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WEAK SOLUTION AND WEAKLY UNIFORMLY BOUNDED SOLUTION OF IMPULSIVE HEAT EQUATIONS CONTAINING "MAXIMUM" TEMPERATURE

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Abstract. In this paper, criteria for the existence of weak solutions and uniformly weak bounded solution of impulsive heat equation containing maximum temperature are investigated and results obtained. An example is given for heat flow system with impulsive temperature using maximum temperature simulator and criteria for the uniformly weak bounded of solutions of the system are obtained.

1 Introduction

Impulsive partial differential equations (IPDEs) are systems that undergo rapid changes in the state variables describing them and are represented by partial differential equations (PDEs)([1] &[12-14]). Theory of impulsive partial differential equations is richly endowed with many applications to real life processes. IPDEs have a lot of applications in engineering, physical and biological sciences. One of practical real life problems which can be solved by IPDEs as an example is the simulation of an impulsive temperature of systems.

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The study on IPDEs started in 1991 in the paper [1] where the population dynamics of single species was studies .Extensive work have been done in the study of quantitative and qualitative properties of IPDEs. There are many publications on oscillation of solutions of hyperbolic and parabolic PDEs (see [2], [3] & [4]).Moreover, interesting results were obtained for the delay PDEs for example ([18]) investigated a class of impulsive parabolic partial differential equations with delay and oscillation criteria obtained for two-points boundary value conditions coupled with the use of Gauss' divergence theorem and impulsive delay inequalities ([18]). The control and oscillation of parabolic systems with impulsive effects has been studied in ([2], [8] & [18]).

Many impulsive systems can be simulated using systems containing "maximum" ([13] & [14]) .It was noted in [14] that impulsive differential equations containing maximum provide rich platform to study the impulsive real life processes which are described by unknown function in the given set ([12], [13] & [14]).

In this paper, we will consider the application of IPDEs to the heat system for which temperature is required to be controlled by impulsive device using maximum simulated temperature. This kind of problem has many applications in engineering, nuclear science and biotechnology etc.

Furthermore, we intend to investigate the conditions for existence of weak solutions and weakly uniform bounded of solution of impulsive heat equations containing "maximum". We make use of Caldron-Zymund (see [6] & [7]) condition and some growth assumptions, Poincare inequality and the Babuska-Lax-Milgram theorem to obtain the existence of weak solution to problem in a given Sobolev space.

2 Preliminary Definitions and Notes

Let sequence of impulsive moments be $\{x_k\}, k = 0,1,2,\ldots$ such that

$0 < x_0 < x_1 < x_2 < \ldots < x_k$ and $\lim_{k\to\infty} x_k = +\infty$.

Let $R^+ = [0,\infty), R = (0,\infty)$ and $R^0 = R^+ - \{x_k\}_1^\infty$

Let $\bar{\Omega}$ be a Lebesque space of integrable functions having one parameter $p \in [1,\infty)$ for $p < \infty$ then we define L^p -norm of a measurable function u as

$$\|u\| = \left[\int \left|u^p\right| dx\right]^{\frac{1}{p}}$$

And for $p = \infty, \|u\|_\infty = ess\sup\{|u(x)| : x \in \Omega\} = \inf \sup_{|N|=0, x\in R-N} |u(x)|$, if $|u|_p < \infty$ we define by

$$L_p(\Omega) = \left\{u \in \Omega : |u|_p < \infty\right\}.$$

We will denote by

$C(\Omega)$ The space of continuous functions defined on Ω and

$PC(\Omega) = \{y(x) : y(x) \in C(\Omega), k = 0,1,2,..., \lim_{x \to x_k+0} y(x) = y(x_k), x, x_k \in \Omega \}$

Let $D(\Omega) = C_0^\infty(\Omega)$ be the set of functions that are infinitely times differentiable in Ω and have compact support i.e. the set $\text{supp}\, f(x) = \overline{\{x \in \Omega : f(x) = 0\}}$ is compact where $\overline{\{.\}}$ is the closure of the set ([12] & [13]).

