International Journal of Analysis and Applications
ISSN 2291-8639
Volume 13, Number 1 (2017), 54-63
http://www.etamaths.com

A GENERALIZED ITERATIVE ALGORITHM FOR HIERARCHICAL FIXED POINTS PROBLEMS AND VARIATIONAL INEQUALITIES

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Abstract. In this paper we propose a method for approximating of the common fixed point in $\bigcap_{n=1}^{\infty} F(T_n)$ where $\{T_n\}$ is a countable family of nonexpansive mappings on a closed convex subset C of a real Hilbert space $\mathcal{H}$. Then, we prove strong convergence theorems with less control conditions for $\{T_n\}$ which solves some variational inequality. The main results improve and extend the corresponding results of "F. Cianciaruso, G. Marino, L. Muglia, and Y. Yao, On a two-step algorithm for hierarchical fixed point problems and variational inequalities, J. Inequal. Appl., 2009 (2009), Article ID 208692" and "Y. Yao, Y.J. Cho, and Y.C. Liou, Iterative algorithms for hierarchical fixed points problems and variational inequalities, Mathematical and Computer Modelling, 52(9) (2010), 1697–1705".

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ with the inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively. Recall that a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and a nonself-mapping $f : C \to H$ is called a $\rho-$contraction on C if there exists a constant $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{x \in C \mid x = Tx\}$. Note that each $\rho-$contraction f has a unique fixed point in C, and for any fixed element $x_0 \in C$, Picard's iteration $x_{n+1} = f^n(x_0)$ converges strongly to a unique fixed point of f. However, a simple example shows that Picard's iteration cannot be used in the case of nonexpansive mappings. One method in [6] used for nonexpansive mappings is to employ a Halpern-type iterative scheme which produces a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \beta_n u + (1 - \beta_n)Tx_n, \ n \geq 1, \end{cases} \tag{1.1}$$

where $u \in C$ is arbitrary and $\{\beta_n\} \subset [0, 1]$.

In this paper, we consider the following variational inequalities problem:

$$\text{Find } x^* \in F(T) \text{ such that } \langle (I - S)x^*, x - x^* \rangle \geq 0, \ \forall x \in F(T), \tag{1.2}$$

where T and S are nonexpansive mappings such that $F(T)$ is nonempty. It is easy to see that x^* is a solution of the variational inequalities (1.2) if and only if it is a fixed point of the nonexpansive mapping $P_{F(T)}S$, where $P_{F(T)}$ stands for the metric projection on the closed convex set $F(T)$.

In 2000, Moudafi [8] introduced a viscosity approximation method for a nonexpansive mapping as follows:

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)Tx_n, \ n \geq 1, \end{cases} \tag{1.3}$$

where f is a contractive mapping and $\{\beta_n\} \subset [0, 1]$. In a real Hilbert space and under certain control conditions, he proved the sequence $\{x_n\}$ defined by (1.3) converges strongly to a fixed point of T which is the unique solution to the variational inequality $\langle (I - f)x^*, x - x^* \rangle \geq 0$ for all $x \in F(T)$.

Received 19th July, 2016; accepted 20th September, 2016; published 3rd January, 2017.
2010 *Mathematics Subject Classification.* 47H09, 47H10.
Key words and phrases. fixed points; iterative algorithms; nonexpansive mappings; variational inequalities; $\rho-$contraction.

Mainge and Moudafi [7] introduced an iterative scheme for approximating a specific solution of a fixed point problem as follows:

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n S x_n + (1-\alpha_n) T x_n, \ n \geq 1, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n, \ n \geq 1, \end{cases} \tag{1.4}$$

where f is a contractive mapping, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and S and T are nonexpansive mappings. They proved that if the sequence $\{x_n\}$ given by scheme (1.4) is bounded, then $\{x_n\}$ strongly convergence to the fixed point of a nonexpansive mapping T with respect to a nonexpansive mapping S under some control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$.

