International Journal of Analysis and Applications
ISSN 2291-8639
Volume 13, Number 1 (2017), 32-40
http://www.etamaths.com

SOME RESULTS ON FIXED POINT THEOREMS IN BANACH ALGEBRAS

DIPANKAR DAS[1], NILAKSHI GOSWAMI[1] AND VISHNU NARAYAN MISHRA[2,3,*]

Abstract. Let X be a Banach algebra and D be a nonempty subset of X. Let (T_1, T_2) be a pair of self mappings on D satisfying some specific conditions. Here we discuss different situations for existence of solution of the operator equation $u = T_1uT_2u$ in D. Similar results are established in case of reflexive Banach algebra X with the subset D. Again, considering a bounded, open and convex subset B in a uniformly convex Banach algebra X with three self mappings T_1, T_2, T_3 on $\overline{B}$, we derive the conditions for existence of solution of the operator equation $u = T_1uT_2u + T_3u$ in B. Application of some of these results to the tensor product is also shown here with some examples.

1. Introduction

In 1988, Dhage initiated application of fixed point theorems in Banach algebras. Many papers ([4], [5], [6]) of Dhage deals with the study of non-linear integral equations via fixed point theorems in Banach algebras. In 2010, Amar et al. [1], introduced a class of Banach algebras satisfying certain sequential conditions and gave applications of non-linear integral equations using fixed point theorems under certain conditions. In 2012, Pathak and Deepmala [24], defined $\mathcal{P}$-Lipschitzian maps and derived some fixed points theorem of Dhage on a Banach algebra with examples. In [12], Kilbas et al. gave many applications in the field of integral equations. In [20] different application of convergent sequence can be seen. Many researchers viz., Mishra et al. ([13], [14], [15], [16], [17], [18]), Deepmala ([8], [9]), Mishra [19] etc., proved some results concerning the existence of solutions for some nonlinear functional-integral equations in Banach algebra and some interesting results.

In 1982, Hadzic [11] proved a generalization of Rzepecki fixed point theorem for the sum of operators in Hausdorff topological vector space. In ([25], [26], [27]), Vijayaraju proved the existance of fixed points for asymptotic 1-set contraction mappings in real Banach spaces and also for the sum of two mappings in reflexive Banach spaces.

In this paper, for a Banach algebra X with a subset D, we take a pair of self-mappings (T_1, T_2) on D and study the conditions under which the operator equation $u = T_1uT_2u$ has a solution in D. An application of the results to the tensor product of Banach algebras is also discussed here with some suitablle examples. Also, give an apllication for nonlinear functional-integral equation.

Preliminaries

Def. 1 [3] Let X be a Banach space and f be a continuous (not necessarily linear) mapping of X into itself. The mapping f is said to be completely continuous if the image under f of each bounded set of X is contained in a compact set.

Def. 2 Let X be a Banach algebra and T_1, T_2 be two self mappings on X. Then T_1,T_2 are said to satisfy the nonvacuous condition if for every sequence $\{x_n\} \subset X$ the operator equation $\lim_{n\to\infty} T_1(u)T_2(x_n) = u$, $u \in X$ has one and only one solution $(x_n)^0$ in X.

Def. 3 [21] T is demiclosed if $\{x_n\} \subset D(T)$, $x_n \to x$ and $T(x_n) \to y$ (weakly) implies $x \in D(T)$ and $Tx = y$.

Def. 4 [21] T is closed if if $\{x_n\} \subset D(T)$, $x_n \to x$ and $T(x_n) \to y$ implies $x \in D(T)$ and $Tx = y$.

Received 13th July, 2016; accepted 19th September, 2016; published 3rd January, 2017.
2010 *Mathematics Subject Classification.* 47B48; 46B28; 47A80; 47H10.
Key words and phrases. Banach algebras; fixed points; projective tensor product.

Def. 5 [22] T is said to be demicompact at a if for any bounded sequence $\{x_n\}$ in D such that $x_n - Tx_n \to a$ as $n \to \infty$, there exists a subsequence x_{n_i} and a point b in D such that $x_{n_i} \to b$ as $i \to \infty$ and $b - T(b) = a$

Def. 6 ([10], [23]) Let $T : D \to D$ be a mapping.
(1) T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that, for any $x, y \in D$

$$\|T^n x - T^n y\| \leqslant L\|x - y\| \; \forall \, n \in \mathbb{N}$$

(2) T is said to be asymptotically nonexpansive if there exists a sequence $b_n \subset [1, \infty)$ with $b_n \to 1$ such that, for any $x, y \in D$

$$\|T^n x - T^n y\| \leqslant b_n\|x - y\| \; \forall \, n \in \mathbb{N}$$

Algebric tensor product: [2] Let X, Y be normed spaces over F with dual spaces X^* and Y^* respectively. Given $x \in X, y \in Y$, Let $x \otimes y$ be the element of $BL(X^*, Y^*; F)$ (which is the set of all bounded bilinear forms from $X^* \times Y^*$ to F), defined by

$$x \otimes y(f, g) = f(x)g(y), \; (f \in X^*, g \in Y^*)$$

The algebraic tensor product of X and Y, $X \otimes Y$ is defined to be the linear span of $\{x \otimes y : x \in X, y \in Y\}$ in $BL(X^*, Y^*; F)$.

