = 0. \tag{3.3}$$

respectively, where $r(t) \ge r_1(t)$ for $t > a$ are given continuous functions. Then exactly one of the following conditions holds:

(1) $x(t)$ has at least one zero between any two zeros of $y(t)$,

(2) $r(t) = r_1(t)$ for all $t > a$, and $x(t)$ is a constant multiple of $y(t)$.

The Sturm's comparison theorem for conformable fractional differential equations deals with functions $x(t)$ and $y(t)$ satisfying equations (3.2) and (3.3). If $r_1(t) \ge r(t)$, then solutions of (3.3) oscillate more rapidly than solutions of (3.2). More precisely, if $x(t)$ is a non-trivial solution of (3.2) for which $x(t_1) = x(t_2)$, $t_1 < t_2$, and $r_1(t) \ge r(t)$ for $t_1 \le t \le t_2$, then $y(t)$ has a zero in $(t_1, t_2]$. Now, we give the following Riccati equations for conformable fractional differential equations;

$$D_\alpha u(t) = u^2(t) + r(t) \tag{3.4}$$

$$D_\alpha v(t) = v^2(t) + r_1(t). \tag{3.5}$$

Assume that $r(t), r_1(t)$ are given continuous functions on (τ_1, τ_2). By the substitutions $u(t) = -\frac{D_\alpha x(t)}{x(t)}$, $v(t) = -\frac{D_\alpha y(t)}{y(t)}$ in (3.4) and (3.5), respectively, we obtain the equations (3.2) and (3.3). In the following theorems we give the comparison theorems for conformable fractional differential equations.

Theorem 8. *Let x and y be non-trivial solutions of (3.2) and (3.3), respectively, such that $x(t)$ does not vanish on $[\tau_1, \tau_2]$, $y(\tau_1) \neq 0$ and the inequality*

$$-\frac{D_\alpha x(\tau_1)}{x(\tau_1)} + \int_{\tau_1}^{t} r(s) d_\alpha s > \left| -\frac{D_\alpha y(\tau_1)}{y(\tau_1)} + \int_{\tau_1}^{t} r_1(s) d_\alpha s \right| \tag{3.6}$$

holds for all t on $[\tau_1, \tau_2]$. Then $y(t)$ does not vanish on $[\tau_1, \tau_2]$ and

$$-\frac{D_\alpha x(t)}{x(t)} > \left| \frac{D_\alpha y(t)}{y(t)} \right|, \ \tau_1 \le t \le \tau_2. \tag{3.7}$$

Proof. Since $x(t)$ does not vanish on $[\tau_1, \tau_2]$, $u(t) = -\frac{D_\alpha x(t)}{x(t)}$ is continuous on $[\tau_1, \tau_2]$ and satisfies the Riccati equation (3.4), which is equivalent to the integral equation. Then by using the (2.5) we have

$$u(t) = u(\tau_1) + \int_{\tau_1}^{t} u^2(s) d_\alpha s + \int_{\tau_1}^{t} r(s) d_\alpha s. \tag{3.8}$$

By the hypothesis (3.6), we get

$$u(t) \ge -\frac{D_\alpha x(\tau_1)}{x(\tau_1)} + \int_{\tau_1}^{t} r(s) d_\alpha s > 0. \tag{3.9}$$

Since $y(\tau_1) \neq 0,\ v(t) = -\frac{D_\alpha y(t)}{y(t)}$ is continuous on some interval $[\tau_1, \delta]\,,\ \tau_1 < \delta \leq \tau_2$. On this interval, the equation (3.4) is well defined and implies the following integral equation

$$v(t) = v(\tau_1) + \int_{\tau_1}^{t} v^2(s) d_\alpha s + \int_{\tau_1}^{t} r_1(s) d_\alpha s. \tag{3.10}$$

Therefore, by using (3.6) and (3.9) in (3.10), we have

$$\begin{aligned} v(t) &\geq v(\tau_1) + \int_{\tau_1}^{t} r_1(s) d_\alpha s \\ &\geq -u(\tau_1) - \int_{\tau_1}^{t} r(s) d_\alpha s \geq -u(t) \end{aligned}$$

and consequently, $u(t) \geq -v(t)$. In order to obtain

$$|v(t)| \leq u(t) \quad \text{on } \tau_1 \leq x \leq \delta, \tag{3.11}$$

it is sufficient to show that $u(t) \geq v(t)$ on this interval. Suppose to the contrary that there exists a point t_0 on $[\tau_1, \delta]$ such that $u(t_0) < v(t_0)$. Thus, since $|v(\tau_1)| \leq u(\tau_1)$ from (3.6) (with $t = \tau_1$) and u and v are continuous on $[\tau_1, \delta]\,$, there exists t_1 in $\tau_1 < t_1 \leq t_0$ such that $v(t_1) = u(t_1)$ and $v(t) \leq u(t)$ for $\tau_1 < t \leq t_1$. Because $u(t) \geq -v(t)$ was establishes, it follows that $|v(t)| \leq u(t)$ for $\tau_1 < t \leq t_1$, and consequently

$$\int_{\tau_1}^{t} v^2(s) d_\alpha s \leq \int_{\tau_1}^{t} u^2(s) d_\alpha s.$$

