International Journal of Analysis and Applications
ISSN 2291-8639
Volume 13, Number 1 (2017), 93-97
http://www.etamaths.com

SOME EQUIVALENCE THEOREMS ON ABSOLUTE SUMMABILITY METHODS

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ABSTRACT. In this paper, we obtained necessary and sufficient conditions for the equivalence of two general summability methods. Some new and known results are also obtained.

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1.1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (t_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \mid \Delta t_{n-1} \mid^k < \infty, \tag{1.3}$$

where

$$\Delta t_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{1.4}$$

If we set $\delta = 0$, then we obtain $\mid \bar{N}, p_n \mid_k$ summability (see [1]). In the special case $p_n = 1$ for all values of n, then $\mid \bar{N}, p_n; \delta \mid_k$ summability is the same as $\mid C, 1; \delta \mid_k$ summability (see [6]). Also if we take $\delta = 0$ and $k = 1$, then we get $\mid \bar{N}, p_n \mid$ summability.
Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |\Delta t_{n-1}|^k < \infty. \tag{1.5}$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |\bar{N}, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n; \delta|_k$ summability. Also, if we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |\bar{N}, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability.

Received 22nd July, 2016; accepted 18th September, 2016; published 3rd January, 2017.
2010 *Mathematics Subject Classification.* 26D15, 40F05, 40G05, 40G99, 46A45.
Key words and phrases. Riesz mean; absolute summability; Hölder inequality; equivalence theorem; Minkowski inequality; infinite series; sequence space.

2. Known Results

We say that two summability methods are equivalent if they sum the same set of series (not necessarily to the same sums). Bor and Thorpe proved the following theorems about the $\mid \bar{N}, p_n \mid_k$ and $\mid \bar{N}, q_n \mid_k$ summability methods.

Theorem 2.1 ([2]). Let (p_n) and (q_n) be positive sequences and $k \geq 1$. In order that $\mid \bar{N}, p_n \mid_k$ should be equivalent to $\mid \bar{N}, q_n \mid_k$ it is sufficient that

$$\frac{q_n P_n}{Q_n p_n} = O(1) \tag{2.1}$$

and

$$\frac{p_n Q_n}{P_n q_n} = O(1) \tag{2.2}$$

hold.

Theorem 2.2 ([4]). Let (p_n) and (q_n) be positive sequences and $k \geq 1$. In order that every $\mid \bar{N}, p_n \mid_k$ summable series be $\mid \bar{N}, q_n \mid_k$ summable it is necessary that (2.1) holds. If (2.2) holds then (2.1) is also sufficient for the conclusion.

Theorem 2.3 ([4]). Let (p_n) and (q_n) be positive sequences and $k \geq 1$. In order that $\left|\bar{N}, p_n\right|_k$ be equivalent to $\mid \bar{N}, q_n \mid_k$ it is necessary and sufficient that (2.1) and (2.2) hold.

3. Main Results

The aim of this paper is to generalize Theorem 2.2 and Theorem 2.3 for the general summability methods. Now, we shall prove the following theorems.

Theorem 3.1. Let $k \geq 1$ and $0 \leq \delta < 1/k$. (φ_n), (p_n) and (q_n) be sequences of positive numbers, and let

$$\sum_{n=v+1}^{m+1} \frac{\varphi_n^{\delta k+k-1} q_n^k}{Q_n^k Q_{n-1}} = O\left\{\varphi_v^{\delta k+k-1} \frac{q_v^{k-1}}{Q_v^k}\right\} \quad as \quad m \to \infty. \tag{3.1}$$

In order that every $\varphi - \mid \bar{N}, p_n; \delta \mid_k$ summable series be $\varphi - \mid \bar{N}, q_n; \delta \mid_k$ summable it is necessary that (2.1) holds. If (2.2) holds then (2.1) is also sufficient for the conclusion.

It should be noted that if we take $\varphi_n = \frac{P_n}{p_n}$, $\delta = 0$ for $\varphi - \mid \bar{N}, p_n; \delta \mid_k$ and $\varphi_n = \frac{Q_n}{q_n}$, $\delta = 0$ for $\varphi - \mid \bar{N}, q_n; \delta \mid_k$, then Theorem 3.1 reduces to Theorem 2.2. In this case condition (8) reduces to

$$\sum_{n=v+1}^{m+1} \frac{q_n}{Q_n Q_{n-1}} = O\left(\frac{1}{Q_v}\right) \quad as \quad m \to \infty, \tag{3.2}$$

which always exists.