Let $D_w^\alpha u$ be weak derivative of order u will denote by $W^{m,p}(\Omega) = \{u \in L_p(\Omega) : D_w^\alpha \in L_p(\Omega), |\alpha| \leq m\}$ If p=2, we write $W^{m,2} = H^m(\Omega)$ and $W_0^{m,p}(\Omega)$ is the closure of $D(\Omega)$ in $H^m(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Remark 1

$W^{m,p}(\Omega)$ is separable for $1 \leq p < \infty$ and uniformly convex (reflexive) for $1 < p < \infty$.

2.1 Sobolev norm and inner products

We define the Sobolev space norm as follows

$$|u|_{k,p} = \sum_{|\alpha| \le k} \|D^{\alpha} u\|_p$$

Remark 2

If $k = 1$ then $|u|_{1,p} = |u|_p + \left|\frac{\partial u}{\partial x_i}\right|_p + \cdots + \left|\frac{\partial u}{\partial x_n}\right|_p$

If $k = 1$ then $|u|_{1,2}^2 = \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|_2^2$

The scalar product in the Sobolev Space is define as

$$\langle u, v \rangle = \int_{\Omega} \left[\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right) + uv \right] dx.$$

2.1 Sobolev Imbedding Properties

Let Ω be bounded domain in R^n with Lipchitz boundary, $n \ge 2$ and $1 \le p < \infty$ then

$$W^{1,p}(\Omega) \subset L^q(\Omega),\ |u|_q \le c_1 |u|_{1,p}$$

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ if } p > N$$

And

$W^{1,p}(\Omega) \subset C^0(\overline{\Omega}) \subset L^{\infty}(\Omega), \max_m |u| \le c_2 |u|_{1,p}$ Provided p>N. The constants c_1 and c_2 are independent of u.

Definition 1

$a(.,.): H \times H \to R$ is said to be H-elliptic or coercive if there exists a constant $c>0$ such that $|a(u,u)| \ge c|u|^2, u \in H$ .The meaning of this is that high value of u will lead to high value of $a(u,u)$.

Definition 2

Consider the cone L_c in $L_2(\Omega)$ such that $L_C = \{v \in V : v \geq 0,\ v \in L_2(\Omega)\}$

And the ball $S_p = \{x \in V : |x| \leq p,\ p = \text{const}\} \subset L^2(\Omega)$.

2.3 Auxiliary Results

The following standard auxiliary results will be used in this paper:

Lemma 1 (Green's theorem)

Let Ω be a domain with Lipchitz boundary, and u and v be smooth functions on $\bar{\Omega}$. Then

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v\, dx = \int_{\partial\Omega} u v n_i ds - \int_{\Omega} u \frac{\partial v}{\partial x_i} dx$$

Where n_i is the i th component of the outer unit normal vector.

2.4.1 Poincare inequality and the Lax-Milgram Theorem

Lemma 2 (Poincare inequality)

Let $\Omega \subset R^n$ be an open set, bounded along some axes, $|\cdot|_{1,p}$ between a norm and is equivalent to the norm $\|\cdot\|_{1,p}$ on $W^{1,p}(\Omega)$

Such that

$$|u|_{0,p} \leq c\,|u|_{1,p}\,,\ \forall\ u \in W_0^{1,p}(\Omega)$$

$$|u|_{0,\Omega} \leq c\left|\frac{\partial u}{\partial x_n}\right|,\ \forall\ u \in H_0(\Omega).$$

2.4.1 Lax-Milgram Theorem

One of most highly utilized Theorem in the classical theory of PDEs is the Lax-Milgram Theorem which is useful for establishing weak solutions to PDEs. There are several versions of the Theorem exist in the literature. The generalized Lax-Milgram theorem has been extended from Hilbert spaces to

Banach spaces (see for example, Ramaswamy [16]) and even applied to elliptic and hyperbolic boundary value problems.

Babuska-Lax-Milgram (BLM) theorem is a generalization of the famous Lax-Milgram theorem which gives the conditions for which a bilinear form can be inverted to show the extremal and uniqueness of a weak solution to a give BVP. The BLM Theorem will be exploited to obtain criteria for weak uniformly bounded properties of impulsive heat equations containing "maximum".

