International Journal of Analysis and Applications
ISSN 2291-8639
Volume 13, Number 1 (2017), 70-81
http://www.etamaths.com

A GENERALIZED AND REFINED PERTURBED VERSION OF OSTROWSKI TYPE INEQUALITIES

M. Z. SARIKAYA[1], H. BUDAK[1,*], S. ERDEN[2] AND A. QAYYUM[3]

Abstract. In this paper, we first obtain a new identity for twice differentiable mappings. Then, we establish generalized and improved perturbed version of Ostrowski type inequalities for functions whose derivatives are of bounded variation or second derivatives are either bounded or Lipschitzian.

1. Introduction

In 1938, Ostrowski first declared his inequality for different differentiable mappings. Ostrowski inequalities appear in most of the domains of Mathematics. Its importance has increased remarkably during the past few years and it is now cosidered as an independent branch of Mathematics. The development of the theory of Ostrowski inequality was initiated by Dragomir. In [6], Dragomir et al. obtained Ostrowski type inequalities for functions whose second derivatives are bounded. During the time, the growing interest for the ostrowski inequalities led to the apparition of several research papers in the area. In this sense, we mention ([6], [8], [16], [17], [19]- [21]). In recent years, modern theory of inequalities is used at large and many efforts devoted to establish several generalizations of the Ostrowski's inequalities for mappings of bounded variation ([1]- [5], [7], [9]- [13], [15], [18]). In this study, we establish some perturbed version of Ostrowski type inequalities for twice differentiable functions whose derivatives are of bounded variation or second derivatives are either bounded or Lipschitzian.

Theorem 1.1. *[14] Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty,\tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

In [9], Dragomir proved the following Ostrowski type inequalitiesfor functions of bounded variation:

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)\tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

The following lemma is required to prove the main theorem.

Received 30th July, 2016; accepted 6th October, 2016; published 3rd January, 2017.

2010 *Mathematics Subject Classification.* 26D07, 26D10, 26D15.

Key words and phrases. Ostrowski inequality; function of bounded variation; Lipschitzian mappings.

Lemma 1.1. *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on (a,b). Then for any $\lambda_i(x)$, $i=1,2,..5$ complex number the following identity holds*

$$\begin{aligned}
&\frac{1}{2(b-a)}\left\{\int_a^{\frac{a+x}{2}}(t-a)^2\left[f''(t)-\lambda_1(x)\right]dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^2\left[f''(t)-\lambda_2(x)\right]dt\right.\\
&+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left[f''(t)-\lambda_3(x)\right]dt\\
&\left.+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^2\left[f''(t)-\lambda_4(x)\right]dt+\int_{\frac{a+2b-x}{2}}^{b}(t-b)^2\left[f''(t)-\lambda_5(x)\right]dt\right\}\\
=\;&A+\frac{1}{48(b-a)}\left\{\left(x-\frac{a+b}{2}\right)^3\left[\lambda_2(x)+16\lambda_3(x)+\lambda_4(x)\right]\right.\\
&\left.-(x-a)^3\left[\lambda_1(x)+\lambda_5(x)\right]-8\left(x-\frac{3a+b}{4}\right)^3\left[\lambda_2(x)+\lambda_4(x)\right]\right\},
\end{aligned}\tag{1.3}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, where A is defined by

$$\begin{aligned}
&A\\
=\;&\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right.\\
&+\left(x-\frac{5a+3b}{8}\right)\left\{f'(a+b-x)-f'(x)\right\}\\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\right].
\end{aligned}\tag{1.4}$$

Proof. Integrating the by parts for each integral, we can easily obtain the required result (1.3). □

Now with the help of above Lemma, we will prove the following inequalities.