By using (3.6),(3.8), (3.10), it follows that

$$\begin{aligned} v(t_1) &= v(\tau_1) + \int_{\tau_1}^{t_1} v^2(s) d_\alpha s + \int_{\tau_1}^{t_1} r_1(s) d_\alpha s \\ &< u(\tau_1) + \int_{\tau_1}^{t_1} u^2(s) d_\alpha s + \int_{\tau_1}^{t_1} r(s) d_\alpha s = u(t_1), \end{aligned}$$

this is a contradiction with the fact that $v(t_1) = u(t_1)$. Hence (3.11) holds on any interval $[\tau_1, \delta]$ of continuity of v, $\tau_1 < \delta \leq \tau_2$, but this implies that v is continuous on the entire interval $[\tau_1, \tau_2]\,$, because $u(t)$ is bounded and $v(t)$ has only poles at its points of discontinuity. Thus (3.11) holds on all of the interval $[\tau_1, \tau_2]$. This result proves (3.7), and since the left member is bounded on $[\tau_1, \tau_2]\,$, $y(t)$ cannot have a zero on this interval. The proof is complete. □

Theorem 9. *Let x and y be non-trivial solutions of (3.2) and (3.3), respectively, such that $x(t)$ does not vanish on $[\tau_1, \tau_2]\,,\ y(\tau_1) \neq 0$ and the inequality*

$$\frac{D_\alpha x(\tau_2)}{x(\tau_2)} + \int_{t}^{\tau_2} r(s) d_\alpha s > \left| \frac{D_\alpha y(\tau_2)}{y(\tau_2)} + \int_{t}^{\tau_2} r_1(s) d_\alpha s \right| \tag{3.12}$$

holds for all t on $[\tau_1, \tau_2]\,$. Then $y(t)$ does not vanish on $[\tau_1, \tau_2]$ and

$$\frac{D_\alpha x(t)}{x(t)} > \left| \frac{D_\alpha y(t)}{y(t)} \right|,\ \tau_1 \leq t \leq \tau_2. \tag{3.13}$$

Proof. Let new functions $x_1, y_1, \lambda, \lambda_1$ be defined on $\tau_1 \leq t \leq \tau_2$ by the following equations

$$x_1(t) = x(\tau_1 + \tau_2 - t) \quad y_1(t) = y(\tau_1 + \tau_2 - t)$$

$$\lambda(t) = r(\tau_1 + \tau_2 - t) \qquad \lambda_1(t) = r_1(\tau_1 + \tau_2 - t).$$

Then $x_1(t)$ does not vanish on $[\tau_1, \tau_2]$, $y_1(\tau_1) = y(\tau_2) \neq 0$ and

$$-\frac{D_\alpha x(\tau_1)}{x(\tau_1)} + \int_{\tau_1}^{\tau_1+\tau_2-t} \lambda(s) d_\alpha s = \frac{D_\alpha x(\tau_2)}{x(\tau_2)} + \int_t^{\tau_2} r(s) d_\alpha s$$

$$-\frac{D_\alpha y(\tau_1)}{y(\tau_1)} + \int_{\tau_1}^{\tau_1+\tau_2-t} \lambda_1(s) d_\alpha s = \frac{D_\alpha y(\tau_2)}{y(\tau_2)} + \int_t^{\tau_2} r_1(s) d_\alpha s.$$

Thus the hypothesis (3.12) is equivalent to the hypothesis (3.6). Since $t \in [\tau_1, \tau_2]$ if and only if $\tau_1 + \tau_2 - t \in [\tau_1, \tau_2]$, and the conclusion (3.13) follows from Theorem 8. □

Theorem 10. *Let $x(t)$ and $y(t)$ be non-trivial solutions of the equations*

$$D_\alpha^2 x(t) - 2b(t) D_\alpha x(t) + r(t)x(t) = 0, \qquad t > 0, \tag{3.14}$$

$$D_\alpha^2 y(t) - 2c(t) D_\alpha y(t) + r_1(t)y(t) = 0, \qquad t > 0, \tag{3.15}$$

respectively, where r and r_1 are continuous functions such that $r(t) \leq r_1(t)$ with the initial conditions:

$$D_\alpha x(t_1) + \sigma x(t_1) = 0 \tag{3.16}$$

$$D_\alpha y(t_1) + \tau y(t_1) = 0 \tag{3.17}$$

where σ and τ are constants. If $b(t)D_\alpha y(t) > c(t)D_\alpha x(t)$, Then between any two consecutive zeroes τ_1 and τ_2 of $x(t)$, there exists at least one zero of $y(t)$ unless $r(t) \equiv r_1(t)$ on $[\tau_1, \tau_2]$.