We state without proof Lax-Milgram theorem:

Lemma 3 (Lax-Milgram Theorem)
Let H be a real Hilbert space and $a(\cdot,\cdot): H \times H \to R$ be a continuous H-elliptic bilinear form and let $f: H \to R$ be a continuous linear form. Then

(a) There exists a unique $u \in H$ such that $a(u,v) = f(v), \ \forall \ u \in H$;
(b) If, in addition, $a(\cdot,\cdot)$ is symmetric and $a(u,v) \geq 0, \ \forall \ v \in H$ then the function $J: H \to R$ defined by $J(v) := \frac{1}{2}a(v,v) - f(v)$ has minimum value at u given as

$$u = \inf_{\forall \in H} J(u) \leq J(u) .$$

In this paper, we will assume that

$$A_1 : a(x_i, \xi) \begin{cases} \text{is measurable in } x \text{ for all } \xi \in L^2(\Omega) \\ \text{continuous in } \xi \text{ for almost all } x \in I \end{cases}$$

A_2 : $a(x,\xi)$ Satisfies Calderon-Zygmund standard bound kind of condition ([6] & [7]) such that

$$\left| a(x,\xi)\frac{\partial \varepsilon}{\partial x} - a(x,\xi)\frac{\partial \xi}{\partial x}\left|\frac{\partial v}{\partial x}\right|\right|$$

$$\leq c * \left(\left|\frac{\partial u_1}{\partial x}\right|^2 \right)^{\frac{1}{2}} \left|\xi_1 - \xi_2\right|^{\alpha}_{W^{\alpha,2}} \leq c_1 \left|\xi_1 - V_2\right|_{L^2(\Omega)}.$$

3 Statement of the Problem

Consider an impulsive heat system with the source f with u= u(x, u) mean temperature with the system was the volume Ω. The system is governed by the following impulsive heat system containing "maximum" temperature:

$$\left.\begin{aligned} & -\sum \frac{\partial}{\partial x_i}\left[a(x_i,u)\frac{\partial u}{\partial x_i}\right] + g(x, \max u) \\ & = f \text{ in } \Omega,\ x \neq x_k,\ k = 0,1,2,\cdots \\ & \Delta x = I\left(x_i, u\right),\ x_0 \leq x_i \leq x,\ x = x_k \\ & \text{with mixed boundary conditions} \\ & u = 0 \text{ on } \Gamma_0,\ \Sigma_i\, a(x_i, y)\frac{\partial u}{\partial x_i} n_i = h \text{ on } \Gamma_1, \Gamma_0 \bigcup \Gamma_1 = \Omega. \end{aligned}\right\} \quad (1)$$

We will make use of the following relationship to obtain results:

Let $u \in D(\Omega)$ satisfies $u = 0$ on Γ_0 then by the Green's theorem, we have

$$\int_{\Omega}\sum_{i} a(x,u)\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}dx = \int_{\Omega} fvdx - \int_{\Omega} fg + \int_{\Gamma_1} hvds + \sum_{i} v_i I(x_i,u). \tag{2}$$

Definition 3

Let $u = u(x)$ such that $u:(0,x_N] \to V$, then it is said to be the weak solution to equation (1) if all $\varphi \in V'$ we have $u \to < Au, \varphi >$ is locally integrable in $(0, x_N]$ and

$$\int_{\Omega} < A(u), v > \frac{d\varphi}{ds}ds = -\int_{\Omega} < A(u), \frac{dv}{ds} > ds, \forall \varphi \in D(\Omega), u \in PC(\Omega), v \in V'.$$

Definition 4

Let $u = u(x,u)$ be weak solution of eq. (1), we introduce the Dini derivative of the Energy function $J(v)$ across the weak solution path of the eq. (1) as

$$D^{+}_{(1)}J(V) = \limsup_{h\to 0}\frac{(J(v+h) - J(v))}{h}.$$

Definition 5

A. The trivial weak solution $u = 0$ of eq. (1) is said to be weakly equi-bounded in the L_c cone if for any $\alpha > 0$ there exists a continuous function $\beta(x_0,\alpha) > 0$ such that if $|x_0| < \alpha$ then $|u(x_0,u)| < \beta$;

B. Uniformly weak bounded if β from above *is* independent of x_0 ;

C. Ultimately weak bounded with the bound N if there exist numbers $\alpha > 0$ and $T = T(x_0,\alpha) > 0$ such that if $|x_0| < \alpha$ then $|u(x_0,u_0)| < N$ for $x \geq x_0 + T$.