2. Inequalities for Functions Whose Second Derivatives are Bounded

Recall the sets of complex-valued functions:

$$\begin{aligned}
&\overline{U}_{[a,b]}(\gamma,\Gamma)\\
:\;&=\left\{f:[a,b]\to\mathbb{C}\,|\,\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}\right)-\overline{\gamma}\right]\geq 0\text{ for almost every }t\in[a,b]\right\}
\end{aligned}$$

and

$$\overline{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}\,\middle|\,\left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\text{ for a.e. }t\in[a,b]\right\}.$$

Proposition 2.1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\overline{U}_{[a,b]}(\gamma,\Gamma)$ and $\overline{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty and closed sets and*

$$\overline{U}_{[a,b]}(\gamma,\Gamma)=\overline{\Delta}_{[a,b]}(\gamma,\Gamma).$$

Let $I_1=\left[a,\frac{a+x}{2}\right]$, $I_2=\left[\frac{a+x}{2},x\right]$ $I_3=[x,a+b-x]$ $I_4=\left[a+b-x,\frac{a+2b-x}{2}\right]$ and $I_5=\left[\frac{a+2b-x}{2},b\right]$.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on (a,b) and $x\in(a,b)$. Suppose that $\gamma_i(x),\Gamma_i(x)\in\mathbb{C}$, $\gamma_i(x)\neq\Gamma_i(x)$, $i=1,2,3,4,5$ and*

$$f''\in\bigcap_{i=1}^{5}\overline{U}_{I_i}(\gamma_i,\Gamma_i)$$

then we have the inequality

$$
\begin{aligned}
& \left| A + \frac{1}{96\left(b-a\right)} \left[\left(x - \frac{a+b}{2} \right)^3 \right. \right. \\
& \times \left[\gamma_2(x) + \Gamma_2(x) + 16\left(\gamma_3(x) + \Gamma_3(x)\right) + \gamma_4(x) + \Gamma_4(x) \right] \\
& - \left(x-a\right)^3 \left[\gamma_1(x) + \Gamma_1(x) + \gamma_5(x) + \Gamma_5(x) \right] \\
& \left. \left. -8\left(x - \frac{3a+b}{4} \right)^3 \left[\gamma_2(x) + \Gamma_2(x) + \gamma_4(x) + \Gamma_4(x) \right] \right] \right| \\
\leq \; & \frac{1}{96\left(b-a\right)} \left\{ \left(x-a\right)^3 \left| \Gamma_1(x) - \gamma_1(x) \right| \right. \\
& + \left[8\left(x - \frac{3a+b}{4} \right)^3 - \left(x - \frac{a+b}{2} \right)^3 \right] \left| \Gamma_2(x) - \gamma_2(x) \right| \\
& +16\left(\frac{a+b}{2} - x \right)^3 \left| \Gamma_3(x) - \gamma_3(x) \right| \\
& + \left[8\left(x - \frac{3a+b}{4} \right)^3 - \left(x - \frac{a+b}{2} \right)^3 \right] \left| \Gamma_4(x) - \gamma_4(x) \right| \\
& \left. + \left(x-a\right)^3 \left| \Gamma_5(x) - \gamma_5(x) \right| \right\},
\end{aligned}
$$

where A is defined as in (1.4).

Proof. Taking the modulus identity (1.3) for $\lambda_i(x) = \frac{\gamma_i(x)+\Gamma_i(x)}{2}$, $i = 1, 2, ..., 5$, since $f'' \in \bigcap_{i=1}^{5} \overline{U}_{I_i}\left(\gamma_i, \Gamma_i\right)$, we have