Projective tensor norm: [2] Given normed spaces X and Y, the projective tensor norm γ on $X \otimes Y$ is defined by

$$\|u\|_\gamma = \inf\{\sum_i \|x_i\|\|y_i\| : u = \sum_i x_i \otimes y_i\}$$

where the infimum is taken over all (finite) representations of u.
The completion of $(X \otimes Y, \gamma)$ is called projective tensor product of X and Y and it is denoted by $X \otimes_\gamma Y$.

Lemma 1: [28] Let X and Y be Banach spaces. Then γ is a cross norm on $X \otimes Y$ and $\|x \otimes y\|_\gamma = \|x\|\|y\|$ for every $x \in X, y \in Y$.

Lemma 2: [2] $X \otimes_\gamma Y$ can be represented as a linear subspace of $BL(X^*, Y^*; F)$ consisting of all elements of the form $u = \sum_i x_i \otimes y_i$ where $\sum_i \|x_i\|\|y_i\| < \infty$. Moreover, $\|u\|_\gamma = \inf\{\sum_i \|x_i\|\|y_i\|\}$ over all such representations of u.

Lemma 3: [2] Let X and Y be normed algebras over $\mathbb{F}$. There exists a unique product on $X \otimes Y$ with respect to which $X \otimes Y$ is an algebra and

$$(a \otimes b)(c \otimes d) = ac \otimes bd \qquad (a, c \in X, b, d \in Y)$$

Lemma 4: [2] Let X and Y be normed algebras over $\mathbb{F}$. Then projective tensor norm on $X \otimes Y$ is an algebra norm.
Clearly, we can conclude that if X and Y are Banach algebras over $\mathbb{F}$ then $X \otimes_\gamma Y$ becomes a Banach algebra.

2. Main Results

Theorem 1: Let D be a non-empty compact convex subset of a Banach Algebra X and let (T_1, T_2) be a pair of self-mappings on D such that

(a) T_1 and T_2 are continuous,
(b) $T_1 u T_2 u \in D$ for all $u \in D$

Then the operator equation $u = T_1 u T_2 u$ has a solution in D.

Proof. We define $J : D \to D$ by $J(u) = T_1 u T_2 u$. Let $\{q_n\}$ be a sequence in D converging to a point q. So, $q \in D$ as D is closed. Now,

$$\begin{aligned}\|J(u) - J(v)\| &= \|T_1 u T_2 u - T_1 v T_2 v\| \\ &\leqslant \|T_1 u - T_1 v\|\|T_2 u\| + \|T_1 v\|\|T_2 u - T_2 v\|\end{aligned}$$

Since T_1 and T_2 are continuous so, J is continuous. By an application of Schauder's fixed point theorem we have fixed point for J. Hence the operator equation $u = T_1 u T_2 u$ has a solution. □

Corollary 1: Let D_X, D_Y and $D_X \otimes D_Y$ be closed, convex and bounded subsets of Banach algebras X, Y and $X \otimes_\gamma Y$ respectively. Let (T_1, T_2) be a pair of self mappings on $D_X \otimes D_Y$ such that

(a) T_1 and T_2 are completely continuous
(b) $T_1 u T_2 u \in D_X \otimes D_Y$ for all $u \in D_X \otimes D_Y$

then the operator equation $u = T_1 u T_2 u$ has a solution in $D_X \otimes D_Y$.

Example 1: Let D_{l^1}, $D_{\mathbb{K}}$ and $D_{l^1} \otimes D_{\mathbb{K}}$ be subsets of Banach algebras l^1, $\mathbb{K}$ and $l^1 \otimes_\gamma \mathbb{K}$ respectively. Define

$$D_{l^1} = \{x \in D_{l^1} : \|x\| \leqslant M_1\} \; and \; D_{\mathbb{K}} = \{y \in D_{\mathbb{K}} : \|y\| \leqslant M_2\}$$

then clearly D_{l^1}, $D_{\mathbb{K}}$ and $D_{l^1} \otimes D_{\mathbb{K}}$ are closed, convex and bounded.

We define $T_1, T_2 : D_{l^1} \otimes_\gamma D_{\mathbb{K}} \to D_{l^1} \otimes_\gamma D_{\mathbb{K}}$ by $T_1(\sum_i a_i \otimes x_i) = \sum_i \{\frac{a_{i_n} x_i}{n}\}_n = T_2(\sum_i a_i \otimes x_i)$, where $a_i = \{a_{i_n}\}_n$. [$l^1 \otimes_\gamma X = l^1(X)$ by [28]].