D. Uniformly ultimate weak bounded with the bound N if $T = T(\alpha)$, in the definition C above.

2.3 Weak formulation

Let $V = \{u \in PC^{/}(\bar{\Omega}) : u = 0 \text{ on } \Gamma_0\}$ where $V^{/} = PC^{/}(\bar{\Omega})$ the dual space of $PC(\Omega)$.

With norm in the Sobolev space $W^{1,2}(\Omega)$, then the space $V = PC(\Omega \bigcup \partial\Omega)$ is a reflexive separable Banach space. Let $A : V \to V'$ and the functional $b \in V'$ by the relations:

$$\langle A(u), v \rangle = \int_{\Omega} \sum_{i} a(x,u) \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x} dx, \quad \forall \ u \in V, v \in D(\Omega) \tag{3}$$

$$\langle b, v \rangle = \langle (b_1 + b_2 + b_3 + b_4) x, v \rangle \tag{4}$$

Where

$b_1 = \int_{\Omega} fv dx$, $b_2 = \int_{\Omega} vg dx$, $b_3 = \int_{\Gamma} fv dx$, and $b_4 = \sum_{i=1} v_i I_i$.

Then, the solvability of the problem can be formulated as:

Find $u \in V$ such that $\langle A(u), u \rangle = \langle b, v \rangle \ \ \forall \in v \in V$. Before we establish the results of solvability, we need to justify the weak formulation of the problem as follows:

By Cauchy-Swartz inequality, we can show that

$$\left| \langle b, v \rangle \right| = \left| \int_{\Omega} fv dx - \int_{\Omega} fv + \int_{\Gamma_1} fv dx + \Sigma_i v_i I_i \right| \tag{5}$$

$$\leq |f|_{L^2(\Omega)} \cdot |v|_{L^2(\Omega)} + |g|_{L^2(\Omega)} |v|_{L^2(\Omega)} + |h|_{L^2(\Omega)} \cdot |v|_{L^2(\Omega)} + c_1 |v|_{L^2(\Omega)}$$

$$\leq c |v|_{L^2(\Omega)}$$

Where $c = \max \left[|f|_{L^2(\Omega)}, |g|_{L^2(\Omega)} |h|_{L^2(\Omega)}, c_1 \right]$ = constant. Therefore $A : V \to V'$ is a bounded linear functional which is well defined.

4 Main Results

We consider theorem on solvability of the impulsive heat equation containing maximum temperature we start with the following theorem:

Theorem 1:

Let V be a Hilbert Space and $A : V \to V$ as operator which strongly monotone, i.e., there exists $\alpha > 0$ such that

$$\left| A(u_1) - A(u_2) \right| \geq \alpha \left| u_1 - u_2 \right|^2, \ \alpha \geq 0, \ u_1, u_2 \in V$$

Then the equation $A(u) = b$ admits a unique solution for each $b \in V$.

Proof:

Let

$$T_\varepsilon(u) = u - \varepsilon \left(A(u) - b \right) \tag{6}$$

For a constant $\varepsilon > 0$, we must show that T_ε is contractive, i.e.

$$\left| T_\varepsilon(u_1) - T_\varepsilon(u_2) \right| < \in \left| u_1 - u_2 \right|, \ u_1, u_2 \in V$$

With a constant c< 1. We proceed as follows:

$$\left|T_{\varepsilon}(u_1) - T(u_2)\right|^2 = \left|u_1 - u_2 - \varepsilon\left(A(u_1) - A(u_2)\right)\right|^2$$

$$= \left\langle u_1 - u_2 - \varepsilon\left(A(u_1) - A(u_2)\right), u_1 - u_2 - \varepsilon\left(A(u_1) - A(u_2)\right)\right\rangle$$

$$= \left|u_1 - u_2\right|^2 - 2\varepsilon\left\langle u_1 - u_2, A(u_1) - A(u_2)\right\rangle + \varepsilon^2\left|A(u_1) - A(u_2)\right|^2$$

$$\leq \left|u_1 - u_2\right|^2\left(1 - 2\varepsilon\alpha + \varepsilon^2 M^2\right).$$

By monotonicity property of A we have $\left|A(u_2) - A(u_1)\right| \geq \alpha\left|u_2 - u_1\right|^2$