Recently, Alimohammady and Dadashi [1] studied the iterative scheme (1.5) for a countable family of nonexpansive mappings $\{T_n\}$ as follows:

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n S x_n + (1-\alpha_n) T_n x_n, \ n \geq 1, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n, \ n \geq 1, \end{cases} \tag{1.5}$$

where f is a contractive mapping, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings. They proved that the iterative scheme (1.5) strongly convergence to a common fixed point of $\{T_n\}$ with respect to a nonexpansive mapping S.

On the other hand, Cianciaruso et al. in [2] studied the sequence generated by the algorithm

$$\begin{cases} x_1 = x \in C \\ y_n = \beta_n S x_n + (1-\beta_n) x_n, \ n \geq 1, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T y_n, \ n \geq 1, \end{cases} \tag{1.6}$$

where f is a contractive mapping, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and S and T are nonexpansive mappings. They proved the sequence $\{x_n\}$ generated by (1.6) strongly converges to the fixed point of a nonexpansive mapping T with respect to a nonexpansive mapping S under some control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$. Also, they show that this fixed point is a unique solution of a variational inequality. Another results about fixed point and variational inequality problems can be found in [3,4,9] and the references therein.

Very recently, Yao et al. in [10] introduced another iterative algorithm and proved some strong convergence results for solving the hierarchical fixed point problem (1.2).

In this paper, inspired and motivated by the above iterative schemes, we introduced and studied a new composite iterative scheme for countable family of nonexpansive mappings T_k ($k \in \mathbb{N}$) with respect to a finite family of nonexpansive mapping $S_k(k \in \{1, 2, ..., N\}$ for some $N \in \mathbb{N})$ as follows:

$$\begin{cases} y_n = \beta_n S_n x_n + (1-\beta_n) x_n, \\ x_{n+1} = P_C(\alpha_n f(x_n) + (1-\alpha_n) T_n y_n), \ \forall n \geq 1, \end{cases} \tag{1.7}$$

where f is a contractive mapping, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $S_n = S_{n \bmod N}$. In particular, if we take $f \equiv 0$, then it is reduced to the iterative scheme:

$$\begin{cases} y_n = \beta_n S_n x_n + (1-\beta_n) x_n, \\ x_{n+1} = P_C((1-\alpha_n) T_n y_n), \ \forall n \geq 1, \end{cases} \tag{1.8}$$

The main results improve and extend the corresponding results of [2, 10]. In particular, It should be noticed that we prove strong convergence theorems with less control conditions for $\{T_n\}$ which solves some variational inequality.

2. Preliminaries

In this section, we recall the well known results and give some useful lemmas that will be used in the next section. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. For every point $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \ \forall y \in C.$$

P_C is called the metric projection of $\mathcal{H}$ onto C. Recall that, P_C is characterized by the following Lemma

Lemma 2.1. *Let $x \in \mathcal{H}$ and $z \in C$ be any points. Then $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0$, $\forall y \in C$.*

Lemma 2.2. *[5] Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in F(T)$.*

Lemma 2.3. *Let $f : C \to \mathcal{H}$ be a contraction with coefficient $\rho \in [0, 1)$ and $T : C \to C$ be a nonexpansive mapping. Then,*
(i) *the mapping $(I - f)$ is strongly monotone with coefficient $(1 - \rho)$ i.e.*

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq (1 - \rho)\|x - y\|^2,\ \forall x, y \in C;$$

(ii) *the mapping $(I - T)$ is monotone that is*

$$\langle x - y, (I - T)x - (I - T)y\rangle \geq 0,\ \forall x, y \in C.$$

Lemma 2.4. [11] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,\ \forall n \geq 0,$$

where $\{\gamma_n\}$ is a subsequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main results

In this section, we prove several strong convergence theorems of the iterative scheme (1.7). Throughout this section, C is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, T_n for each $n \in \mathbb{N}$ and S_n for each $n = 1, 2, ..., N$ are nonexpansive mappings of C into itself such that $F := \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and $f : C \to \mathcal{H}$ be a $\rho-$contraction (possibly nonself) with $\rho \in [0, 1)$.