$$
\begin{aligned}
& \left| A + \frac{1}{96\left(b-a\right)} \left[\left(x - \frac{a+b}{2} \right)^3 \right. \right. \\
& \times \left[\gamma_2(x) + \Gamma_2(x) + 16\left(\gamma_3(x) + \Gamma_3(x)\right) + \gamma_4(x) + \Gamma_4(x) \right] \\
& - \left(x-a\right)^3 \left[\gamma_1(x) + \Gamma_1(x) + \gamma_5(x) + \Gamma_5(x) \right] \\
& \left. \left. -8\left(x - \frac{3a+b}{4} \right)^3 \left[\gamma_2(x) + \Gamma_2(x) + \gamma_4(x) + \Gamma_4(x) \right] \right] \right| \\
\leq \; & \frac{1}{2\left(b-a\right)} \left\{ \int_a^{\frac{a+x}{2}} \left(t-a\right)^2 \left| f''(t) - \frac{\gamma_1(x)+\Gamma_1(x)}{2} \right| dt \right. \\
& + \int_{\frac{a+x}{2}}^{x} \left(t - \frac{3a+b}{4} \right)^2 \left| f''(t) - \frac{\gamma_2(x)+\Gamma_2(x)}{2} \right| dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)^{2}\left|f''(t)-\frac{\gamma_3(x)+\Gamma_3(x)}{2}\right|dt\\
&+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^{2}\left|f''(t)-\frac{\gamma_4(x)+\Gamma_4(x)}{2}\right|dt\\
&\left.+\int_{\frac{a+2b-x}{2}}^{b}(t-b)^{2}\left|f''(t)-\frac{\gamma_5(x)+\Gamma_5(x)}{2}\right|dt\right\}\\
\leq\ & \frac{1}{96\,(b-a)}\left\{(x-a)^{3}\left|\Gamma_1(x)-\gamma_1(x)\right|\right.\\
&+\left[8\left(x-\frac{3a+b}{4}\right)^{3}-\left(x-\frac{a+b}{2}\right)^{3}\right]\left|\Gamma_2(x)-\gamma_2(x)\right|\\
&+16\left(\frac{a+b}{2}-x\right)^{2}\left|\Gamma_3(x)-\gamma_3(x)\right|\\
&\left[8\left(x-\frac{3a+b}{4}\right)^{3}-\left(x-\frac{a+b}{2}\right)^{3}\right]\left|\Gamma_4(x)-\gamma_4(x)\right|\\
&\left.+(x-a)^{3}\left|\Gamma_5(x)-\gamma_5(x)\right|\right\}.
\end{aligned}
$$

This completes the proof. □

Remark 2.1. *If we choose $x=a$ in Theorem 2.1, we obtain the inequality*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_{a}^{b}f(t)\,dt-\frac{f(a)+f(b)}{2}\right.\\
&\left.-(b-a)\frac{f'(b)-f'(a)}{8}-\frac{(b-a)^{2}}{48}\left(\gamma_3(x)+\Gamma_3(x)\right)\right|\\
\leq\ & \frac{(b-a)}{48}\left|\Gamma_3(x)-\gamma_3(x)\right|
\end{aligned}
$$

which was given by Sarikaya et al. in [15].

Corollary 2.1. *Under assumption of Theorem 2.1 with $x=\frac{a+b}{2}$, we have*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_{a}^{b}f(t)\,dt-\frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+f\left(\frac{a+3b}{4}\right)\right.\right.\\
&\left.+\frac{1}{8}(b-a)\left\{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right\}\right]\\
&-\frac{(b-a)^{2}}{768}\left[\gamma_1(x)+\Gamma_1(x)+\gamma_2(x)+\Gamma_2(x)\right.\\
&\left.\left.+\gamma_4(x)+\Gamma_4(x)+\gamma_5(x)+\Gamma_5(x)\right]\right|\\
\leq\ & \frac{(b-a)^{2}}{768}\left[\left|\Gamma_1(x)-\gamma_1(x)\right|+\left|\Gamma_2(x)-\gamma_2(x)\right|\right.\\
&\left.+\left|\Gamma_4(x)-\gamma_4(x)\right|+\left|\Gamma_5(x)-\gamma_5(x)\right|\right].
\end{aligned}
$$

Corollary 2.2. *Under assumption of Theorem 2.1 with* $x=\frac{3a+b}{4}$*, we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right.\right.\\
&+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\\
&\left.-\frac{1}{8}(b-a)\left\{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right\}\right]\\
&+\frac{(b-a)^2}{6144}\left[\gamma_1(x)+\Gamma_1(x)+\gamma_2(x)+\Gamma_2(x)\right.\\
&\left.\left.+16\left(\gamma_3(x)+\Gamma_3(x)\right)+\gamma_4(x)+\Gamma_4(x)+\gamma_5(x)+\Gamma_5(x)\right]\right|\\
\leq\ &\frac{(b-a)^2}{6144}\left[\left|\Gamma_1(x)-\gamma_1(x)\right|+8\left|\Gamma_2(x)-\gamma_2(x)\right|+16\left|\Gamma_4(x)-\gamma_4(x)\right|\right.\\
&\left.+8\left|\Gamma_4(x)-\gamma_4(x)\right|+\left|\Gamma_5(x)-\gamma_5(x)\right|\right].
\end{aligned}$$

3. Inequalities for Mappings of Bounded Variation

In this section, we establish some inequalities for function whose second derivatives are of bounded variation.