Proof. Let τ_1 and τ_2 with $0 < \tau_1 < \tau_2$ be consecutive zeroes of $x(t)$. Assume $x(t) > 0$ on $[\tau_1, \tau_2]$ (if not, consider $-x(t)$ or $-y(t)$ which have these properties). Consequently, by arguments in the proof of Theorem 6,

$$D_\alpha x(\tau_1) > 0, \qquad \text{and} \qquad {}_a D_\alpha x(\tau_2) < 0.$$

Suppose that $y(t)$ does not have a zero on $[\tau_1, \tau_2]$. Let $y(t) > 0$ on $[\tau_1, \tau_2]$. Multiplying the equation satisfied by $x(t)$, with $y(t)$, and vice versa, and then subtract the two equations we get

$$\left[y(t)D_\alpha^2 x(t) - x(t)D_\alpha^2 y(t)\right] + 2\left[cx(t)D_\alpha y(t) - by(t)D_\alpha x(t)\right] + x(t)y(t)(r - r_1) = 0.$$

Rewriting the last equation as

$$D_\alpha \left[y(t)D_\alpha x(t) - x(t)D_\alpha y(t)\right] = -2\left[cx(t)D_\alpha y(t) - by(t)D_\alpha x(t)\right] - x(t)y(t)(r - r_1)$$

Integrating on both sides of the last equation from τ_1 to τ_2, we obtain

$$y(t)D_\alpha x(t) - x(t)D_\alpha y(t)|_{\tau_1}^{\tau_2} = -2I_\alpha \left[cx(t)D_\alpha y(t) - by(t)D_\alpha x(t)\right] - I_\alpha(x(t)y(t)(r - r_1))$$

The left hand side of equation (7) is non-positive. The right hand side is strictly positive unless $r(t) \equiv r_1(t)$ on $[\tau_1, \tau_2]$. Thus, if $r(t) \neq r(t)$ on $[\tau_1, \tau_2]$, we arrive at a contradiction. This finishes the proof of theorem. □

Now in order to test the result of Theorem 6, we present a numerical example of equations 3.14 and 3.15.

Example 1. *Let us consider the coupled of equations*

$$D_\alpha^2 x(t) + \frac{3}{2}\cot(\sqrt{t} + \pi) D_\alpha x(t) + x(t) = 0, \qquad t > 0, \tag{3.18}$$

$$D_\alpha^2 y(t) + \frac{3}{2}\varepsilon_1 \cot(\sqrt{t} + \pi) D_\alpha y(t) + \varepsilon_1 y(t) = 0, \qquad t > 0, \tag{3.19}$$

where $\varepsilon_1 \in \mathbb{R}$. *And* $x(t)$ *and* $y(t)$ *be non-trivial solutions of the equations such that*

$$x(t) = \cos(\sqrt{t} + \pi) \tag{3.20}$$

$$y(t) = \cos(\sqrt{\varepsilon_1 t}) \tag{3.21}$$

According to Figure 1 we can say that the solution $y(t)$ *oscillates faster than the solution* $x(t)$ *whenever*

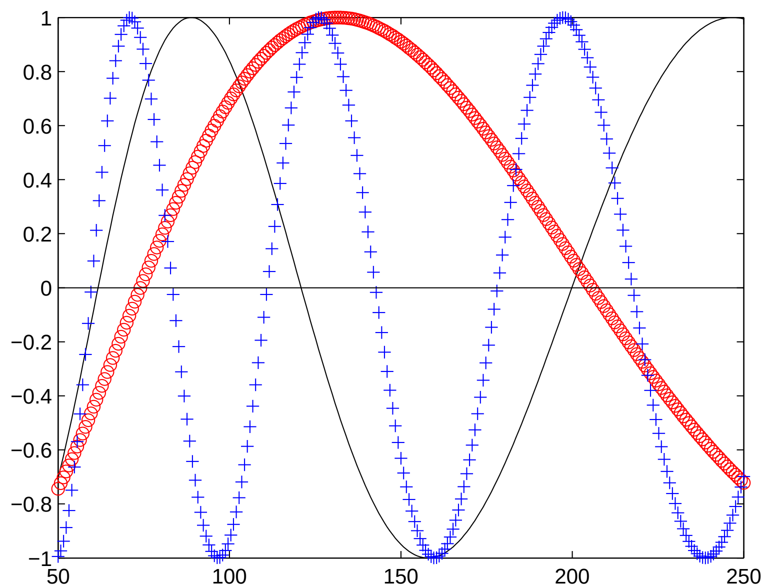

FIGURE 1. Function $x(t)$ (black), function $y(t)$ with $\varepsilon_1 = 0.3$ (red) and $\varepsilon_1 = 5$ (blue).