Theorem 3.1. *Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ which satisfy in conditions*

$$\text{(C1)} \lim_{\mathrm{n}\to\infty} \alpha_{\mathrm{n}} = 0,\ \sum_{\mathrm{n}=1}^{\infty} \alpha_{\mathrm{n}} = \infty,$$

$$\text{(C2)} \lim_{\mathrm{n}\to\infty} \frac{\beta_{\mathrm{n}}}{\alpha_{\mathrm{n}}} = 0.$$

Then the sequence $\{x_n\}$ generated by (1.7) converges strongly to a point $z \in F$, which is the unique solution of the variational inequality:

$$\langle (I - f)z, x - z\rangle \geq 0,\ \forall x \in F. \tag{3.1}$$

In particular, if $f = 0$, then $\{x_n\}$ generated by (1.8) converges in norm to the minimum norm common fixed point z of T_n, $n \in \mathbb{N}$, namely, the point z is the unique solution to the quadratic minimization problem:

$$z = arg \min_{x \in \bigcap_{n=1}^{\infty} F(T_n)} \|x\|^2. \tag{3.2}$$

Proof. First, we claim that $\{x_n\}$ is bounded. Indeed, take an arbitrary fixed $u \in F = \bigcap_{n=1}^{\infty} F(T_n)$ and using (C2), we can assume, without loss of generality, that $\beta_n \le \alpha_n$ for all $n \ge 1$. From (1.7), we have

$$\begin{aligned}
\|x_{n+1} - u\| &= \|P_C(\alpha_n f(x_n) + (1-\alpha_n)T_n y_n) - P_C(u)\| \\
&\le \alpha_n \|f(x_n) - f(u)\| + \alpha_n \|f(u) - u\| + (1-\alpha_n)\|T_n y_n - u\| \\
&\le \alpha_n \rho \|x_n - u\| + \alpha_n \|f(u) - u\| + (1-\alpha_n)\|y_n - u\| \\
&\le \alpha_n \rho \|x_n - u\| + \alpha_n \|f(u) - u\| + (1-\alpha_n)\beta_n \|S_n x_n - u\| + (1-\alpha_n)(1-\beta_n)\|x_n - u\| \\
&\le (1-(1-\rho)\alpha_n)\|x_n - u\| + (1-\rho)\alpha_n \left\{ \frac{\|f(u)-u\| + \|S_n u - u\|}{1-\rho} \right\} \\
&\le \max\left\{ \|x_n - u\|, \frac{\|f(u)-u\| + \|S_n u - u\|}{1-\rho} \right\} \\
&\le \max_{1 \le K \le N} \left\{ \|x_1 - u\|, \frac{\|f(u)-u\| + \|S_k u - u\|}{1-\rho} \right\},
\end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded and so are the sequences $\{f(x_n)\}$, $\{y_n\}$, $\{T_n x_n\}$, $\{T_n y_n\}$ and $\{S_n x_n\}$. Now, we prove that $x_n \to z$ where, $z = P_F f(z)$. From Lemma 2.1 and set $u_n := \alpha_n f(x_n) + (1-\alpha_n)T_n y_n$, we get

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \langle P_C(u_n) - u_n, P_C(u_n) - z\rangle + \langle u_n - z, x_{n+1} - z\rangle \\
&\le \langle u_n - z, x_{n+1} - z\rangle \\
&= \alpha_n \langle f(x_n) - f(z), x_{n+1} - z\rangle + (1-\alpha_n)\langle T_n y_n - z, x_{n+1} - z\rangle + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\
&\le \alpha_n \rho \|x_n - z\|\|x_{n+1} - z\| + (1-\alpha_n)\|T_n y_n - z\|\|x_{n+1} - z\| \\
&\quad + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle, \qquad (3.3)
\end{aligned}$$

and hence by the definition of $\{y_n\}$, we have

$$\begin{aligned}
\|T_n y_n - z\| &\le \|T_n y_n - T_n x_n\| + \|T_n x_n - x_n\| + \|x_n - z\| \\
&\le \|y_n - x_n\| + \|T_n x_n - x_n\| + \|x_n - z\| \\
&\le \beta_n \|S_n x_n - x_n\| + \|T_n x_n - x_n\| + \|x_n - z\|. \qquad (3.4)
\end{aligned}$$

Also, we have

$$\|T_n x_n - x_n\| \le \|T_n x_n - T_n z\| + \|z - x_n\| \le 2\|x_n - z\|. \qquad (3.5)$$

