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CHEBYSHEV RATIONAL APPROXIMATIONS FOR THE ROSENAU-KDV-RLW EQUATION ON THE WHOLE LINE

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Abstract. In this paper, we consider the use of a modified Chebyshev rational approximations for the Rosenau-KdV-RLW equation on the whole line with initial-boundary values. It is shown that the proposed scheme leads to optimal error estimates. Furthermore, the stability and convergence of the proposed schemes are proved. The fully discrete Chebyshev pseudo-spectral scheme is constructed. Numerical results confirm well with the theoretical results. The idea and techniques presented in this paper will be useful to solve many other problems.

1. Introduction

The application of spectral methods for approximating solutions of partial differential equations in unbounded domains has achieved great success and popularity in recent years. As a case in point, we can refer to the book by Shen et al. [32] and a more recent research paper by Foroutan et al. [18]. In general, spectral methods used for solving partial differential equations on unbounded domains can be classified into three families.

The first family is to use spectral methods associated with some orthogonal systems such as the Hermite spectral method and Laguerre spectral method (see e.g. Parand and Taghavi [22], Guo [12] and Parand et al. [21]).

The second family replaces infinite domain with [-L,L] and semi-infinite interval with [0,L] by choosing L,

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sufficiently large. This method is named the domain truncation [5].

The third family, that is used in this paper too, is based on rational approximations. For example, Boyd [6,7] and Christov [8] developed some spectral methods on infinite intervals by using mutually orthogonal systems of rational functions. This family of spectral schemes is efficient specially for solving boundary value problems; see also [14–17,34]. Overall, rational spectral methods are highly flexible, but it is hard to obtain convergence results and error estimates for those rational spectral methods. To this end, we apply convergence and error estimates in the sense of Guo [11,13]. The purpose of this paper is to develop and analyze the modified Chebyshev rational spectral methods for Rosenau-KdV-RLW equation

$$v_t - v_{xxt} + v_{xxxxt} + v_{xxx} + v_x + vv_x = 0, \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

with the initial condition:

$$v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.2}$$

and the boundary conditions:

$$\lim_{|x|\to\infty} v(x,t) = \lim_{|x|\to\infty} v_x(x,t) = \lim_{|x|\to\infty} v_{xx}(x,t) = 0, \quad t \in [0,T], \tag{1.3}$$

where $v_0(x)$ is a known smooth function and $\Omega = (-\infty, \infty)$.

The nonlinear wave is one of the most widely researched areas. The dynamics of wave behaviors can be described by several models. Some of these well-known models are Korteweg-de Vries (KdV) equation, regularized long wave (RLW) equation, and Rosenau equation. In the following section, we address a short review of these important wave models.

Korteweg-de Vries (KdV) equation as one of the well-known equations in mathematics and physics:

$$v_t + v_{xxx} + 6vv_x = 0.$$

This equation has been applied in many various fields and its application for describing wave propagation and interaction has been studied widely. There are many numerical methods that can be used to solve KdV equation such as the modified Legendre rational spectral methods applied on the semi-infinite interval [14], explicit scheme [20], Petrov-Galerkin method on the half line [36], finite-difference method [9], and solitary wave solution [1,3].

The regularized long-wave (RLW) equation (also known as Benjamin-Bona-Mahony equation) which was first introduced by Peregrine [23] to describe the development of an undular bore is presented as follows:

$$v_t - v_{xxt} + v_x + vv_x = 0.$$

The RLW equation was well studied numerically and theoretically in the literature. For instance, Biswas [2] has introduced an analytical solution of the RLW equation with power-law nonlinearity. On the other hand, Islam et al. [19] investigated the meshfree method for the numerical solution of the RLW equation.

Since the case of wave-wave and wave-wall interactions cannot be described by the KdV equation, Rosenau [29, 30] proposed an equation known as the Rosenau equation to over come this shortcoming of the KdV equation:

$$v_t + v_{xxxxt} + v_x + vv_x = 0.$$

The Rosenau equation has been the subject of several analytical and numerical studies [4,24,27] and references therein. Recently, the Rosenau-KdV-RLW equation was proposed in [25] as a conjunction of Rosenau-KdV and Rosenau-RLW equations both of which are well studied and explained with regard to shallow water waves. Also in this paper the results of Rosenau-KdV-RLW equation have been reported without considering the effects of perturbation.

For theoretical investigations, Razborova et al. [26] explored the dynamics of perturbed soliton solutions to the Rosenau-KdV-RLW equation with power-low nonlinearity. Solutions of the perturbated RosenauKdV-RLW equation are obtained [31]. Soliton perturbation theory was applied to obtain the adiabatic parameter dynamics of these solitary waves [28].