It is also remarked that if we take $\varphi_n = \frac{Q_n}{q_n}$ and $q_n = 1$ for all values of n, then the condition (8) fulfils.

We need the following lemma for the proof of Theorem 3.1.

Lemma 3.2 ([3]). Let $k \geq 1$ and $A = (a_{nv})$ be an infinite matrix. In order that $A \in (l^k, l^k)$ it is necessary that

$$a_{nv} = O(1) \quad for\ all\ n, v \geq 0. \tag{3.3}$$

4. Proof of Theorem 3.1.

Firstly we prove sufficiency. Let (t_n) denote $(\bar{N}, p_n)$ mean of the series $\sum a_n$. Then, by definition, we have

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v. \tag{4.1}$$

If the series $\sum a_n$ is summable $\varphi - \left|\bar{N}, p_n; \delta\right|_k$, then

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \left|\Delta t_{n-1}\right|^k < \infty. \tag{4.2}$$

Since,

$$\begin{aligned}\Delta t_{n-1} &= \left(-\frac{1}{P_{n-1}}+\frac{1}{P_n}\right)\sum_{v=0}^{n} P_{v-1}a_v \\ &= -\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v, \quad n\geq 1, \quad (P_{-1}=0), \end{aligned} \tag{4.3}$$

we have

$$P_{n-1}a_n = -\frac{P_nP_{n-1}}{p_n}\Delta t_{n-1} + \frac{P_{n-1}P_{n-2}}{p_{n-1}}\Delta t_{n-2}. \tag{4.4}$$

That is

$$a_n = -\frac{P_n}{p_n}\Delta t_{n-1} + \frac{P_{n-2}}{p_{n-1}}\Delta t_{n-2}. \tag{4.5}$$

If (T_n) denotes the $(\bar{N}, q_n)$ mean of the series $\sum a_n$, similarly we have that

$$T_n = \frac{1}{Q_n}\sum_{v=0}^{n} q_v s_v = \frac{1}{Q_n}\sum_{v=0}^{n}(Q_n - Q_{v-1})a_v. \tag{4.6}$$

Hence

$$\Delta T_{n-1} = -\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n} Q_{v-1}a_v, \quad n\geq 1, \quad (Q_{-1}=0). \tag{4.7}$$

Since

$$a_v = -\frac{P_v}{p_v}\Delta t_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta t_{v-2},$$

by (15), we have that

$$\begin{aligned}\Delta T_{n-1} &= -\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n} Q_{v-1}\left(-\frac{P_v}{p_v}\Delta t_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta t_{v-2}\right) \\ &= \frac{q_nP_n}{p_nQ_n}\Delta t_{n-1} + \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1} Q_{v-1}\frac{P_v}{p_v}\Delta t_{v-1} - \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1} Q_v\frac{P_{v-1}}{p_v}\Delta t_{v-1} \\ &= \frac{q_nP_n}{p_nQ_n}\Delta t_{n-1} + \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{\Delta t_{v-1}}{p_v}\left(Q_{v-1}P_v - Q_vP_{v-1}\right).\end{aligned}$$

Also,

$$\begin{aligned}Q_{v-1}P_v - Q_vP_{v-1} &= Q_{v-1}P_v - Q_v\left(P_v - p_v\right) = Q_{v-1}P_v - Q_vP_v + p_vQ_v \\ &= \left(Q_{v-1}-Q_v\right)P_v + p_vQ_v = -q_vP_v + p_vQ_v,\end{aligned}$$

so that

$$\begin{aligned}\Delta T_{n-1} &= \frac{q_nP_n}{Q_np_n}\Delta t_{n-1} - \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{p_v}q_v\Delta t_{v-1} + \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1} Q_v\Delta t_{v-1} \\ &= T_{n,1} + T_{n,2} + T_{n,3}.\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\left|T_{n,r}\right|^k < \infty, \quad for \quad r = 1,2,3. \tag{4.8}$$

Firstly, by using (6) and (12), we have

$$\begin{aligned}\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left|T_{n,1}\right|^k &= \sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left|\frac{q_nP_n}{Q_np_n}\Delta t_{n-1}\right|^k\\ &= O(1)\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left|\Delta t_{n-1}\right|^k=O(1)\quad as\quad m\to\infty.\end{aligned}$$