Let $f:[a,b]\to\mathbb{C}$ be a twice differentiable function on I°(I° is the interior of I) and $[a,b]\subset I^{\circ}$.Then, from (1.3), we have for

$$\lambda_1(x)=f''(a),$$
$$\lambda_2(x)=\frac{f''\left(\frac{a+x}{2}\right)+f''(x)}{2},$$
$$\lambda_3(x)=\frac{f''(x)+f''(a+b-x)}{2},$$
$$\lambda_4(x)=\frac{f''(a+b-x)+f''\left(\frac{a+2b-x}{2}\right)}{2},$$
$$\lambda_5(x)=f''(b),$$

$$\begin{aligned}
&\frac{1}{2(b-a)}\left\{\int_a^{\frac{a+x}{2}}(t-a)^2\left[f''(t)-f''(a)\right]dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^2\right.\\
&\times\left[f''(t)-\frac{f''\left(\frac{a+x}{2}\right)+f''(x)}{2}\right]dt\\
&+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left[f''(t)-\frac{f''(x)+f''(a+b-x)}{2}\right]dt\\
&+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^2\left[f''(t)-\frac{f''(a+b-x)+f''\left(\frac{a+2b-x}{2}\right)}{2}\right]dt\\
&\left.+\int_{\frac{a+2b-x}{2}}^{b}(t-b)^2\left[f''(t)-f''(b)\right]dt\right\}
\end{aligned}$$

$$\begin{aligned}
&= A+\frac{1}{48\left(b-a\right)}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^{3}\right. && (3.1)\\
&\quad\times\left\{f''\left(\frac{a+x}{2}\right)+17\left(f''\left(x\right)+f''\left(a+b-x\right)\right)+f''\left(\frac{a+2b-x}{2}\right)\right\}\\
&\quad-\left(x-a\right)^{3}\left[f''\left(a\right)+f''\left(b\right)\right]-4\left(x-\frac{3a+b}{4}\right)^{3}\\
&\quad\left.\times\left\{f''\left(\frac{a+x}{2}\right)+f''\left(x\right)+f''\left(a+b-x\right)+f''\left(\frac{a+2b-x}{2}\right)\right\}\right]
\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, where A is defined as in (1.4).

Theorem 3.1. *Let $f:[a,b]\to\mathbb{C}$ be a twice differentiable function on I° (I° is the interior of I) and $[a,b]\subset I^{\circ}$. If the second derivative f'' is of bounded variation on $[a,b]$, then we have*

$$\begin{aligned}
&\left|A+\frac{1}{48\left(b-a\right)}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^{3}\right.\right. && (3.2)\\
&\times\left\{f''\left(\frac{a+x}{2}\right)+17\left(f''\left(x\right)+f''\left(a+b-x\right)\right)+f''\left(\frac{a+2b-x}{2}\right)\right\}\\
&-\left(x-a\right)^{3}\left[f''\left(a\right)+f''\left(b\right)\right]-4\left(x-\frac{3a+b}{4}\right)^{3}\\
&\left.\left.\times\left\{f''\left(\frac{a+x}{2}\right)+f''\left(x\right)+f''\left(a+b-x\right)+f''\left(\frac{a+2b-x}{2}\right)\right\}\right]\right|\\
\leq\ &\frac{1}{48\left(b-a\right)}\left\{\left(x-a\right)^{3}\bigvee_{a}^{\frac{a+x}{2}}\left(f''\right)\right.\\
&+\left[8\left(x-\frac{3a+b}{4}\right)^{3}-\left(x-\frac{a+b}{2}\right)^{3}\right]\bigvee_{\frac{a+x}{2}}^{x}\left(f''\right)\\
&+8\left(\frac{a+b}{2}-x\right)^{3}\bigvee_{x}^{a+b-x}\left(f''\right)\\
&+\left[8\left(x-\frac{3a+b}{4}\right)^{3}-\left(x-\frac{a+b}{2}\right)^{3}\right]\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}\left(f''\right)\\
&\left.+\left(x-a\right)^{3}\bigvee_{\frac{a+2b-x}{2}}^{b}\left(f''\right)\right\},
\end{aligned}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, where A is defined as in (1.4).