The remainder of the paper is organized as follow. In section 2, we first review some basic results on Chebyshev rational functions. Some orthogonal projections with their properties are also given in this section as they play an important role in the error analysis. In section 3, we will discuss some basic techniques employed for stability of the spectral methods in infinite domains. In section 4, we use the results in the previous sections to validate the convergence of proposed scheme and derive error estimates. In section 5, we construct the fully-discrete Chebyshev pseudo-spectral scheme, and obtain the optimum error estimate of approximation solutions. Numerical results are shown in section 6. Finally, the final section gives some concluding remarks.

2. Modified Chebyshev rational functions

This section addresses the basic notions and working tools concerning orthogonal modified Chebyshev rational functions. More specifically, it presents some properties of the modified Chebyshev rational functions. The well-known Chebyshev polynomials are orthogonal in the interval [-1,1] with respect to the weight function $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ and can be calculated through the following recurrence formula:

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, 3, ...$$

The new basis functions denoted by $\mathcal{R}_n(x)$, are defined by [14] in interval $\Omega = (-\infty, \infty)$.

$$\mathcal{R}_n(x) = \frac{1}{\sqrt{x^2+1}} T_n(\frac{x}{\sqrt{x^2+1}}), \quad n = 0, 1, 2, ... \ .$$

$\mathcal{R}_n(x)$ is the eigenfunction of the singular Sturm-Liouville problem

$$(x^2+1)^{\frac{1}{2}}\frac{d}{dx}\big((x^2+1)\frac{d}{dx}((x^2+1)^{\frac{1}{2}}w(x))\big)+n^2w(x)=0, \quad x\in\Omega,\ \ n=0,1,2,\dots\ . \tag{2.1}$$

and satisfies the following recurrence relation:

$$\begin{aligned}
\mathcal{R}_0(x) &= \frac{1}{\sqrt{x^2+1}},\\
\mathcal{R}_1(x) &= \frac{x}{x^2+1},\\
\mathcal{R}_{n+1}(x) &= (\frac{2x}{\sqrt{x^2+1}})\mathcal{R}_n(x)-\mathcal{R}_{n-1}(x), \quad n=1,2,3,\dots\ .
\end{aligned}$$

$\{\mathcal{R}_n(x)\}_{n\geq 1}$ are orthogonal with respect to the weight function $\chi(x)=1$ in the interval $(-\infty,\infty)$, with the orthogonality property:

$$\int_\Omega \mathcal{R}_n(x)\mathcal{R}_m(x)\chi(x)dx=\frac{\pi}{2}c_n\delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker function and $c_0=2, c_n=1$ for $n\geq 1$. For $1\leq p\leq\infty$, we define the space $L^p(\Omega)$ and its norm $\|w\|_{L^p(\Omega)}$ as usual. In particular $\|w\|_\infty=\|w\|_{L^\infty(\Omega)}$. For any nonnegative integer m, we define the Sobolev space as follows:

$$H^m(\Omega)=\{w:\frac{d^kw}{dx^k}\in L^2(\Omega), \quad 0\leq k\leq m\}.$$

Equipped with the inner product, the semi-norm, and the norm are defined as follow:

$$\begin{aligned}
(v,w)_m &= \sum_{k=0}^{m}(\frac{d^kv}{dx^k},\frac{d^kw}{dx^k}),\\
\mid w\mid_m &= \|\frac{d^mw}{dx^m}\|,\\
\|w\|_m &= (w,w)_m^{\frac{1}{2}}.
\end{aligned}$$

For any real $r>0$, we define the space $H^r(\Omega)$ and its norm $|w|_r$ by space interpolation. To describe approximation results, we introduce a sequence of Hilbert spaces $\{H^r_{Z_s}\}_{s\geq 1}$. For simplicity, let $\partial_x w(x)=\frac{\partial}{\partial x}w(x)$, etc. Let A be the Sturm-Liouville operator in (2.1), namely

$$Aw(x)=-(x^2+1)^{\frac{1}{2}}\frac{d}{dx}\big((x^2+1)\frac{d}{dx}((x^2+1)^{\frac{1}{2}}w(x))\big).$$