Substituting (3.4) and (3.5) into (3.3) to obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le (\alpha_n \rho + 3(1-\alpha_n))\|x_n - z\|\|x_{n+1} - z\| + (1-\alpha_n)\beta_n \|S_n x_n - x_n\|\|x_{n+1} - z\| \\
&\quad + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\
&\le \frac{\alpha_n \rho + 3(1-\alpha_n)}{2}\left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) + (1-\alpha_n)\beta_n \|S_n x_n - x_n\|\|x_{n+1} - z\| \\
&\quad + \alpha_n \langle f(z) - z, x_{n+1} - z\rangle
\end{aligned}$$

So

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \left(1 - \frac{2\alpha_n(\rho-3)+4}{\alpha_n(\rho-3)+1}\right)\|x_n - z\|^2 + \frac{2(1-\alpha_n)\beta_n}{\alpha_n(3-\rho)-1}\|S_n x_n - x_n\|\|x_{n+1} - z\| \\
&\quad + \frac{2\alpha_n}{\alpha_n(3-\rho)-1}\langle f(z) - z, x_{n+1} - z\rangle \\
&= (1-\gamma_n)\|x_n - z\|^2 + \delta_n
\end{aligned}$$

which $\gamma_n = \frac{2\alpha_n(\rho-3)+4}{\alpha_n(\rho-3)+1}$ and $\delta_n = \frac{2(1-\alpha_n)\beta_n}{\alpha_n(3-\rho)-1}\|S_n x_n - x_n\|\|x_{n+1} - z\| + \frac{2\alpha_n}{\alpha_n(3-\rho)-1}\langle f(z) - z, x_{n+1} - z\rangle$. Then, Lemma 2.4 implies that $x_n \to z$ as $n \to \infty$.

In particular, if $f = 0$, then $\{x_n\}$ generated by (1.8) converges strongly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$ such that z is the unique solution of the variational inequality

$$\langle z, x - z\rangle \ge 0, \ \forall x \in F,$$

and hence, for each $x \in \bigcap_{n=1}^{\infty} F(T_n)$

$$\|z\|^2 \le \langle z, x\rangle \le \|z\|\|x\|.$$

Then for each $x \in \bigcap_{n=1}^{\infty} F(T_n)$, $\|z\|^2 \le \|x\|^2$, that is, z is the unique solution to the quadratic minimization problem (3.2). □

Corollary 3.2. *Let S , T be nonexpansive mapping of C with $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ which satisfy in conditions* (C1) and (C2). Then the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C(\alpha_n f(x_n) + (1-\alpha_n)T y_n), \ \forall n \ge 1, \end{cases} \tag{3.6}$$

converges strongly to a point $z \in F(T)$, which is the unique solution of the variational inequality:

$$\langle (I-f)z, x-z\rangle \ge 0, \ \forall x \in F(T). \tag{3.7}$$

In particular, if $f = 0$, then $\{x_n\}$ generated by (1.8) converges in norm to the minimum norm fixed point z of T, namely, the point z is the unique solution to the quadratic minimization problem:

$$z = arg \min_{x\in F(T)} \|x\|^2.$$

Proof. It is sufficient that assume $S_n = S$ and $T_n = T$ in Theorem 3.1. □

Remark 3.3. It is worth to mention that Yao et al. in [10] proved that the sequence $\{x_n\}$ generated by (3.6) converges strongly to a point $z \in F(T)$, which is the unique solution of the variational inequality (3.7) under control conditions (C1), (C2) and the following conditions

$$\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0, \lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\beta_n} = 0 \in (0,\infty); \tag{3.8}$$

or

$$\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty, \ \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty; \tag{3.9}$$

But Corollary 3.2 proves that the sequence $\{x_n\}$ converges strongly under control conditions (C1) and (C2) and it does not require conditions (3.8) and (3.9) for convergence.