Proof. From (3.1), we find that

$$\begin{aligned}
&\left|A+\frac{1}{48\left(b-a\right)}\left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^{3}\right.\right.\\
&\times\left\{f''\left(\frac{a+x}{2}\right)+17\left(f''\left(x\right)+f''\left(a+b-x\right)\right)+f''\left(\frac{a+2b-x}{2}\right)\right\}\\
&-\left(x-a\right)^{3}\left[f''\left(a\right)+f''\left(b\right)\right]-4\left(x-\frac{3a+b}{4}\right)^{3}\\
&\left.\left.\times\left\{f''\left(\frac{a+x}{2}\right)+f''\left(x\right)+f''\left(a+b-x\right)+f''\left(\frac{a+2b-x}{2}\right)\right\}\right]\right|
\end{aligned}$$

$$\leq \frac{1}{2(b-a)}\left\{\int_a^{\frac{a+x}{2}}(t-a)^2\left|f''(t)-f''(a)\right|dt\right.$$
$$+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^2\left[f''(t)-\frac{f''\left(\frac{a+x}{2}\right)+f''(x)}{2}\right]dt$$
$$+\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left[\left|f''(t)-\frac{f''(x)+f''(a+b-x)}{2}\right|\right]dt$$
$$+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^2\left|f''(t)-\frac{f''(a+b-x)+f''\left(\frac{a+2b-x}{2}\right)}{2}\right|dt$$
$$\left.+\int_{\frac{a+2b-x}{2}}^{b}(t-b)^2\left|f''(t)-f''(b)\right|dt\right\}.$$

Since f'' is of bounded variation on $[a,b]$, we get

$$\left|f''(t)-f''(a)\right|\leq\bigvee_a^t(f'')$$

for $t\in\left[a,\frac{a+x}{2}\right]$

$$\left|f''(t)-\frac{f''\left(\frac{a+x}{2}\right)+f''(x)}{2}\right|\leq\frac{1}{2}\bigvee_{\frac{a+x}{2}}^{x}(f'')<\bigvee_{\frac{a+x}{2}}^{x}(f'')$$

for $t\in\left[\frac{a+x}{2},x\right]$

$$\left|f''(t)-\frac{f''(x)+f''(a+b-x)}{2}\right|\leq\frac{1}{2}\bigvee_{x}^{a+b-x}(f'')$$

for $t\in[x,a+b-x]$

$$\left|f''(t)-\frac{f''(a+b-x)+f''\left(\frac{a+2b-x}{2}\right)}{2}\right|\leq\frac{1}{2}\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f'')<\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f'')$$

for $t\in\left[a+b-x,\frac{a+2b-x}{2}\right]$

$$\left|f''(t)-f''(b)\right|\leq\bigvee_t^b(f'')$$

for $t\in\left[\frac{a+2b-x}{2},b\right]$.

Thus, using the elementary analysis operations, we deduce desired inequality (3.2) which completes the proof. □

Remark 3.1. *If we choose $x=a$ in (3.2), then we get the result proved by Sarikaya et al. [15].*

Corollary 3.1. *Under assumption of Theorem 3.1 with $x = \frac{a+b}{2}$, we have the inequality*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{4} \left[f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+b}{2}\right) + f\left(\frac{a+3b}{4}\right) \right.\right. \\
&+ \frac{1}{8}(b-a) \left\{ f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right\} \Bigg] \\
&- \frac{(b-a)}{384} \left[f''(a) + f''(b) + f''\left(\frac{a+b}{2}\right) \right. \\
&\left.\left. + \frac{1}{2} \left[f''\left(\frac{a+3b}{4}\right) + f''\left(\frac{3a+b}{4}\right) \right] \right] \right| \\
\leq\ & \frac{1}{384} \bigvee_a^b (f'').
\end{aligned}$$

4. Inequalities for Lipschitzian Mappings

In this section we obtain some inequalities for function whose second derivatives are Lipschitzian. We say that the function $g : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ if

$$|g(t) - g(s)| \leq L\,|t - s|$$

for any $t, s \in [a,b]$.