For any even integer $r\geq 0$,

$$H^r_{Z_s}(\Omega)=\{w:w\ \ \textit{is a measurable on}\ \ \Omega\ \ \textit{and}\ \ \|w\|_{r,Z_s}<\infty\},$$

where $\|w\|_{r,Z_0}=\|A^{\frac{r}{2}}w\|$, and for $s\geq 1$,

$$\|w\|_{r,Z_s}=\|(x^2+1)\partial_x((x^2+1)^{\frac{1}{2}}w))\|_{r-1,Z_{s-1}}.$$

We define these spaces and their norms by space interpolation. Now let N be any positive integer. In this case, we have:

$$\Re_N = Span\{\mathcal{R}_0, \mathcal{R}_1, ..., \mathcal{R}_N\}.$$

The $L^2(\Omega)$-orthogonal projection

$$L_N : L^2(\Omega) \longrightarrow \Re_N$$

is defined in the following way:

$$(L_N w - w, \phi) = 0\,, \quad \forall \phi \in \Re_N.$$

The $H^m(\Omega)$-orthogonal projection

$$L_N^m : H^m(\Omega) \longrightarrow \Re_N$$

is defined as:

$$(L_N^m w - w, \phi)_m = 0\,, \quad \forall \phi \in \Re_N.$$

Let C denote a generic positive constant independent of any function and N. We have the following results:

Theorem 2.1. *For any $w \in H^r_{Z_m}(\Omega)$ and $0 \leq m \leq r$,*

$$\|L_N^m w - w\|_m \leq C N^{m-r} \|w\|_{r,Z_m}.$$

Theorem 2.2. *Let $0 \leq \mu \leq m - \frac{1}{2}$, with a positive integer m. Then for any $w \in H^m_{Z_m}(\Omega) \cap H_0^m(\Omega)$,*

$$\|L_N^m w\|_{\mu,\infty} \leq C \|w\|_{r,Z_m}.$$

The above two theorems were proved in [17].

In order to analyze the modified Chebyshev rational approximation of the equation (1.1), we need another orthogonal projection. Let

$$H_0^m(\Omega) = \{w : w \in H^m(\Omega) \ \ and \ \ \partial_x^k w(0) = 0, \quad for \ \ 0 \leq k \leq m\},$$

and

$$\Re_N^{m,0} = \Re_N \cap H_0^m(\Omega).$$

We denote in particular $\Re_N^0 = \Re_N^{1,0}$ and define the orthogonal projection

$$L_N^{m,0} : H_0^m(\Omega) \longrightarrow \Re_N^{m,0}$$

by

$$(L_N^{m,0} w - w, \phi)_m = 0\,, \quad \forall \phi \in \Re_N^{m,0}.$$

Theorem 2.3. *For any $w \in H^r_{Z_m}(\Omega) \cap H^m_0(\Omega)$ and $0 \le m \le r$,*

$$\|L^{m,0}_N w - w\|_m \le CN^{m-r}\|w\|_{r,Z_m}.$$

Proof. Let

$$v(x) = \int_0^x (z^2+1)\partial_z\big((z^2+1)^{\frac{1}{2}} w(z)\big)dz,$$

and

$$\varphi(x) = \frac{1}{\sqrt{x^2+1}} \int_0^x \frac{1}{(z^2+1)} L^{m-1,0}_{N-1}\partial_z v(z)dz.$$

Clearly $\varphi \in \Re^{m,0}_N$. The desired result follows from the same argument as in the proof of Theorem 4 [17]. □

3. Stability of the Rosenau-KdV-RLW equation

This section discusses the application of modified Chebyshev rational approximations for the Rosenau-KdV-RLW equation (1.1) and presents the stability results of the proposed approach. A weak form of equation (1.1) is to find $v \in H^2_0(\Omega)$ such that for any $w \in H^2_0(\Omega)$,

$$\begin{cases} (v_t, w) + (v_{xt}, w_x) + (v_{xxt}, w_{xx}) - (v_{xx}, w_x) + (v_x, w) - \frac{1}{2}(v^2, w_x) = 0 \\ v(x,0) = v_o(x), \qquad x \in \Omega. \end{cases} \tag{3.1}$$

The Chebyshev rational spectral scheme of (3.1) is to find $v_N(t) \in \Re^0_N$ such that for any $\phi \in \Re^0_N$,

$$\begin{cases} (v_{Nt}, \phi) + (v_{Nxt}, \phi_x) + (v_{Nxxt}, \phi_{xx}) - (v_{Nxx}, \phi_x) + (v_{Nx}, \phi) - \frac{1}{2}(v_N^2, \phi_x) = 0 \\ v_N(x,0) = L^2_N v_o(x), \qquad x \in \Omega. \end{cases} \tag{3.2}$$