Theorem 3.4. *Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ which satisfy in conditions* (C1),

$$(\mathrm{C2'}) \lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \tau \in (0,\infty);$$

$$(\mathrm{C3}) \lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|}{\alpha_n \beta_n} = 0;$$

(C4) there exist a constant $K > 0$ such that $\frac{1}{\alpha_n}|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}| \le K$;

$$(\mathrm{C5}) \sum_{n=1}^{\infty} \sup\left\{\frac{\|T_n z - T_{n-1} z\|\|}{\alpha_n \beta_n}, z \in B\right\} < \infty \text{ for any bounded subset } B \text{ of } C.$$

Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C(\alpha_n f(x_n) + (1-\alpha_n)T_n y_n), \ \forall n \ge 1, \end{cases} \tag{3.10}$$

converges strongly to a point $x^* \in F$, which is the unique solution of the variational inequality

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, y - x^*\rangle \ge 0, \ \forall y \in F. \tag{3.11}$$

Proof. At first, we show that uniqueness of the solution to the variational inequality (3.11) in $F(T)$. In fact, suppose that x^* and $\tilde{x}$ satisfy in (3.11).Then, since $\tilde{x}$ satisfy in (3.11), for $y = x^*$, it follows that

$$\langle (I-f)\tilde{x}, \tilde{x}-x^*\rangle \leq \tau\langle (I-S)\tilde{x}, x^*-\tilde{x}\rangle. \tag{3.12}$$

Similarly, we have

$$\langle (I-f)x^*, x^*-\tilde{x}\rangle \leq \tau\langle (I-S)x^*, \tilde{x}-x^*\rangle. \tag{3.13}$$

By (3.12), (3.13) and Lemma 2.3, we get

$$\begin{array}{rcl} (1-\rho)\|\tilde{x}-x^*\|^2 & \leq & \langle (I-f)\tilde{x}-(I-f)x^*, \tilde{x}-x^*\rangle \\ & = & \langle (I-f)\tilde{x}, \tilde{x}-x^*\rangle - \langle (I-f)x^*, \tilde{x}-x^*\rangle \\ & \leq & \tau\langle (I-S)\tilde{x}, x^*-\tilde{x}\rangle + \tau\langle (I-S)x^*, \tilde{x}-x^*\rangle \\ & = & -\tau\langle (I-S)\tilde{x}-(I-S)x^*, \tilde{x}-x^*\rangle \\ & \leq & 0. \end{array}$$

Hence, $x^* = \tilde{x}$. We can assume from (C2$'$), without loss of generality, that $\beta_n \leq (\tau+1)\alpha_n$ for all $n \geq 1$. By a similar argument as that of Theorem 3.1, we have

$$\begin{array}{rcl} \|x_{n+1}-u\| & \leq & (1-(1-\rho)\alpha_n)\|x_n-u\| + \alpha_n\|f(u)-u\| + (1-\alpha_n)\beta_n\|Su-u\| \\ & \leq & (1-(1-\rho)\alpha_n)\|x_n-u\| + \alpha_n\|f(u)-u\| + \alpha_n(\tau+1)\|Su-u\| \\ & = & (1-(1-\rho)\alpha_n)\|x_n-u\| + (1-\rho)\alpha_n\left[\dfrac{\|f(u)-u\|}{1-\rho} + \dfrac{(\tau+1)\|Su-u\|}{1-\rho}\right] \\ & \leq & \max\left\{\|x_n-u\|,\ \dfrac{\|f(u)-u\|}{1-\rho} + \dfrac{(\tau+1)\|Su-u\|}{1-\rho}\right\}, \end{array}$$

which implies that the sequence $\{x_n\}$ is bounded. Set $u_n = \alpha_n f(x_n) + (1-\alpha_n)T_n y_n$, then we have

$$\begin{array}{rcl} \|x_{n+1}-x_n\| & = & \|P_C(u_n)-P_C(u_{n-1})\| \leq \|u_n-u_{n-1}\| \\ & \leq & \alpha_n\|f(x_n)-f(x_{n-1})\| + |\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-T_{n-1}y_{n-1}\| \\ & & +(1-\alpha_n)\|T_n y_n - T_{n-1}y_{n-1}\| \\ & \leq & \alpha_n\rho\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-T_{n-1}y_{n-1}\| + (1-\alpha_n)\|y_n-y_{n-1}\| \\ & & +\|T_n y_{n-1}-T_{n-1}y_{n-1}\| \end{array} \tag{3.14}$$