To show that T_1 is compact:

Let $T_{1m} : D_{l^1} \otimes_\gamma D_{\mathbb{K}} \to D_{l^1} \otimes_\gamma D_{\mathbb{K}}$ be defined by

$$T_{1m}(\sum_i a_i \otimes x_i) = \sum_i \{a_{i_1} x_i, \frac{a_{i_2} x_i}{2}, \frac{a_{i_3} x_i}{3}, ..., \frac{a_{i_m} x_i}{m}, 0, 0, 0, ...\}$$

Then each T_{1m} is linear, bounded and compact [7]. Also,

$$\begin{aligned}
\|(T_{1m} - T_1)(\sum_i a_i \otimes x_i)\| &= \| \sum_i \{a_{i_1} x_i, \frac{a_{i_2} x_i}{2}, \frac{a_{i_3} x_i}{3}, ..., \frac{a_{i_m} x_i}{m}, 0, 0, 0, ...\} \\
&\quad - \sum_i \{a_{i_1} x_i, \frac{a_{i_2} x_i}{2}, \frac{a_{i_3} x_i}{3}, ..., \frac{a_{i_m} x_i}{m}, \frac{a_{i_{m+1}} x_i}{m+1}, ...\}\| \\
&= \| \sum_i \{0, 0, ..., 0, \frac{a_{i_{m+1}} x_i}{m+1}, \frac{a_{i_{m+2}} x_i}{m+2}, ...\}\| \\
&\leqslant \sum_i \sum_{j=m+1}^{\infty} \frac{1}{j} |a_{ij}|.|x_i| < \frac{1}{m+1} \sum_i \sum_{j=m+1}^{\infty} |a_{ij}|.|x_i| \\
&\leqslant \frac{1}{m+1} \sum_i \sum_{j=1}^{\infty} |a_{ij}|.|x_i| = \frac{1}{m+1} \sum_i \|a_i\|.|x_i|
\end{aligned}$$

So, taking the projective tensor norm,

$$\|(T_{1m} - T_1)(\sum_i a_i \otimes x_i)\| < \frac{1}{m+1} \| \sum_i a_i \otimes x_i\|$$

Therefore, $T_{1m} \to T_1$ and so, T_1 is compact. Similarly, T_2 is compact. Since every compact operator in Banach space is completely continuous, so T_1 and T_2 are completely continuous. Then, by **Corollary 1**, the operator equation has a solution.

Theorem 2: Let X be a non-empty Banach Algebra and let T_1, T_2 be three self mappings on X such that

(a) S is a homomorphism and it has a unique fixed point
(b) $T_1 S = S T_1$ and $T_2 S = S T_2$

then the unique fixed point of S is a solution of the operator equation $u = T_1 u T_2 u$ in X.

Proof. Defne $J : X \to X$ by $J(u) = T_1 u T_2 u$. Let a be the unique fixed point of S. Now,

$$J(S(u)) = T_1(S(u)) T_2(S(u)) = S(T_1(u)) S(T_2(u)) = S(T_1 u T_2 u) = S(Ju)$$

Hence, $S(Ja) = J(S(a)) = Ja$ so $Ja = a$ as S has unique fixed point. Hence the operator equation $u = T_1 u T_2 u$ has a solution. □

Example 2: Given a closed and bounded interval $I = \left[\frac{1}{10}, \frac{10}{10}\right]$ in $\mathbb{R}^+$ the set of real numbers, consider the nonlinear functional integral equation (in short FIE)

$$x(s) = [x(\alpha(s))]^2[q(s) + \int_0^s g(t, x(\beta(t)))dt]^2 \tag{2.1}$$

for all $s \in I$, where $\alpha, \beta : I \to I, q : I \to \mathbb{R}^+$ and $g : I \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous.
By a solution of the FIE (1) we mean a continuous function $x : I \to \mathbb{R}^+$ that satisfies FIE (1) on I. Let $X = C(I, \mathbb{R}^+)$ be a Banach algebra of all continuous real-valued functions on I with the norm $\|x\| = \sup_{s\in I} |x(s)|$. We shall obtain the solution of FIE (1) under some suitable conditions on the functions involved in (1).
Suppose that the function g satisfy the condition $|g(s, x)| \leqslant 1 - q, \|q\| < 1$ for all $s \in I$ and $x \in \mathbb{R}^+$.
Consider the two mappings $T_1, T_2 : X \to X$ defined by

$$T_1x(s) = [x(\alpha(s))]^2,\ s \in I \quad and \quad T_2x(s) = [q(s) + \int_0^s g(t, x(\beta(t)))dt]^2, s \in I$$

Then the FIE (1) is equivalent to the operator equation $x(s) = T_1x(s)T_2x(s),\ s \in I$. Let $S : X \to X$ defined by $S(y) = \sqrt{y},\ y \in X$, where $\sqrt{y}(t) = \sqrt{y(t)}, (positive\ squareroot)\ t \in I$. Clearly, S is a homomorphism and it has a unique fixed point 1, where $1(s) = 1,\ s \in I$. It is obvious that $T_1S = ST_1$ and $T_2S = ST_2$. So, 1 is a solution of FIE(1).