Therefore,

$$\left|A(u_1) - A(u_2)\right| = \left|\left[\sum_i\left(\frac{\partial}{\partial x_i}\left(a(x_i, x_2)\right)\frac{\partial u_2}{\partial x_i} - a(x_i, u_1)\frac{\partial u_1}{\partial x_i}\right)\right]\right|$$

But

$$\frac{\partial}{\partial x_i}\frac{a(x_i, u_2)}{\partial x} = \frac{\partial a}{\partial x_i}\frac{\partial u_2}{\partial x} - \frac{a\partial u}{\partial x_i} - \frac{\partial u}{\partial x_i}\frac{\partial u_1}{\partial x_x} - \frac{\partial a}{\partial x_i}\frac{\partial u_1}{\partial x}. \tag{7}$$

Hence

$$\left|A(u_1) - A(u_2)\right| \leq 2L\left|\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x_i}\right|$$

$$\leq 2L\left[\left|u_2 - u_1\right| + \left|\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x}\right|\right]$$

$$= m\left|u_1 - u_2\right|_V.$$

If we take $1 - 2\varepsilon\alpha + m^2\varepsilon^2 < 1 \Leftrightarrow \varepsilon^2 - 2\varepsilon\alpha < 0$ since $\varepsilon > 0,\ \varepsilon - 2\alpha > 0 \Leftrightarrow 0 < \alpha < \dfrac{\varepsilon}{2}$.Therefore, if $\alpha = \min\left(1, \dfrac{\varepsilon}{2}\right)$.Then T is contractive and V is closed and bounded and T is Lipchitz then T is a continuous map. V

is closed and convert then by Banach fixed point theorem there exists a fix point to T_ε such that

$$T_\varepsilon(u) = u$$

But

$$T_\varepsilon u = u - \varepsilon(Au - b) = u$$

$$\Rightarrow Au - b = 0$$

u is also the fixed point of A which is in fact, the solution to the functional equation $A(u) = b$. □

Let us drop the assumption of Calderon-Zygmund condition ([6] & [7]), use the idea of monotonicity of $a(x,u)$ and the Lax-Milgram theorem to show that the solution e.q. (1) exists uniquely as the solution of abstract minimization problem.

Corollary 1

Let

$$a(u,v) = \int_\Omega \sum_{i=1}^{n} a(x_i, u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx - \int_\Omega gv\,dx$$

$$b^* = \int_\Omega fv\,d\alpha + \int_\Gamma hv\,ds + \sum_{i=1}^{n} v_i I_i.$$

And define the Energy function to the eq. (1) as

$$J(v) = \frac{1}{2}\int_\Omega a(x,u)\frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i}dv - \frac{1}{2}\int_\Omega gv\,dx \tag{8}$$

Assume that there exists a constant $m > 0$ such that the following growth property is satisfied

$$\sum_{1\le\{i,j\}\le m} a\left(x_i,\xi_j\right)\xi_i\xi_j \ge m\left|\xi\right|^2,\ \forall\ \xi,\xi_i,\xi_j \in R^n,\ x_i \in R^n.$$

Then there exists a unique solution u to the eq. (1) such that $u = \inf_{v\in V} J(v)$.

Proof:

We can justify the existence of weak solution to the eq. (1) as follows:

$$\int_\Omega \sum a(x,u)\frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i} - \int_\Omega gv\,dx = b^* \tag{9}$$

Where $b \in L_2(\Omega)$, clearly $a: H\times H \to V = H_0(\Omega)$ is bilinear form, by the Poincare inequality and monotonicity assumption on it. Hence by Lax-Milgram theorem, there exists a uniform solution to the eq. (1) such that

$$u = \min_{v\in H_0(\Omega)} J(v)$$

Where

$$J(v) = \frac{1}{2}\int_\Omega \sum_i a(x_i,u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} - \frac{1}{2}\int_\Omega gv\,dx - b^*.$$

Theorem 2

Let the following conditions be satisfied:

There exists constants $\eta^*, N, m > 0$ such that

H_1:

$$N = \sup_{|x|+|u|<1}\left|a\left(x,u\right)\right| < \infty \text{ and } \sup_{x,u\in V}\left|g\left(x,\max u\right)\right| < \infty$$

Such that

$$\eta^* = \frac{1}{2}\min\left[N\left\|u\right\|_{L_2}, M\left\|v\right\|_{L_2}\right].$$

Then the weak solution u = 0 of the eq. (1) is uniformly weakly bounded.