Also by definition of $\{y_n\}$, we get

$$\begin{array}{rl} \|y_n-y_{n-1}\| & \leq \beta_n\|Sx_n-Sx_{n-1}\| + (1-\beta_n)\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|Sx_{n-1}-x_{n-1}\| \\ & \leq \|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|Sx_{n-1}-x_{n-1}\|. \end{array} \tag{3.15}$$

Set, $M = \max\{\sup\|f(x_{n-1})-T_{n-1}y_{n-1}\|, \|Sx_{n-1}-x_{n-1}\|\}$ and substituting (3.15) in (3.14) we have

$$\begin{array}{rl} \|x_{n+1}-x_n\| & \leq \|u_n-u_{n-1}\| \\ & \leq (1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\| + M\left[|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}|\right] \qquad (3.16) \\ & \quad +\|T_n y_{n-1}-T_{n-1}y_{n-1}\| \\ & \leq (1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\| + M(\tau+1)\alpha_n\left[\dfrac{|\alpha_n-\alpha_{n-1}|}{\alpha_n} + \dfrac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\ & \quad +\alpha_n\left[\sup\left\{\dfrac{\|T_n z-T_{n-1}z\|}{\alpha_n}, z\in B\right\}\right]. \end{array}$$

From (C1), (C3), (C5) and Lemma 2.4, we can deduce that $\|x_{n+1}-x_n\| \to 0$. By (3.16) and (C4) we have,

$$\begin{aligned}
\frac{\|x_{n+1}-x_n\|}{\beta_n} &\le \frac{\|u_n-u_{n-1}\|}{\beta_n} \\
&\le (1-(1-\rho)\alpha_n)\frac{\|x_n-x_{n-1}\|}{\beta_n} + M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\
&\quad + \left[\frac{\|T_n y_{n-1}-T_{n-1}y_{n-1}\|}{\beta_n}\right] \\
&= (1-(1-\rho)\alpha_n)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + (1-(1-\rho)\alpha_n)\left[\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right]\|x_n-x_{n-1}\| \\
&\quad + M\alpha_n\left[\frac{|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|}{\alpha_n\beta_n}\right] + \alpha_n\left[\frac{\|T_n y_{n-1}-T_{n-1}y_{n-1}\|}{\alpha_n\beta_n}\right] \\
&\le (1-(1-\rho)\alpha_n)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + \alpha_n\left[\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right]\frac{1}{\alpha_n}\|x_n-x_{n-1}\| \\
&\quad + M\alpha_n\left[\frac{|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|}{\alpha_n\beta_n}\right] + \alpha_n\sup_{z\in B}\left\{\frac{\|T_n z-T_{n-1}z\|}{\alpha_n\beta_n}\right\} \\
&\le (1-(1-\rho)\alpha_n)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + \alpha_n K\|x_n-x_{n-1}\| \\
&\quad + M\alpha_n\left[\frac{|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|}{\alpha_n\beta_n}\right] + \alpha_n\sup_{z\in B}\left\{\frac{\|T_n z-T_{n-1}z\|}{\alpha_n\beta_n}\right\}.
\end{aligned}$$

Again, (C1), (C3), (C5) and Lemma 2.4 imply that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n}=0,\ \lim_{n\to\infty}\frac{\|u_n-u_{n-1}\|}{\beta_n}=0,$$

and hence by (C2′) we get

$$\lim_{n\to\infty}\frac{\|u_n-u_{n-1}\|}{\alpha_n}=0.$$

It follows from (C1) and (C2′) that $\beta_n \to 0$ and by (3.10), $\|y_n-x_n\| \to 0$ and $\|x_{n+1}-T_n y_n\| \to 0$. Then,

$$\begin{aligned}
\|x_n-T_n x_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T_n y_n\|+\|T_n y_n-T_n x_n\| \\
&\le \|x_n-x_{n+1}\|+\|x_{n+1}-T_n y_n\|+\|y_n-x_n\| \to 0 \ as\ n\to\infty,
\end{aligned}$$

therefore, we have

$$\begin{aligned}
\|x_n-Tx_n\| &\le \|x_n-T_n x_n\|+\|T_n x_n-Tx_n\| \\
&\le \|x_n-T_n x_n\|+\sup\{\|T_n z-Tz\|,\ z\in\{x_n\}\} \to 0 \ as\ n\to\infty.
\end{aligned}$$