Theorem 3: Let D be a non-empty compact convex subset of a Banach Algebra X and let $T_1, T_2 : D \to D$ be two continuous self maps such that T_1 and T_2 satisfies nonvacuous condition, then there exists a solution of the operator equation $u = T_1uT_2u$ in D.

Proof. We define $J : D \to D$ by $J(x_n) = (x_n)^0$. First we show that J is continuous. Let $\{y_n\}_n$ be a sequence in D such that $y_n \to y$ as $n \to \infty$. Since, T_1 and T_2 satisfies nonvacuous condition so we have

$$J(y_n) = (y_n)^0 = \lim_{n\to\infty} T_1(y_n)^0T_2(y_n)$$
$$\Rightarrow \lim_{n\to\infty} J(y_n) = \lim_{n\to\infty} T_1(\lim_{n\to\infty} J(y_n))T_2(y_n)$$

So, $\lim_{n\to\infty} J(y_n)$ is a solution of the equation $\lim_{n\to\infty} T_1(u)T_2(x_n) = u,\ u \in X$. Now,

$$\lim_{n\to\infty} J(y_n) = (\lim_{n\to\infty} y_n)^0 = (y)^0 = J(y)$$

Therefore, J is continuous. For $u \in D,\ Ju = u^0 = T_1(u^0)T_2(u)$. Clearly, we get J has a fixed point by Schauder's theorem, say α in D. Therefore, $\alpha = J(\alpha) = T_1\alpha T_2\alpha$. Thus, α is a solution of the equation $u = T_1uT_2u \in D$ □

Theorem 4: Let D be a nonempty closed bounded and convex subset of a weakly compact Banach algebra X. Let $T_1 : D \to D$ and $T_2 : D \to X$ be two mappings such that

(a) T_1 satisfies asymptotically nonexpansive mapping and $\lim_{n\to\infty}[\sup \|T_1x - T_1^nx\| :\ x \in D] = 0$
(b) T_2 is completely continuous and $M = \|T_2(D)\| < 1$
(c) $I - T_1 \diamond T_2$ is demiclosed and $T_1^nuT_2v \in D$ for $u, v \in D\ and\ n \in \mathbb{N}$

then there exists a solution of the operator equation $u = T_1uT_2u(= (T_1 \diamond T_2)u)$ in D.

Proof. First we show that $I - T_1 \diamond T_2$ is closed. Let $c \in \overline{I - T_1 \diamond T_2}$. Then there exists a sequence $\{c_n\} \subseteq I - T_1 \diamond T_2$ such that $c_n \to c$ as $n \to \infty$. Since $c_n \in I - T_1 \diamond T_2$ so $c_n = (I - T_1 \diamond T_2)z_n$ for some $z_n \in X$. Since X is weakly compact so for every sequence $\{z_n\}$ in D there exists weakly convergence subsequence $\{z_{n_i}\}$ i.e., $z_{n_i} \to z$ as $n \to \infty$. Now,

$$z_{n_i} - T_1 \diamond T_2z_{n_i} \to c\ as\ n \to \infty$$

Since $I - T_1 \diamond T_2$ is demiclosed so $c = (I - T_1 \diamond T_2)z$. Therefore $c \in I - T_1 \diamond T_2$. Hence $I - T_1 \diamond T_2$ is closed.

For $u, v \in D$, we define $J_n : D \to D$ by $J_n(u) = q_n T_1^n u T_2 v$. where $q_n = \dfrac{(1-\frac{1}{n})}{b_n}$ and $\{b_n\} \to 1$ *as* $n \to \infty$. Now,

$$\|J_n(u) - J_n(p)\| = \|q_n T_1^n u T_2 v - q_n T_1^n p T_2 v\| = q_n \|T_2 v\| \|T_1^n u - T_1^n v\|$$
$$\leqslant q_n b_n M \|u - p\| = (1 - \frac{1}{n}) M \|u - p\| \leqslant M \|u - p\|$$

Since J_n is contraction and so it has unique fixed point $K_n(v) \in D$ (say), where $K_n(v) = J_n(K_n(v)) = q_n T_1^n(K_n(v)) T_2 v$. Now, for any $v, y \in D$ we have

$$\|K_n(v) - K_n(y)\| = \|q_n T_1^n (K_n v) T_2 v - q_n T_1^n (K_n y) T_2 y\|$$
$$\leqslant q_n \|T_1^n(K_n v) - T_1^n(K_n y)\| \|T_2 v\| + q_n \|T_1^n(K_n y)\| \|T_2 v - T_2 y\| \tag{2.2}$$

For fixed $a \in D$, we have

$$\|T_1^n(u)\| = \|T_1^n(u) - T_1^n(a) + T_1^n(a)\|$$
$$\leqslant b_n \|u - a\| + \|T_1^n(a)\| = d(say) < \infty$$