Theorem 4.1. *Let $f : [a,b] \to \mathbb{C}$ be a twice differantiable function on (a,b). If the second derivative f'' is a Lipschitzian mapping with the constant $L > 0$,then we have the inequality*

$$\begin{aligned}
&\left| A + \frac{1}{48(b-a)} \left[\left(x - \frac{a+b}{2} \right)^3 \right.\right. \\
&\times \left[f''\left(\frac{3a+b}{4}\right) + 16 f''\left(\frac{a+b}{2}\right) + f''\left(\frac{a+3b}{4}\right) \right] \\
&- (x-a)^3 \left[f''(a) + f''(b) \right] \\
&\left.\left. - 8\left(x - \frac{3a+b}{4} \right)^3 \left[f''\left(\frac{3a+b}{4}\right) + f''\left(\frac{a+3b}{4}\right) \right] \right] \right| \\
\leq\ & \frac{L}{128(b-a)} \left\{ 2(x-a)^4 + sgn\left(\frac{3a+b}{4} - x \right) \right. \\
&\times \left[16 \left(x - \frac{3a+b}{4} \right)^4 - \left(x - \frac{a+b}{2} \right)^4 \right] \\
&\left. + 31 \left(x - \frac{a+b}{2} \right)^4 + 16 \left(x - \frac{3a+b}{4} \right)^4 \right],
\end{aligned} \tag{4.1}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$, where A is defined as in (1.4).

Proof. If we take the $\lambda_1 = f''(a)$, $\lambda_2 = f''\left(\frac{3a+b}{4}\right)$, $\lambda_3 = f''\left(\frac{a+b}{2}\right)$, $\lambda_4 = f''\left(\frac{a+3b}{4}\right)$ and $\lambda_5 = f''(b)$ in equality (1.3), we have

$$
\begin{aligned}
&\frac{1}{2(b-a)}\left\{\int_a^{\frac{a+x}{2}} (t-a)^2\left[f''(t)-f''(a)\right]dt+\right. \qquad (4.2)\\
&\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^2\left[f''(t)-f''\left(\frac{3a+b}{4}\right)\right]dt\\
&+\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left[f''(t)-f''\left(\frac{a+b}{2}\right)\right]dt\\
&+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^2\left[f''(t)-f''\left(\frac{a+3b}{4}\right)\right]dt\\
&\left.+\int_{\frac{a+2b-x}{2}}^{b}(t-b)^2\left[f''(t)-f''(b)\right]dt\right\}\\
=\;&\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right.\\
&+\left(x-\frac{5a+3b}{8}\right)\left\{f'(a+b-x)-f'(x)\right\}\\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\right]\\
&+\frac{1}{48(b-a)}\left[\left(x-\frac{a+b}{2}\right)^3\right.\\
&\times\left[f''\left(\frac{3a+b}{4}\right)+16f''\left(\frac{a+b}{2}\right)+f''\left(\frac{a+3b}{4}\right)\right]\\
&-(x-a)^3\left[f''(a)+f''(b)\right]\\
&\left.-8\left(x-\frac{3a+b}{4}\right)^3\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]\right]
\end{aligned}
$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Since f'' is Lipschitzian, taking the madulus in (4.2), we have

$$
\begin{aligned}
&\left|A+\frac{1}{48(b-a)}\left[\left(x-\frac{a+b}{2}\right)^3\right.\right.\\
&\times\left[f''\left(\frac{3a+b}{4}\right)+16f''\left(\frac{a+b}{2}\right)+f''\left(\frac{a+3b}{4}\right)\right]\\
&-(x-a)^3\left[f''(a)+f''(b)\right]
\end{aligned}
$$

$$
\begin{aligned}
& -8\left(x-\frac{3a+b}{4}\right)^{3}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]\Bigg]\Bigg| \\
\leq\ & \frac{L}{128\,(b-a)}\left\{2\,(x-a)^{4}+sgn\left(\frac{3a+b}{4}-x\right)\right. \\
& \times\left[16\left(x-\frac{3a+b}{4}\right)^{4}-\left(x-\frac{a+b}{2}\right)^{4}\right] \\
& \left.+31\left(x-\frac{a+b}{2}\right)^{4}+16\left(x-\frac{3a+b}{4}\right)^{4}\right] \\
\leq\ & \frac{L}{2\,(b-a)}\left\{\int_{a}^{\frac{a+x}{2}}(t-a)^{3}\,dt+\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|^{3}dt+\int_{x}^{a+b-x}\left|\frac{a+b}{2}-t\right|^{3}dt\right. \\
& \left.+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(\frac{a+3b}{4}-t\right)^{3}dt+\int_{\frac{a+2b-x}{2}}^{b}(b-t)^{3}\,dt\right\}.
\end{aligned}
$$