By demiclosedness principle, Lemma 2.2, we obtain $w_w(x_n)\subseteq F(T)=\bigcap_{n=1}^{\infty}F(T_n)$. Also

$$\|y_n-T_n y_n\| \le \|y_n-x_n\|+\|x_n-x_{n+1}\|+\|x_{n+1}-T_n y_n\| \to 0.$$

From (3.10), we have

$$x_{n+1}=P_C(u_n)-u_n+\alpha_n f(x_n)+(1-\alpha_n)(T_n y_n-y_n)+(1-\alpha_n)[\beta_n Sx_n+(1-\beta_n)x_n],$$

and hence

$$\begin{aligned}
x_n-x_{n+1} &= x_n-P_C(u_n)+u_n-\alpha_n f(x_n)-(1-\alpha_n)(T_n y_n-y_n)-(1-\alpha_n)\beta_n Sx_n-(1-\alpha_n)(1-\beta_n)x_n \\
&= u_n-P_C(u_n)+\alpha_n(I-f)x_n+(1-\alpha_n)(I-T_n)y_n+(1-\alpha_n)\beta_n(I-S)x_n.
\end{aligned}$$

Set $v_n=\frac{x_n-x_{n+1}}{(1-\alpha_n)\beta_n}$. Hence, we obtain

$$v_n=\frac{1}{(1-\alpha_n)\beta_n}(u_n-P_C[u_n])+\frac{\alpha_n}{(1-\alpha_n)\beta_n}(I-f)x_n+\frac{1}{\beta_n}(I-T_n)y_n+(I-S)x_n.$$

For any $z \in \bigcap_{n=1}^{\infty} F(T_n)$ we have

$$\begin{aligned}
\langle v_n, x_n - z\rangle &= \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C(u_n), x_n - z\rangle + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n, x_n - z\rangle \\
&\quad + \frac{1}{\beta_n}\langle (I-T_n)y_n, x_n - z\rangle + \langle (I-S)x_n, x_n - z\rangle \\
&= \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n) + P_C(u_n) - z\rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n - (I-f)z + (I-f)z, x_n - z\rangle \\
&\quad + \frac{1}{\beta_n}\langle (I-T_n)y_n - (I-T_n)z, x_n - z\rangle + \langle (I-S)x_n - (I-S)z + (I-S)z, x_n - z\rangle \\
&= \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C(u_n), P_C(u_n) - z\rangle \\
&\quad + \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n)\rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n - (I-f)z, x_n - z\rangle + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle \\
&\quad + \frac{1}{\beta_n}\langle (I-T_n)y_n - (I-T_n)z, x_n - y_n\rangle + \frac{1}{\beta_n}\langle (I-T_n)y_n - (I-T_n)z, y_n - z\rangle \\
&\quad + \langle (I-S)x_n - (I-S)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle
\end{aligned} \tag{3.17}$$

By Lemma 2.3, we obtain

$$\begin{aligned}
\langle v_n, x_n - z\rangle &\geq \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n)\rangle \\
&\quad + \frac{\alpha_n(1-\rho)}{(1-\alpha_n)\beta_n}\|x_n - z\|^2 + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle \\
&\quad + \frac{1}{\beta_n}\langle (I-T_n)y_n - (I-T_n)z, x_n - y_n\rangle + \langle (I-S_n)z, x_n - z\rangle \\
&\geq \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n)\rangle \\
&\quad + \frac{\alpha_n(1-\rho)}{(1-\alpha_n)\beta_n}\|x_n - z\|^2 + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle \\
&\quad + \langle (I-T_n)y_n, x_n - Sx_n\rangle + \langle (I-S)z, x_n - z\rangle
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\|x_n - z\|^2 &\leq \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}[\langle v_n, x_n - z\rangle - \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n)\rangle \\
&\quad - \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle - \langle (I-T_n)y_n, x_n - Sx_n\rangle - \langle (I-S)z, x_n - z\rangle] \\
&\leq \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\left[\langle v_n, x_n - z\rangle - \langle (I-T_n)y_n, x_n - Sx_n\rangle - \langle (I-S)z, x_n - z\rangle\right] \\
&\quad + \frac{\|u_n - u_{n-1}\|}{\alpha_n(1-\rho)}\|u_n - P_C(u_n)\| - \frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle
\end{aligned}$$