From equation (2), we have

$$\|K_n(v) - K_n(y)\| \leqslant \frac{d q_n}{1 - M} \|T_2 v - T_2 y\|$$

So, K_n is completely continuous as T_2 is completely continuous. By Schauder's fixed point theorem K_n has a fixed point x_n, say in D. Hence $x_n = K_n x_n = J_n(x_n) = q_n T_1^n(x_n) T_2 x_n$. Now,

$$x_n - T_1^n x_n T_2 x_n = (q_n - 1) T_1^n x_n T_2 x_n \to 0 \text{ } as \text{ } n \to \infty \tag{2.3}$$

$$\|x_n - T_1 x_n T_2 x_n\| \leqslant \|x_n - T_1^n x_n T_2 x_n\| + \|T_1^n x_n T_2 x_n - T_1 x_n T_2 x_n\|$$
$$= \|x_n - T_1^n x_n T_2 x_n\| + \|T_2 x_n\| \|T_1^n x_n - T_1 x_n\|$$
$$\to 0 \text{ } as \text{ } n \to \infty \text{ (by (3) and condition (a))}$$

So, $0 \in I - T_1 \diamond T_2$ as $I - T_1 \diamond T_2$ is closed. Hence there exists a point r such that $0 = (I - T_1 \diamond T_2) r$. Hence the theorem follows. □

Theorem 5: Let D be a nonempty closed bounded and convex subset of a reflexive Banach algebra X. Let $T_1 : D \to D$ and $T_2 : D \to X$ be two mappings such that

(a) T_1 satisfies uniformly $L-$Lipschitzian mapping and $\lim_{n\to\infty}[\sup \|T_1 x - T_1^n x\| : \ x \in D] = 0$
(b) T_2 is completely continuous and $M = \|T_2(D)\|$ such that $LM < 1$
(c) $I - T_1 \diamond T_2$ is demiclosed and $T_1^n u T_2 v \in D$ for $u, v \in D$ *and* $n \in \mathbb{N}$

then there exists a solution of the operator equation $u = T_1 u T_2 u (= (T_1 \diamond T_2) u)$ in D.

Theorem 6: Let D_X, D_Y and $D_X \otimes D_Y$ be closed bounded and convex subsets of a Banach algebras X, Y and $X \otimes_\gamma Y$ respectively. Let (T_1, T_2) be a pair of self mappings on $D_X \otimes D_Y$ such that

(a) T_1 satisfies uniformly $L-$Lipschitzian mapping and $\lim_{n\to\infty}[\sup \|T_1 x - T_1^n x\| : \ x \in D_X \otimes D_Y] = 0$
(b) T_2 is completely continuous and $M = \|T_2(D_X \otimes D_Y)\|$ such that $LM < 1$
(c) if $\{x_n\} \subset D_X \otimes D_Y$ with $x_n - T_1 x_n T_2 x_n \to \ 0$ *as* $n \to \infty$ then there exists $b \in D_X \otimes D_Y$ such that $0 = (I - T_1 \diamond T_2) b$ and $T_1^n u T_2 v \in D_X \otimes D_Y$ for $u, v \in D_X \otimes D_Y$ *and* $n \in \mathbb{N}$

then there exists a solution of the operator equation $u = T_1 u T_2 u$ in $D_X \otimes D_Y$.

Example 3: Let D_{l^1}, $D_{\mathbb{R}}$ and $D_{l^1} \otimes D_{\mathbb{R}}$ be subsets of Banach algebras l^1, $\mathbb{R}$ and $l^1 \otimes_\gamma \mathbb{R}$ respectively. Define

$$D_{l^1} = \{x \in D_{l^1} : \|x\| \leqslant 1\} \text{ } and \text{ } D_{\mathbb{R}} = \{y \in D_{\mathbb{R}} : \|y\| \leqslant 1\}$$

then clearly D_{l^1}, $D_{\mathbb{R}}$ and $D_{l^1} \otimes D_{\mathbb{R}}$ are bounded closed and convex.

We define $T_1 : D_{l^1} \otimes_\gamma D_{\mathbb{R}} \to D_{l^1} \otimes_\gamma D_{\mathbb{R}}$ is defined by

$$\begin{aligned} T_1(\sum_i a_i \otimes x_i) &= T_1(\sum_i \{(a_{i_n})x_i\}_n) \\ &= T_1(u), (say) = -u \end{aligned}$$

where if $u = \{y_1, y_2, ...\}$ then $-u = \{-y_1, -y_2, ...\}$.
It is easy to see that T_1 satisfies uniformly $L-$Lipschitzian (where $L = 1$) whether n is odd or even. But

$$\lim_{n\to\infty}[\sup \|T_1x - T_1^n x\| : \ x \in D_{l^1} \otimes D_{\mathbb{R}}] = 0$$

only when n is odd. Hence condition (a) of **Theorem 6** is satisfied.
Now, let $T_2 : D_{l^1} \otimes_\gamma D_{\mathbb{R}} \to D_{l^1} \otimes_\gamma D_{\mathbb{R}}$ be defined by $T_2(\sum_i a_i \otimes x_i) = \frac{1}{2}\sum_i\{\frac{a_{i_n}x_i}{n}\}_n$, where $a_i = \{a_{i_n}\}_n$.
Clearly, condition (b) of **Theorem 6** is satisfied with $M = \|T_2(D_{l^1} \otimes D_{\mathbb{R}})\| \leqslant \frac{1}{2}$ hence $LM < 1$. Proceeding as in **Theorem 4**, we have, for $\{x_n\} \subset D_{l^1} \otimes D_{\mathbb{R}}$