Since the initial value term cannot be exactly evaluated, here we consider how stable the numerical solution of (3.2) depending on the initial value term. We now analyze the stability of (3.2) in the sense of Guo [11,13]. To this end, we suppose that v_N has the error $\tilde{v}_N$. Then for any $\phi \in \Re^0_N$ and $t \in [0,T]$, we derive the following equation from (3.2) as follows:

$$\begin{cases} (v_{Nt}(t) + \tilde{v}_{Nt}(t), \phi) + (v_{Nxt}(t) + \tilde{v}_{Nxt}(t), \phi_x) + (v_{Nxxt}(t) + \tilde{v}_{Nxxt}(t), \phi_{xx}) \\ -(v_{Nxx}(t) + \tilde{v}_{Nxx}(t), \phi_x) + (v_{Nx}(t) + \tilde{v}_{Nx}(t), \phi) - \frac{1}{2}((v_N(t) + \tilde{v}_N(t))^2, \phi_x) = 0 \\ \tilde{v}_N(0) = \tilde{v}_{N,0}. \end{cases} \tag{3.3}$$

It can be demonstrated that $\tilde{v}_N$ must satisfy the following relation:

$$\begin{cases} (\tilde{v}_{Nt}(t), \phi) + (\tilde{v}_{Nxt}(t), \phi_x) + (\tilde{v}_{Nxxt}(t), \phi_{xx}) - (\tilde{v}_{Nxx}(t), \phi_x) + (\tilde{v}_{Nx}(t), \phi) \\ \qquad -\frac{1}{2}(\tilde{v}_N(t) + 2v_N(t)\tilde{v}_N(t), \phi_x) = 0, \quad \forall \phi \in \Re^0_N,\ 0 \le t \le T \\ \tilde{v}_N(0) = \tilde{v}_{N,0}. \end{cases}$$

By taking $\phi = 2\tilde{v}_N$ in (3.4), it follows that

$$\frac{d}{dt}\|\tilde{v}_N(t)\|_2^2 = 2A(t), \tag{3.4}$$

where $A(t) = (v_N(t)\tilde{v}_N(t), \tilde{v}_{Nx}(t))$. Then we have:

$$|A(t)| \leq \|v_N(t)\|_\infty \|\tilde{v}_N(t)\| |\tilde{v}_N(t)|_1 \leq \frac{1}{2}\|v_N(t)\|_\infty^2 \|\tilde{v}_N(t)\|^2 + \frac{1}{2}|\tilde{v}_N(t)|_1^2. \tag{3.5}$$

To describe the error, let $\rho(\vartheta, t) = \|\vartheta_N(t)\|_2^2$. Also let

$$|||w|||_\infty = sup_{0\leq t\leq T}\|w(t)\|_\infty,$$

and

$$M(w) = (1 + |||w|||_\infty).$$

By substituting (3.5) into (3.4) and integrating the resulting inequality for t, we find that

$$\|\tilde{v}_N(t)\|_2^2 \leq M(v_N)\int_0^t \|\tilde{v}_N(\tau)\|_2^2 d\tau + \rho(\tilde{v}_{N,0}, t). \tag{3.6}$$

Applying the Gronwall Lemma to the above inequality, we draw the following conclusion.

Theorem 3.1. *Let v_N be the solution of (3.2), and $\tilde{v}_N$ be its error induced by $\tilde{v}_{N,0}$. Then for all $0 \leq t \leq T$,*

$$\|\tilde{v}_N(t)\|_2^2 \leq \rho(\tilde{v}_{N,0}, t)e^{M(v_N)t}. \tag{3.7}$$

4. Convergence analysis and error estimate for the Rosenau-KdV-RLW equation

In this section, we will discuss some basic techniques use to estimate error bounds for spectral methods in infinite domains. Furthermore, to show that the convergence results of the spectral methods for the Rosenau-KdV-RLW equation (1.1), we use the modified Chebyshev rational approximations. For this purpose, we first present an a priori estimate:

Lemma 4.1. *If for $r \geq 2$, $v_0 \in H^r_{Z_2}(\Omega) \cap H^2_0(\Omega)$ then for any $t \in [0, T]$,*

$$\|v_N(t)\|_2^2 \leq 2\|v_0\|_2^2 + C_0 N^{4-2r}\|v_0\|_{r,Z_2}^2. \tag{4.1}$$

Proof. Taking $\phi = v_N$ in (3.2), we have

$$\frac{1}{2}\partial_t(\|v_N\|^2 + \|v_{Nx}\|^2 + \|v_{Nxx}\|^2) - (v_{Nxx}, v_{Nx}) + (v_{Nx}, v_N) - \frac{1}{2}(v_N^2, v_{Nx}) = 0. \tag{4.2}$$

Clearly

$$(v_{Nxx}, v_{Nx}) = \frac{1}{2}\int_\Omega \partial_x v_{Nx}^2(x,t)dx = 0, \tag{4.3}$$

$$(v_{Nx}, v_N) = \frac{1}{2}\int_\Omega \partial_x v_N(x,t)dx = 0, \tag{4.4}$$

$$(v_N^2, v_{Nx}) = \frac{1}{3}\int_\Omega \partial_x v_N^3(x,t)dx = 0. \tag{4.5}$$