Proof:
By Theorem 1, there exists unique weak solution to the eq. (1) such that

$$u = u(x, u_0) = \min_{v \in V} J(v)$$

Where

$$J(v) = \frac{1}{2}\int \sum_i a(x_i, u)\frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i}dx - \frac{1}{2}\int_\Omega gv\,dx - b^*.$$

Therefore,

$$\left|u\right| \le \left|\min_{v \in J} J(v)\right|$$

$$\le \frac{1}{2}\left|\min\left[\int_\Omega a(x,u)\frac{\partial u}{\partial x_1}\cdot\frac{\partial v}{\partial x_1}dx - \frac{1}{2}\int gv\,dx\right]\right|$$

$$\le \frac{1}{2}\min\left[N\left\|u\right\|_{L_2} + M\left\|v\right\|_{L_2}\right] = \eta^*.$$

Given $\alpha > 0$ such that $|x| < \alpha$ and choose $\beta > \eta^*$ then $|u(x,u)| \le \beta$, therefore, the weak solution $u = u(x,u)$ of eq. (1) is uniformly weakly bounded.

Lemma 4
Let the following conditions be satisfied:

H_1: There exists k_i, $i = 1, 2, 3$ such that $|f| \le k_1 v$, $|h^*| \le k_2 v$ and $|I_1| \le k_3 v$

H_1: There exists at least one v such that

$\frac{1}{2}\int\sum_i a(x_i,u)\frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i}dx \geq \frac{1}{2}\int_\Omega gv\,dx > b^*$ where $b^* = \int_\Omega fv\,dx + \int_{\Gamma_1} h^* v\,dx + \sum_i v_i I_i$.

Then $J(v) > 0, J(x_k + \Delta x_k) \leq J(x_k)$ and $D^+ J(v) < 0, \forall\ v \in V$.

Where $\{x_k\}, k = 1, 2, \ldots$ are impulsive points for the system.

Proof:

For J (v) > 0 it is straight forward using the conditions in H_1 & H_2 of the hypotheses of the lemma

$$J(v) = \frac{1}{2}\int_\Omega \sum_i a(x_i,u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} - \int_\Omega gv\,dx$$

$$= \left[\int_\Omega fv\,dx + \int_\Gamma h^* v\,ds + \sum_{i=1}^n v_i I_i\right]$$

From H_1,

$$\int_\Omega fv\,dx + \int_{\Gamma_1} h^* v\,ds + \sum_{i=1}^n v_i I_i$$

$$\leq k_1\int_\Omega v^2\,dx + k_2\int_\Gamma v^2\,ds + k_3\sum v_i^2 \leq 0.$$

From H_2, it follows that $J(v) > 0$.

Furthermore,

$$D^+ J(v) = \limsup_{h\to 0^+}\frac{J(v+h) - J(v)}{h}$$

$$= -\frac{1}{2}\int\left(g + 2 + 2h^*\right)dx - \sum I_1 < 0$$

Since $g \geq 0,\ h^* \geq 0$. Hence the proof. □

Remark 2
We note that $u = u(x,u)$ will be uniformly weak bounded in the cone $L_C = \{v \in L_2 : v \geq 0\}$ given $\alpha > 0$ such that $|u_0| < \alpha$ if we choose $\beta > 0$ such that $\eta^* < \beta$ note that η^* is independent of α.

Without loss of generality, we will consider the following one-dimensional example:

Example 1
Consider one dimensional heat flow system give as

$$-c_p \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} - g(x, \max u) = f(x,u), x \neq x_k, k = 0,1,2,\ldots$$

$$\Delta u = e^{-\lambda x_k} I(x_k)$$

$$u = 0 \text{ on } \partial\Omega$$

c_p is the thermal heat capacity of system and $\lambda > 0$ some impulsive thermal heat constant. It easy to see that $\varphi(x,u) = -c_p x + ku + \varphi, \varphi(u)$ $v = a(x, u)$ is transform the eqn. (1) into eqn. (10). $\varphi(u)$ is purely function of u.