$$x_n - T_1x_nT_2x_n \to \ 0 \ as \ n \to \infty.$$

Now we can take b as the constant sequence $\{0, 0, 0, ...\}$ for which $0 = (I - T_1 \diamond T_2)b$ and $T_1^n uT_2v \in D_{l^1} \otimes D_{\mathbb{R}}$ for $u, v \in D_{l^1} \otimes D_{\mathbb{R}}$. So, the condition (c) of **Theorem 6** is satisfied. Hence the operator equation $u = T_1uT_2u$ has a solution.

Theorem 7: Let B be the bounded, open and convex subset with $0 \in B$ in a uniformly convex Banach algebra X. Let (T_1, T_2, T_3) be three self mappings on $\overline{B}$ such that

(a) T_1 satisfies uniformly $L-$Lipschitzian mapping on $\overline{B}$ and $\lim_{n\to\infty}[\sup \|T_1x - T_1^n x\| : \ x \in B] = 0$
(b) T_1 is demicompact on $\overline{B}$ and $M = \|T_2(B)\|$ such that $LM < 1$
(c) T_2, T_3 are completely continuous and $T_1^n uT_2v + T_3v \in B$ for $u, v \in B$ *and* $n \in \mathbb{N}$

then there exists a solution of the operator equation $u = T_1uT_2u + T_3u(= (T_1 \diamond T_2)u + T_3u)$ in B.

Proof. Since T_2 is a completely continuous, it is demicompact on $\overline{B}$. Also T_1 is demicompact by (b). So for a sequence $\{c_n\} \in \overline{B}$ such that $c_n - T_1c_n \to a$, $c_n - T_2c_n \to b$ as $n \to \infty$ in $\overline{B}$, there exists subsequence $\{c_{n_k}\}$ such that $c_{n_k} \to c$ as $k \to \infty$, where $c \in \overline{B}$.
Since T_1, T_2 and T_3 are continuous so $T_1c_{n_k} \to T_1c$, $T_2c_{n_k} \to T_2c$ and $T_3c_{n_k} \to T_3c$. Now we show that $I - T_1 \diamond T_2 - T_3$ is closed.
Let $z \in \overline{I - T_1 \diamond T_2 - T_3}$. Then for $\{z_n\} \subseteq (I - T_1 \diamond T_2 - T_3)c_n$ such that $z_n \to z$ as $n \to \infty$. We have as in **Theorem 4**,

$$c_{n_k} - T_1 \diamond T_2c_{n_k} - T_3c_{n_k} \to z \ as \ n \to \infty$$

Since $I - T_1 \diamond T_2 - T_3$ is continuous so $c \in I - T_1 \diamond T_2 - T_3$. Hence $I - T_1 \diamond T_2 - T_3$ is closed.
Define $J_n : \overline{B} \to \overline{B}$ by $J_n(u) = q_n(T_1^n uT_2v + T_3v)$, where $\{q_n\} \to 1 \ as \ n \to \infty$. Now,

$$\|J_n(u) - J_n(p)\| \leqslant q_nLM\|u - p\|$$

Since J_n is contraction and so it has unique fixed point $K_nv \in \overline{B}$ (say)
$K_nv = J_n(K_nv) = q_n(T_1^n(K_nv)T_2v + T_3v)$. Now, for any $v, y \in \overline{B}$ we have

$$\|K_n(v) - K_n(y)\| \leqslant q_n\|T_1^n(K_nv) - T_1^n(K_ny)\|\|T_2v\| + q_n\|T_1^n(K_ny)\|\|T_2v - T_2y\| + \|T_3v - T_3y\| \tag{2.4}$$

For fixed $a \in \overline{B}$, we have

$$\|T_1^n(u)\| \leqslant L\|u - a\| + \|T_1^n(a)\| = d(say) < \infty$$

From equation (4), we have

$$\|K_n(v) - K_n(y)\| \leqslant \frac{dq_n}{1 - LM}\|T_2v - T_2y\| + \frac{q_n}{1 - LM}\|T_3v - T_3y\|$$

So, K_n is completely continuous as T_2 and T_3 are completely continuous. By Schauder's fixed point theorem K_n has a fixed point x_n, say in $\overline{B}$. Hence $x_n = K_n x_n = J_n(x_n) = q_n(T_1^n(x_n)T_2x_n + T_3(x_n))$. Now,

$$x_n - T_1^n x_n T_2 x_n - T_3 x_n = (q_n - 1)(T_1^n x_n T_2 x_n + T_3 x_n) \to 0 \; as \; n \to \infty \tag{2.5}$$

$$\begin{aligned}\|x_n - T_1 x_n T_2 x_n - T_3 x_n\| &\leqslant \|x_n - T_1^n x_n T_2 x_n - T_3 x_n\| + \|T_2 x_n\| \|T_1^n x_n - T_1 x_n\| \\ &\to 0 \; as \; n \to \infty \text{ (by (5) and condition (a))}\end{aligned}$$

Since, $0 \in I - T_1 \diamond T_2 - T_3$ and $I - T_1 \diamond T_2 - T_3$ is closed. Hence there exists a point r such that $0 = (I - T_1 \diamond T_2 - T_3)r$. Hence the theorem follows. □

If $0 \notin B$ in the above **Theorem 7**.