Now, applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|T_{n,2}\right|^k &= \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{p_v}q_v\Delta t_{v-1}\right|^k\\ &\leq \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\frac{q_n^k}{Q_n^kQ_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}q_v\left|\Delta t_{v-1}\right|\right\}^k\\ &\leq \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\frac{q_n^k}{Q_n^kQ_{n-1}}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k q_v\left|\Delta t_{v-1}\right|^k\right\}\\ &\quad\times\left\{\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}q_v\right\}^{k-1}\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k q_v\left|\Delta t_{v-1}\right|^k\sum_{n=v+1}^{m+1}\frac{\varphi_n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}}\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k q_v\left|\Delta t_{v-1}\right|^k\varphi_v^{\delta k+k-1}\frac{q_v^{k-1}}{Q_v^k}\\ &= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k+k-1}\left|\Delta t_{v-1}\right|^k\\ &= O(1)\quad as\quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1.
Finally, as in $T_{n,2}$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|T_{n,3}\right|^k &= \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}Q_v\Delta t_{v-1}\right|^k\\ &\leq \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\frac{q_n^k}{Q_n^kQ_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{Q_v}{q_v}q_v\left|\Delta t_{v-1}\right|\right\}^k\\ &\leq \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\frac{q_n^k}{Q_n^kQ_{n-1}}\left\{\sum_{v=1}^{n-1}\left(\frac{Q_v}{q_v}\right)^k q_v\left|\Delta t_{v-1}\right|^k\right\}\\ &\quad\times\left\{\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}q_v\right\}^{k-1}\\ &= O(1)\sum_{v=1}^{m}\left(\frac{Q_v}{q_v}\right)^k q_v\left|\Delta t_{v-1}\right|^k\sum_{n=v+1}^{m+1}\frac{\varphi_n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}}\\ &= O(1)\sum_{v=1}^{m}\left(\frac{Q_v}{q_v}\right)^k q_v\left|\Delta t_{v-1}\right|^k\varphi_v^{\delta k+k-1}\frac{q_v^{k-1}}{Q_v^k}\\ &= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k+k-1}\left|\Delta t_{v-1}\right|^k\\ &= O(1)\quad as\quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1.
Therefore, we get

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \left|T_{n,r}\right|^k = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1, 2, 3.$$

This completes the proof of sufficiency of Theorem 3.1.
For the proof of the necessity, we consider the series to series version of (2) i.e. for $n \geq 1$, let

$$b_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad c_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v.$$

A simple calculation shows that for $n \geq 1$

$$c_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{b_v}{p_v} \left(Q_{v-1} P_v - Q_v P_{v-1}\right) + \frac{q_n P_n}{p_n Q_n} b_n.$$

From this we can write down at once the matrix A that transforms $\left(\varphi_n^{\frac{\delta k+k-1}{k}} b_n\right)$ into $\left(\varphi_n^{\frac{\delta k+k-1}{k}} c_n\right)$. Thus every $\varphi - \left|\bar{N}, p_n; \delta\right|_k$ summable series is $\varphi - \left|\bar{N}, q_n; \delta\right|_k$ summable if and only if $A \in (l^k, l^k)$. By Lemma 3.2, it is necessary that the diagonal terms of A must be bounded, which gives that (6) must hold.

Theorem 3.2. Let (p_n) and (q_n) be positive sequences satisfying the condition (8), $k \geq 1$ and $0 \leq \delta < 1/k$. In order that $\varphi- \mid \bar{N}, p_n; \delta \mid_k$ be equivalent to $\varphi- \mid \bar{N}, q_n; \delta \mid_k$ it is necessary and sufficient that (6) and (7) hold.

It should be remarked that if we set $\varphi_n = \frac{P_n}{p_n}$, $\delta = 0$ for $\varphi- \mid \bar{N}, p_n; \delta \mid_k$ and $\varphi_n = \frac{Q_n}{q_n}$, $\delta = 0$ for $\varphi- \mid \bar{N}, q_n; \delta \mid_k$, then Theorem 3.2 reduces to Theorem 2.3.

Proof of Theorem 3.2. Interchange the roles of (p_n) and (q_n) in Theorem 3.1.

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