$\varepsilon_1 > 1$. *Otherwise the solution* $y(t)$ *oscillates slower than the solution* $x(t)$ *with* $\varepsilon_1 < 1$ *and one of its zeros will not lie between two consecutive zeros of* $x(t)$.

By means of the transformation

$$u(t) = -\frac{D_\alpha x(t)}{x(t)}, \; v(t) = -\frac{D_\alpha y(t)}{y(t)}$$

equations (3.14) and (3.15) are transformed into Riccati equations

$$D_\alpha u(t) = u^2(t) + 2b(t)u(t) + r(t) \tag{3.22}$$

$$D_\alpha v(t) = v^2(t) + 2e(t)v(t) + r_1(t). \tag{3.23}$$

and the initial conditions

$$-\frac{D_\alpha x(\tau_1)}{x(\tau_1)} = \sigma \; , \quad -\frac{D_\alpha y(\tau_1)}{y(\tau_1)} = \tau$$

for (3.14) and (3.15), become initial values

$$u(\tau_1) = \sigma, \quad v(\tau_1) = \tau \tag{3.24}$$

for (3.22) and (3.23). The differential equations (3.14) and (3.15) subject to (3.24) can be written as equivalent integral equations

$$u(t) = \sigma + \int_{\tau_1}^{t} u^2(s)d_\alpha s + \int_{\tau_1}^{t} 2b(s)u(s)d_\alpha s + \int_{\tau_1}^{t} r(s)d_\alpha s \tag{3.25}$$

$$v(t)=\tau+\int_{\tau_1}^{t} v^2(s)d_\alpha s+\int_{\tau_1}^{t} 2e(s)v(s)d_\alpha s+\int_{\tau_1}^{t} r_1(s)d_\alpha s. \tag{3.26}$$

It is obvious from these equations that if $\tau\geq\sigma\geq 0,\ e(t)\geq b(t)\geq 0$ and

$$\int_{\tau_1}^{t} r_1(s)d_\alpha s\geq\int_{\tau_1}^{t} r(s)d_\alpha s$$

on an interval $[\tau_1,\tau_2]$, then $v(t)\geq u(t)\geq 0$ as long as $v(t)$ can be continued on $[\tau_1,\tau_2]$. Since the singularities of $u(t)$ and $v(t)$ correspond to the zeros of $x(t)$ and $y(t)$, respectively, these observations lead to the following comparison theorem for (3.14) and (3.15).

Theorem 11. *Suppose x is a non-trivial solutions of (3.14) satisfying $-\frac{D_\alpha x(\tau_1)}{x(\tau_1)}=\sigma\geq 0$, $x(\tau_2)=0$. If*

i) $e(t)\geq b(t)\geq 0$ for $\tau_1\leq t\leq\tau_2$

ii) $\int_{\tau_1}^{t} r_1(s)d_\alpha s\geq\int_{\tau_1}^{t} r(s)d_\alpha s\geq 0,$ for $\tau_1\leq t\leq\tau_2$,

then every solution of (3.15) satisfying $-\frac{D_\alpha y(\tau_1)}{y(\tau_1)}=\sigma$ has a zero in $(\tau_1,\tau_2]$.

We note that the integral equations (3.25) and (3.26) can be written as follows

$$u(t)=\sigma+\int_{\tau_1}^{t}(u(s)+b(s))^2\,d_\alpha s+\int_{\tau_1}^{t}\left(r(s)-b^2(s)\right)d_\alpha s$$

$$v(t)=\tau+\int_{\tau_1}^{t}(v(s)+e(s))^2\,d_\alpha s+\int_{\tau_1}^{t}\left(r_1(s)-e^2(s)\right)d_\alpha s.$$

This formulation shows that the condition ii) of Theorem 11 can be replaced by

$$\int_{\tau_1}^{t}\left(r_1(s)-e^2(s)\right)d_\alpha s\geq\int_{\tau_1}^{t}\left(r(s)-b^2(s)\right)d_\alpha s\geq 0.$$

4. Concluding remarks

In this investigation, the aim was to present some inequalities for conformable fractional integrals through the instrument of the Sturm's comparison and separation theorems. Since the obtained results are general forms of earlier works they would help for the future studies.

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