Theorem 8: Let B be the bounded, open and convex subset in a uniformly convex Banach algebra X. Let (T_1, T_2, T_3) be three self mappings on $\overline{B}$ such that

(a) there exists $r \in B$ such that $r = T_1 c + T_2 c$ for some $c \in B$
(b) all the conditions of above **Theorem 7**

then there exists a solution of the operator equation $u = T_1 u T_2 u + T_3 u (= (T_1 \diamond T_2)u + T_3 u)$ in B.

Proof. Suppose that $K = B - r = \{x - r : \; x \in B\}$. Since B is open and bounded, so is K, and $\overline{K} = \overline{B} - r$ and $0 \in K$.
Define (T_1, T_2, T_3) are three self maps on $\overline{K}$ by
$T_1(c-r) = T_1 c - r$, $T_2(c-r) = T_2 c - r$ and $T_3(c-r) = T_3 c - r$. Hence (T_1, T_2, T_3) are three continuous self mappings in $\overline{K}$ and $I - T_1 \diamond T_2 - T_3$ is closed in $\overline{K}$. Then
(i) T_1 satisfies uniformly $L-$Lipschitzian mapping and

$$\lim_{n\to\infty}[\sup \|T_1(x-r) - T_1^n(x-r)\| : \; x - r \in K] = 0$$

(ii) Since T_1 is demicompact in $\overline{B}$, so T_1 is demicompact in $\overline{K}$. Also, $LM < 1$.
Similarly, since T_2 and T_3 are completely continuous in $\overline{B}$, so T_2 and T_3 are completely continuous in $\overline{K}$.
(iii) Clearly $T_1^n(u-r)T_2(v-r) + T_3(v-r) \in K$ for $u - r, v - r \in K \; and \; n \in \mathbb{N}$.
Hence all the conditions of **Theorem 7** satisfied so, there exists a solution $m - r$ such that

$$m - r = T_1(m-r)T_2(m-r) + T_3(m-r) \tag{2.6}$$

$$\begin{aligned}T_1(a-r)T_2(a-r) + T_3(a-r) &= [(T_1(a) - r][T_2(a) - r] + T_3(a) - r \\ &= T_1 a T_2 a + T_3 a - r - [r(T_1 a + T_2 a) - r^2]\end{aligned}$$

without loss of generality if $r = T_1 m + T_2 m$, $m \in B$ we have

$$T_1(m-r)T_2(m-r) + T_3(m-r) = T_1(m)T_2(m) + T_3(m) - r$$

Then from equation (6) we have a solution of the equation $u = T_1 u T_2 u + T_3 u$. □

Theorem 9: Let B be the bounded, open and convex subset in a uniformly convex Banach algebra X. Let (T_1, T_2, T_3) be three self mappings on $\overline{B}$ such that

(a) T_1 satisfies uniformly $L-$Lipschitzian mapping on $\overline{B}$, there exists $r \in D$ such that $\lim_{n\to\infty}[\sup \|T_1(x) - T_1^n(x)\| : \; x \in B] = 0$ and $r = T_1 c + T_2 c$ for some $c \in B$
(b) T_2 and T_3 are completely continuous $M = \|T_2(B)\|$ such that $LM < 1$ and $T_1^n u T_2 v + T_3 v \in B$ for $u, v \in B \; and \; n \in \mathbb{N}$
(c) if $\{x_n\} \in \overline{B}$ with $x_n - T_1 x_n T_2 x_n - T_3 x_n \to \; 0 \; as \; n \to \infty$ then there exists $b \in \overline{B}$ such that $0 = (I - T_1 \diamond T_2 - T_3)b$.

then there exists a solution of the operator equation $u = T_1 u T_2 u + T_3 u (= (T_1 \diamond T_2)u + T_3 u)$ in B.

Acknowledgement

The authors are extremely grateful to the anonymous learned referee(s) for their keen reading, valuable suggestion and constructive comments for the improvement of the manuscript. The corresponding author VNM acknowledges that this project was supported by the Cumulative Professional Development Allowance (CPDA), SVNIT, Surat, Gujarat, India. All the authors carried out the proof of theorems in this manuscript. Vishnu Narayan Mishra conceived the study and participated in its design and coordination.