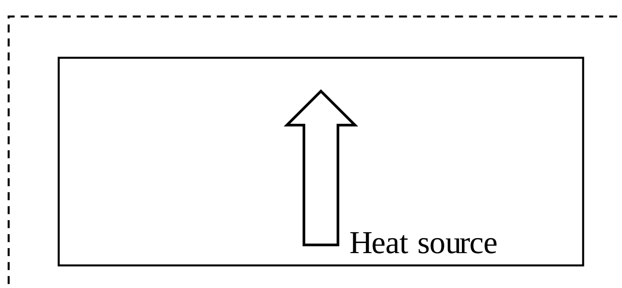

Fig.1: Impulsive heat compartment

The energy function of the system is

$$J(v) = \frac{1}{2}\int_{\Omega}\left(-c\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial x^2}\right)\frac{\partial v}{\partial x}dx - \frac{1}{2}\int_{\Omega} vg\,dx - \int_{\Omega} vf\,dx$$

Let investigate the heat distribution in the system in the following theorem

Theorem 2

Suppose that the following conditions are satisfied:

$g_1(x, \max u) = \dfrac{\alpha \max u}{1 - x^2}$ and $g_2(x, \max u) = \dfrac{\alpha x}{1 - x \max u + x^2(\max u)^2}$ Such that

(i) $-c_p\dfrac{\partial u}{\partial x} + k\dfrac{\partial^2 u}{\partial x^2} \geq 0$ in Ω

(ii) $\dfrac{\partial v}{\partial x} \geq 0$ in Ω

(iii) $\dfrac{1}{2}g + f \leq 0$ in Ω

Then $J(v) \geq 0$ in $\overline{\Omega} = \Omega \bigcup \partial\Omega$ and the temperature u is such that

(iv) $0 < u < \dfrac{2}{\alpha}(x^2 - 1)f, x > 1$

And

(v) $0 < u \leq \dfrac{2 - \alpha x}{2x(1 + x)f} \leq 1$

Furthermore, the weak solution to the eq. (1) will be weakly uniformly bounded if $h^* < 0$ in Γ.

Proof

Let u be the weak solution to the model if all the conditions (i-iii) are satisfied then $J(v) \geq 0$ in Ω and $\int_{\Gamma} hvdx = 0$ then $J(v) \geq 0$ in $\overline{\Omega}$.

From (iii) $\frac{1}{2}g + f = \frac{\alpha \max u}{2(1-x^2)} + f \leq 0$. it implies that $\max u \leq \frac{2}{\alpha}(x^2-1)f$ for $x > 1$.

But the temperature u is always such that

$0 \leq u \leq \max u$, therefore $u < \frac{2}{\alpha}(x^2-1)$ .By similar argument

$\frac{1}{2}g + f = \frac{\alpha x}{2(1 - x\max u - x^2(\max u)^2)} + f \leq 0$.

This implies that

$$-\alpha x \geq \left(-2 + 2x\max u + 2x^2(\max u)^2\right)$$

$$\geq \left(-2 + 2xu + 2x^2 u\right) f.$$

Since $y^2 \leq y$ for $0 \leq y < 1, y \in R^+$ hence $(\max u)^2 \leq \max u$ if $0 \leq u \leq \max u < 1$.

Therefore $0 \leq u \leq \max u < \frac{2 - \alpha x}{2x(1+x)f} < 1$ .Hence $J(v) \geq 0$ for every positive source value $f, \alpha > 0, x \geq 0$.

Therefore

$$D^+_{(1)}J(v) = \lim_{h \to 0^+} \frac{J(v+h) - J(v)}{h}$$

$$= -\frac{1}{2}\int_\Omega g\,dx - \int_\Omega f dx + \int_\Gamma h^* ds - \sum_i I(x_i)$$

Which will be negative definite if $h^* < 0$ in Γ .Since g, f and I_i are nonnegative in Ω and assume that $J(x_k + \Delta x_k) \leq J(x_k)$. where x_k are the impulse points for the system.

Therefore, the weak solution u of the impulsive temperature model is ultimately bounded in Ω for given source f .The function containing maximum temperature for the system can be kept at bounded threshold value using impulsive heat source to control the temperature of compartments like incubators, swimming pools, nuclear reactors etc.

5. Conclusion

Theory of impulsive partial differential equations is richly endowed with many applications to real life problems. In this paper we considered its application to heat system for which temperature is required to be control by impulsive device using maximum simulated temperature. This kind of problem has many applications in engineering, nuclear science and biotechnology etc. which be further explored.

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