Hence

$$\partial_t(\|v_N\|^2 + \|v_{Nx}\|^2 + \|v_{Nxx}\|^2) = 0, \tag{4.6}$$

which implies

$$\|v_N(t)\|^2 + \|v_{Nx}(t)\|^2 + \|v_{Nxx}(t)\|^2 = \|v_N(0)\|^2$$
$$+\|v_{Nx}(0)\|^2 + \|v_{Nxx}(0)\|^2 = \|L_N^2 v_0\|_2^2. \tag{4.7}$$

Next, according to Theorem 2.2 for $v_0 \in H_0^2(\Omega)$, we have

$$\|L_N^{2,0} v_0\|_2^2 \leq 2\|v_0\|_2^2 + 2\|v_0 - L_N^{2,0} v_0\|_2^2 \leq 2\|v_0\|_2^2 + C_0 N^{4-2r}\|v_0\|_{r,Z_2}^2. \tag{4.8}$$

Finally, a combination of (4.7) and (4.8) leads to the desired result. □

Therefore, the modified Chebyshev rational approximations are well suited for numerical approximation of the Rosenau-KdV-RLW equation. Indeed, we have the following result concerning the convergence and error estimate for (3.2).

Theorem 4.1. *If for $r \geq 2$,*

$$v \in L^\infty(0,T;W^{1,\infty}(\Omega)) \cap H^2(0,T;H_{Z_2}^r(\Omega)),$$

and $H_{Z_2}^r(\Omega)$, then we have

$$\|v - v_N\|_2^2 \leq C_2(v) N^{2-r}, \quad \forall t \in (0,T], \tag{4.9}$$

where $C_2(v)$ is positive constant depending only on the norms of v and v_0 in the spaces mentioned above.

Proof. For simplicity, let

$$\xi = v - L_N^{2,0} v \quad \textit{and} \quad \eta = v_N - L_N^{2,0} v. \tag{4.10}$$

Obviously, $v_N - v = \eta - \xi$. Hence, by (3.1) and (3.2) we obtain that for any $\phi \in \Re_N^0$,

$$(\eta_t,\phi) + (\eta_{xt},\phi_x) + (\eta_{xxt},\phi_{xx}) - (\eta_{xx},\phi_x) + (\eta_x,\phi) + (v_N v_{Nx} - v v_x,\phi) \tag{4.11}$$
$$= (\xi_t,\phi) + (\xi_{xt},\phi_x) + (\xi_{xxt},\phi_{xx}) - (\xi_x,\phi_{xx}) - (\xi,\phi_x).$$

Take $\phi = \eta$ in (4.11). It can be shown that

$$(\eta_{xx},\eta_x) = 0 \ , \quad (\eta_x,\eta) = 0, \tag{4.12}$$

and it is clear that

$$(\eta_{xxt},\eta_x) + (\eta_{xt},\eta_x) + (\eta_t,\eta) = \frac{1}{2}\partial_t\|\eta\|_2^2. \tag{4.13}$$

Next, by Theorems 2.2 and 2.3 we obtain

$$\begin{aligned}(v_N v_{Nx} - v v_x, \eta) &= (v_N(\eta_x - \xi_x) + v_x(\eta - \xi), \eta)\\ &\le \|\eta\|^2 + \|v_N(\eta_x - \xi_x)\|^2 + \|v_x(\eta - \xi)\|^2\\ &\le \|\eta\|^2 + C\|v_N\|_\infty^2\|\eta_x - \xi_x\|^2 + C\|v_x\|_\infty^2\|\eta - \xi\|^2\\ &\le \|\eta\|^2 + CN^{4-2r}(\|v\|_{1,\infty}^2 + \|v\|_{r,Z_2}^2). \end{aligned} \tag{4.14}$$

Using Theorem 2.3 again yields

$$(\xi_t, \eta) + (\xi_{xt}, \eta_x) + (\xi_{xxt}, \eta_{xx}) \le \|\eta\|_2^2 + \|\xi_t\|_2^2 \le \|\eta\|_2^2 + CN^{4-2r}\|\partial_t v\|_{r,Z_2}^2, \tag{4.15}$$

$$(\xi_x, \eta_{xx}) \le \|\eta_{xx}\|^2 + \|\xi_x\|^2 \le \|\eta_{xx}\|^2 + CN^{4-2r}\|v\|_{r,Z_2}^2, \tag{4.16}$$

$$(\xi, \eta_x) \le \|\eta_x\|^2 + \|\xi\|^2 \le \|\eta_x\|^2 + CN^{4-2r}\|v\|_{r,Z_2}^2. \tag{4.17}$$