References

[1] Afif Ben Amar, Soufiene Chouayekh, Aref Jeribi, New fixed point theorems in Banach algebras under weak topology features and applications to nonlinear integral equations, Journal of Functional Analysis 259 (2010), 2215-2237.
[2] F. F. Bonsal and J. Duncan, Complete Normed algebras, Springer-Verlag, Berlin Heidelberg New York, 1973.
[3] Felix. E. Browder On a Generalization of the Schauder Fixed Point Theorem, Duke Mathematical Journal, 26 (2) (1959), 291-303.
[4] B.C. Dhage On a fixed point theorem in Banach algebras with applications, Applied Mathematics Letters 18 (2005), 273-280.
[5] B.C. Dhage, On some variants of Schauders fixed point principle and applications to nonlinear integral equations, J. Math. Phys. Sci. 25 (1988) 603-611.
[6] B.C. Dhage, On Existance Theorems for Nonlinear Integral Equations in Banach Algebra via Fixed Point Techniques, East Asian Math. J. 17(2001), 33-45.
[7] D. Das, N. Goswami, Some Fixed Point Theorems on the Sum and Product of Operators in Tensor Product Spaces, IJPAM, 109(2016) 651-663.
[8] Deepmala and H.K. Pathak, On solutions of some functional-integral equations in Banach algebra, Research J. Science and Tech. 5 (3) (2013), 358-362.
[9] Deepmala, A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications, Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur 492 010, Chhatisgarh, India, 2014.
[10] K. Goebeli and W. A. Kirk, A Fixed Point Theorem for Asymptotically Nonexpansive Mappings, Proc. of the American Math. Soc. 35 (1) (1972), 171-174.
[11] Olga Hadzic, A Fixed Point Theorem For The Sum of Two Mappings, Proc. Amer. Math. Soc. 85 (1) (1982), 37-41,
[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Math. Std., vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.
[13] L.N. Mishra, H.M. Srivastava, M. Sen, On existence results for some nonlinear functional-integral equations in Banach algebra with applications, Int. J. Anal. Appl., 11 (1) (2016), 1-10.
[14] L.N. Mishra, S.K. Tiwari, V.N. Mishra, Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces, J. of App. Anal. and Comp., 5 (4) 2015, 600-612.
[15] L.N. Mishra, M. Sen, On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order, Applied Mathematics and Computation 285 (2016), 174-183.
[16] L. N. Mishra, R. P. Agarwal, M. Sen, Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving Erdélyi-Kober fractional integrals on the unbounded interval, Progress in Fractional Differentiation and Applications 2 (3) (2016), 153-168.
[17] L.N. Mishra, M. Sen, R.N. Mohapatra, On existence theorems for some generalized nonlinear functional-integral equations with applications, Filomat, in press.
[18] L.N. Mishra, S.K. Tiwari, V.N. Mishra, I.A. Khan, Unique Fixed Point Theorems for Generalized Contractive Mappings in Partial Metric Spaces, Journal of Function Spaces, 2015 (2015), Article ID 960827, 8 pages.
[19] V.N. Mishra, Some Problems on Approximations of Functions in Banach Spaces, Ph.D. Thesis, Indian Institute of Technology, Roorkee 247 667, Uttarakhand, India, 2007.
[20] M. Mursaleen, H.M. Srivastava, S.K. Sharma, Generalized statistically convergent sequences of fuzzy numbers, J. Intelligent Fuzzy Systems 30 (2016), 1511-1518.
[21] W. V. Petryshyn and T. S. Tucker, On the Functional Equations Involving Nonlinear Generalized P-Compact Operators, Transactions of the American Mathematical Society, 135 (1969), 343-373
[22] W. V. Petryshyn and T. E. Williamson, JR. Strong and Weak Convergence of the Sequence of Successive Approximations for Quasi-Nonexpansive Mappings, J. of Math. Ana. and Appl. 43 (1973), 459-497.
[23] J. Schu, Iterative Construction of Fixed Points of Asymptotically Nonexpansive Mappings, J. of Math. Ana. and Appl. 158 (1991) , 407-413
[24] H.K. Pathak and Deepmala, Remarks on some fixed point theorems of Dhage, Applied Mathematics Letters, 25 (11) (2012), 1969-1975.
[25] P. Vijayaraju, A fixed point theorem for a sum of two mappings in reflexive Banach spaces, Math. J. Toyoma Univ. 14 (1991), 41-50.
[26] P. Vijayaraju, Iterative Construction of Fixed Points of Asymptotic 1-Set Contraction in Banach Spaces, Taiwanese J. of Math. 1 (3) (1997), 315-325.

[27] P. Vijayraju, Fixed Point Theorems For a Sum of Two Mappings in Locally Convex Spaces, Int. J. Math. and Math. Sci., 17(4) (1994), 681-686.

[28] A. Raymond Ryan, Introduction to Tensor Product of Banach Spaces, London, Springer -Verlag, 2002.

[1] Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India

[2] Applied Mathematics and Humanities Department, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Dumas Road, Surat 395 007, Gujarat, India

[3] L. 1627 Awadh Puri Colony Beniganj, Phase -III, Opposite - Industrial Training Institute (I.T.I.), Ayodhya Main Road Faizabad 224 001, Uttar Pradesh, India

[*] Corresponding author: vishnunarayanmishra@gmail.com, vishnu_narayanmishra@yahoo.co.in