Since $v_n \to 0, (I-T_n)y_n \to 0$, $\frac{\|u_n - u_{n-1}\|}{\alpha_n} \to 0$ and $\omega_w(x_n) \subseteq F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, then every weak cluster point of $\{x_n\}$ is also a strong cluster point. It follows from the boundedness of the sequence $\{x_n\}$ that there exists a subsequence $\{x_{n_k}\}$ converging to a point $x' \in \mathcal{H}$. For all $z \in F(T)$, it follows from (3.17) that

$$\begin{aligned}\langle (I-f)x_{n_k}, x_{n_k}-z\rangle &= \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)x_{n_k}, x_{n_k}-z\rangle\\ &\quad -\frac{1}{\alpha_{n_k}}\langle u_{n_k}-P_C(u_{n_k}), P_C(u_{n_{k-1}})-z\rangle - \frac{(1-\alpha_{n_k})}{\alpha_{n_k}}\langle (I-T_{n_k})y_{n_k}, x_{n_k}-z\rangle\\ &\leq \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)z, x_{n_k}-z\rangle\\ &\quad -\frac{1}{\alpha_{n_k}}\langle u_{n_k}-P_C(u_{n_k}), P_C(u_{n_{k-1}})-P_C(u_{n_k})\rangle\\ &\quad -\frac{(1-\alpha_{n_k})}{\alpha_{n_k}}\langle (I-T_{n_k})y_{n_k}-(I-T_{n_k})z, x_{n_k}-y_{n_k}\rangle\\ &\leq \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)z, x_{n_k}-z\rangle\\ &\quad +\frac{\|u_{n_k}-u_{n_{k-1}}\|}{\alpha_{n_k}}\|u_{n_k}-P_C(u_{n_k})\| - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-T_{n_k})y_{n_k}, x_{n_k}-Sx_{n_k}\rangle.\end{aligned}$$

Letting $k\to\infty$, we obtain

$$\langle (I-f)x', x'-z\rangle \leq -\tau\langle (I-S)z, x'-z\rangle.$$

Thus x' is a solution of the variational inequality (3.11) and since (3.11)has the unique solution, it follows that $\omega_w(x_n)=\omega_s(x_n)=\{x^*\}$ and this ensures that $x_n\to x^*$ as $n\to\infty$. □

Corollary 3.5. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *which satisfy in conditions* (C1), (C3), (C4), (C5) and

$$(\mathrm{C2}')\ \lim_{\mathrm{n}\to\infty}\frac{\beta_{\mathrm{n}}}{\alpha_{\mathrm{n}}}=1;$$

Then the sequence $\{x_n\}$ defined by (1.8) converges strongly to a point $x^*\in F$, which is the unique solution of the variational inequality

$$\langle (2I-S)x^*, y-x^*\rangle \geq 0,\ \forall y\in F. \tag{3.18}$$

Proof. It is sufficient that assume $f=0$ and $\tau=1$ in Theorem 3.4. □

Corollary 3.6. *Let* $S,T:C\to C$ *be two nonexpansive mappings with* $F(T)\neq\emptyset$. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *which satisfy in conditions* (C1), (C2$''$), (C3) and (C4). Then the sequence $\{x_n\}$ defined by (3.6) converges strongly to a point $x^*\in F(T)$, which is the unique solution of the variational inequality

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, y-x^*\rangle \geq 0,\ \forall y\in F(T).$$

Proof. It is sufficient that assume $T_n=T$ in Theorem 3.4. □

Acknowledgment

Vahid Dadashi and Somayeh Amjadi are supported by Sari Branch, Islamic Azad University.

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