In addition, Theorems 2.1 and 2.3 lead to

$$\|\eta(0)\|_2^2 \le \|v_0 - L_N^2 v_0\|_2^2 + \|v_0 - L_N^{2,0} v_0\|_2^2 \le CN^{4-2r}\|v_0\|_{r,Z_2}^2. \tag{4.18}$$

Hence, by inserting (4.12)–(4.17) in to (4.11), we obtain

$$\partial_t\|\eta\|_2^2 \le \|\eta\|_2^2 + C_1(v)N^{4-2r}(\|v\|_{r,Z_2}^2 + \|\partial_t v\|_{r,Z_2}^2), \tag{4.19}$$

where $C_1(v)$ is a positive constant depending only on
$\|v\|_{L^\infty(0,T,H^2_{Z_2}(\Omega)\cap W^{1,\infty}(\Omega))}$. Substituting (4.18) into (4.20) and integrating the result with respect to t, we obtain:

$$\|\eta\|_2^2 \le \int_0^t \|\eta(s)\|_2^2 ds + C_2(v)N^{4-2r}, \tag{4.20}$$

where $C_2(v)$ is a positive constant depending only on $C_1(v)$,
$\|v\|_{H^2(0,T,H^r_{Z_2}(\Omega))}$ and $\|v_0\|_{H^r_{Z_2}(\Omega))}$. The desired result follows from the Gronwall inequality and Theorem 2.3. □

5. Fully-discretization scheme

In this section, we describe the numerical implementation and present some numerical results. At first, we introduce the operator of interpolation at the Chebyshev- Gauss rational nodes

$$\{x_{N,j} = cot(\frac{\pi(2j+1)}{2N+2})\}_{0\le j\le N},$$

denoted by $I_N v \in \Re_N^0$ and

$$I_N v(x_{N,j}) = v(x_{N,j}), \qquad 0 \le j \le N.$$

The following Theorem can be found in [17].

Theorem 5.1. *For any $v \in H^r_{Z_1}(\Omega)$ and $0 \le \mu \le 1 \le r$,*

$$\|I_N v - v\|_\mu \le CN^{\mu-r+1}\|w\|_{r,Z_1}.$$

Let τ be the mesh size in t and set $t_k = k\tau$ $(k = 0, 1, ..., n+1 = [\frac{T}{\tau}])$. For simplicity, we denote $v^k(x) := v(x, t_k)$ by v^k and

$$v^k_{\hat{t}} = \frac{v^{k+1} - v^{k-1}}{2\tau}, \qquad \hat{v}^k_t = \frac{v^{k+1} + v^{k-1}}{2}. \tag{5.1}$$

The fully-discretization Chebyshev pseudo-spectral method for (1.1)-(1.3) is to find $v^k_N \in \Re^0_N$, such that

$$v^k_{N\hat{t}} - \frac{\partial^2}{\partial x^2} v^k_{N\hat{t}} + \frac{\partial^4}{\partial x^4} v^k_{N\hat{t}} + \frac{\partial^3}{\partial x^3}\hat{v}^k_N + \frac{\partial}{\partial x}\hat{v}^k_N + I_N(v^k_N \frac{\partial}{\partial x}\hat{v}^k_N) = 0. \tag{5.2}$$

$$v^0_N(x) = I_N v_0(x). \tag{5.3}$$

The equations (5.2) and (5.3) are satisfied at Chebyshev-Gauss rational nodes $x_\ell, \ell = 0, 1, ..., N$. Let

$$v^k - v^k_N = (v^k - L^{2,0}_N v^k) + (L^{2,0}_N v^k - v^k_N) = \xi^k + \eta^k.$$

Note that $(\xi^k, w) = 0,\ \forall w \in \Re^0_N$. Taking the inner product of (1.1) with w and subtracting (5.2) from (1.1), we get

$$\begin{aligned}(\eta^k_{\hat{t}}, w) + (\frac{\partial \eta^k_{\hat{t}}}{\partial x}, \frac{\partial w}{\partial x}) - (\frac{\partial^2 \eta^k_{\hat{t}}}{\partial x^2}, \frac{\partial^2 w}{\partial x^2}) + (\frac{\partial \hat{\eta}^k}{\partial x}, \frac{\partial^2 w}{\partial x^2}) - (\hat{\eta}^k, \frac{\partial w}{\partial x}) \\ +(I_N(v^k \frac{\partial \hat{v}^k}{\partial x} - v^k_N \frac{\partial \hat{v}^k_N}{\partial x}), w) = (\Gamma^k, w),\end{aligned} \tag{5.4}$$

where Γ^k is truncation error

$$\begin{aligned}\Gamma^k = (v^k_{\hat{t}} - \frac{\partial v^k}{\partial t}) + \frac{\partial^2}{\partial x^2}(\frac{\partial v^k}{\partial t} - v^k_{\hat{t}}) + \frac{\partial^4}{\partial x^4}(v^k_{\hat{t}} - \frac{\partial v^k}{\partial t}) + \frac{\partial^3}{\partial x^3}(v^k - \hat{v}^k) \\ + \frac{\partial}{\partial x}(v^k - \hat{v}^k) + (v^k \frac{\partial \hat{v}^k}{\partial x} - I_N(v^k_N \frac{\partial \hat{v}^k_N}{\partial x})).\end{aligned} \tag{5.5}$$

By applying Taylor's theorem, Theorems 2.3 and 5.1, we get

$$\begin{aligned}\Gamma^k = \frac{\tau^2}{12}(\frac{\partial^3 v}{\partial t^3}(t^k_1) + \frac{\partial^3 v}{\partial t^3}(t^k_2)) + \frac{\tau^2}{12}\frac{\partial^2}{\partial x^2}(\frac{\partial^3 v}{\partial t^3}(t^k_3) + \frac{\partial^3 v}{\partial t^3}(t^k_4) + \frac{\tau^2}{12}\frac{\partial^4}{\partial x^4}(\frac{\partial^3 v}{\partial t^3}(t^k_5) \\ + \frac{\partial^3 v}{\partial t^3}(t^k_6) + \frac{\tau^2}{4}\frac{\partial^3}{\partial x^3}(\frac{\partial^2 v}{\partial t^2}(t^k_7) + \frac{\partial^2 v}{\partial t^2}(t^k_2)) + \frac{\tau^2}{4}\frac{\partial}{\partial x}(\frac{\partial^2 v}{\partial t^2}(t^k_9) + \frac{\partial^2 v}{\partial t^2}(t^k_{10})) \\ +(v^k \frac{\partial \hat{v}^k}{\partial x} - I_N(v^k_N \frac{\partial \hat{v}^k_N}{\partial x})).\end{aligned}$$

where $t^{k-1} \le t^k_\ell \le t^{k+1}, \ell = 1, ..., 10$. Taking $w = \hat{\eta}^k$ in equation (5.5), we have

$$\begin{aligned}(\Gamma^k, \hat{\eta}^k) = \frac{\|\eta^{k+1}\|^2 - \|\eta^{k-1}\|^2}{4\tau} + \frac{\|\eta^{k+1}_x\|^2 - \|\eta^{k-1}_x\|^2}{4\tau} + \frac{\|\eta^{k+1}_{xx}\|^2 - \|\eta^{k-1}_{xx}\|^2}{4\tau} \\ +(\frac{\partial \hat{\eta}^k}{\partial x}, \frac{\partial^2 \hat{\eta}^k}{\partial x^2}) - (\hat{\eta}^k, \frac{\partial \hat{\eta}^k}{\partial x}) + F^k.\end{aligned} \tag{5.6}$$

where

$$F^k = (I_N(v^k \frac{\partial \hat{v}^k}{\partial x} - v_N^k \frac{\partial \hat{v}_N^k}{\partial x}), \hat{\eta}^k).$$

Clearly,

$$(\frac{\partial \hat{\eta}^k}{\partial x}, \frac{\partial^2 \hat{\eta}^k}{\partial x^2}) = 0, \qquad (\hat{\eta}^k, \frac{\partial \hat{\eta}^k}{\partial x}) = 0. \tag{5.7}$$

In the forthcoming discussions, we denote by C a positive constant independent of τ and N, and estimate F^k of (5.6). By applying Taylor's Theorem, Cauchy-Schwartz inequality and algebraic inequality, we deduce that

$$|F^k| \leq C(N^{-2r} + \|\eta^k\|^2 + \|\hat{\eta}^k\|^2 + \|\hat{\eta}^k\|_1^2). \tag{5.8}$$

Furthermore, by Theorem 2.3 and the Cauchy-Schwartz inequality, we obtain

$$|(\Gamma^k, \hat{\eta}^k)| \leq C(N^{-2r} + \tau^4 + \|\hat{\eta}^k\|^2). \tag{5.9}$$

Inserting (5.7)-(5.9) into (5.6), we get

$$\begin{aligned} \frac{\|\eta^{k+1}\|^2 - \|\eta^{k-1}\|^2}{4\tau} + \frac{\|\eta_x^{k+1}\|^2 - \|\eta_x^{k-1}\|^2}{4\tau} + \frac{\|\eta_{xx}^{k+1}\|^2 - \|\eta_{xx}^{k-1}\|^2}{4\tau} \\ \leq C(N^{-2r} + \tau^4 + \|\eta^k\|^2 + \|\hat{\eta}^k\|^2 + \|\hat{\eta}^k\|_1^2). \end{aligned} \tag{5.10}$$

Obviously, we can verify from (5.1) that

$$\|\hat{v}^k\|^2 \leq \frac{1}{2}(\|v^{k+1}\|^2 + \|v^{k-1}\|^2).$$

By summing up (5.10) for $k = 1, 2, ..., n$, we get

$$\|\eta^n\|_2^2 \leq C(\|\eta^0\|_2^2 + \tau^4 + N^{-2r}) + C\tau \sum_{k=1}^{n-1} \|\eta^n\|_2^2.$$

Note that $\|\eta^0\|_2^2 = 0$. Tanks to the discrete Gronwall inequality, we get

$$C(\tau^4 + N^{-2r}) \leq Me^{CT}.$$

$$\|\eta^n\|_2^2 \leq C(\tau^4 + N^{-2r}) e^{C(n+1)\tau}.$$

where M is the positive constant. Therefore, we have the following result.

Theorem 5.2. *Let v be the solution of (1.1) and $v \in H^r_{Z_m}(\Omega) \cap H^m_0(\Omega)$, $0 \leq m \leq r$. Also, let v_N be the solution of (5.2) in $\Re_N^0$ and τ is sufficiently small. Then there exist constant M, independent of τ and N such that for $k = 1, 2, ..., n-1$,*

$$\|v^{k+1} - v_N^{k+1}\|_2 \leq M(\tau^2 + N^{-r}).$$

6. Numerical experiments

In this section, we will conduct some numerical experiments to verify the theoretical results obtained in the previous sections. We report $\|E\|_\infty$ and the $\|E\|_2$ errors of the solution that are defined as

$$\|E\|_\infty = \max_{0\leq j\leq N} |v(x_j,t) - v_N(x_j,t)|,$$

and

$$\|E\|_2 = \frac{\sum_{j=0}^{N} |v(x_j,t) - v_N(x_j,t)|^2}{\sum_{j=0}^{N} |v(x_j,t)|^2},$$

Where $v_N(x_j,t)$ is the solution of numerical scheme (5.2) and (5.3), while $v(x_j,t)$ is the exact solution of (1.1)-(1.3). Consider the initial boundary value problem of the Rosenau-KdV-RLW equation (1.1)-(1.3) with the exact solution as follow [35]:

$$v(x,t) = \frac{5}{456}(25 - 13\sqrt{457}) sech^4 [\frac{1}{\sqrt{288}}\sqrt{-13+\sqrt{457}}(x - (\frac{241+13\sqrt{457}}{266})t)],$$

and the initial condition is set as

$$v_0(x) = \frac{5}{456}(25 - 13\sqrt{457}) sech^4 [(\frac{1}{\sqrt{288}}\sqrt{-13+\sqrt{457}})x].$$

$\|E\|_\infty$ and $\|E\|_2$ errors for various of N with $T = 20$ and $\tau = 0.1$, are reported in Table 1. We can observe from the table, that the results from the present study are in good agreement with the exact solutions.

N	$\|E\|_\infty$	$\|E\|_2$
10	6.008×10^{-2}	9.863×10^{-2}
20	4.439×10^{-3}	8.746×10^{-3}
30	3.540×10^{-5}	5.298×10^{-5}
40	3.516×10^{-5}	5.255×10^{-5}
50	3.510×10^{-5}	5.140×10^{-5}

Table 1. Norm infinity and norm relative of errors for $\tau = 0.1, T = 20$ and several values of N

Table 2 gives the errors between numerical solutions and exact solutions. We can see that when we use smaller time and temporal mesh, numerical solutions are almost the same as the exact solutions.

τ	$\|E\|_\infty$	$\|E\|_2$
0.1	1.439×10^{-5}	3.611×10^{-5}
0.05	7.160×10^{-6}	6.228×10^{-6}
0.025	5.343×10^{-7}	8.390×10^{-7}
0.0125	1.905×10^{-7}	3.110×10^{-7}
0.00625	1.777×10^{-8}	1.943×10^{-8}

Table 2. Norm infinity and norm relative of errors for $N = 35, T = 10$ and several values of τ

7. Conclusion

In this paper, we proposed the modified Chebyshev rational approximations for the Rosenau-KdV-RLW equation on the infinite intervals. The corresponding spectral scheme was constructed and its convergence was proved. To show the performance of the modified Chebyshev rational approximations, an a prior estimate was derived. We proved that the numerical solutions tend to weak solutions of (1.1) in a suitable sense. The numerical results demonstrate that the suggested method possesses high-order accuracy for the Rosenau-KdV-RLW equation on the whole line with analytical solutions.

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