If we calculate the above five integrals, then we obtain the inequality (4.1). Thus proof is completed. □

Corollary 4.1. *Under assumption of Theorem 4.1 with $x=a$, we get the inequality*

$$
\begin{aligned}
& \left|\frac{1}{b-a}\int_{a}^{b}f(t)\,dt-\frac{f(a)+f(b)}{2}\right. \\
& \left.-(b-a)\frac{f'(b)-f'(a)}{8}-\frac{(b-a)^{2}}{24}f''\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{1}{64}(b-a)^{3}L.
\end{aligned}
$$

Corollary 4.2. *Under assumption of Theorem 4.1 with $x=\frac{a+b}{2}$, we get the inequality*

$$
\begin{aligned}
& \left|\frac{1}{b-a}\int_{a}^{b}f(t)\,dt-\frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+f\left(\frac{a+3b}{4}\right)\right.\right. \\
& \left.+\frac{1}{8}(b-a)\left\{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right\}\right] \\
& +\frac{(b-a)^{2}}{384}\left[f''(a)+f''(b)+f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right] \\
\leq\ & \frac{1}{512}(b-a)^{3}L.
\end{aligned}
$$

Corollary 4.3. *Under assumption of Theorem 4.1 with $x=\frac{3a+b}{4}$, we get the inequality*

$$\begin{aligned}
& \left| \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right)\right.\right. \\
& \left. -\frac{1}{8}(b-a)\left\{ f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right)\right\}\right] \\
& -\frac{1}{3072}(b-a)^2\left[f''(a) + f''\left(\frac{3a+b}{4}\right) + 16 f''\left(\frac{a+b}{2}\right)\right. \\
& \left.\left. + f''\left(\frac{a+3b}{4}\right) + f''(b)\right]\right| \\
\leq\ & \frac{17}{2^{14}}(b-a)^3 L.
\end{aligned}$$

References

[1] H. Budak and M. Z. Sarikaya, *A new Ostrowski type inequality for functions whose first derivatives are of bounded variation*, Moroccan J. Pure Appl. Anal., 2(1)(2016), 1–11.

[2] H. Budak and M.Z. Sarikaya, *A companion of Ostrowski type inequalities for mappings of bounded variation and some applications*, RGMIA Research Report Collection, 19(2016), Article ID 24.

[3] H. Budak, M.Z. Sarikaya and A. Qayyum, *Improvement in companion of Ostrowski type inequalities for mappings whose first derivatives are of bounded variation and application*, RGMIA Research Report Collection, 19(2016), Article ID 25.

[4] H. Budak, M.Z. Sarikaya and S.S. Dragomir, *Some perturbed Ostrowski type inequality for twice differentiable functions*, RGMIA Research Report Collection, 19(2016), Article ID 47.

[5] H. Budak and M. Z. Sarikaya, *Some perturbed Ostrowski type inequality for functions whose first derivatives are of bounded variation*, RGMIA Research Report Collection, 19 (2016), Article ID 54.

[6] S. S. Dragomir and N.S. Barnett, *An Ostrowski type inequality for mappings whose second derivatives are bounded and applications*, RGMIA Research Report Collection, 1(2)(1998) , $69-77$.

[7] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bulletin of the Australian Mathematical Society, 60(1) (1999), 495-508.

[8] S. S. Dragomir and A. Sofo, *An integral inequality for twice differentiable mappings and application*, Tamkang J. Math., 31(4) 2000, 257-266.

[9] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Mathematical Inequalities & Applications, 4(1) (2001), 59–66.

[10] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, International Journal of Nonlinear Analysis and Applications, 5(1) (2014), 89-97.

[11] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation,* Asian-European Journal of Mathematics, 8(4)(2015,), Article ID 1550069. DOI:10.1142/S1793557115500692

[12] S. S. Dragomir, *Perturbed Companions of Ostrowski's Inequality for Functions of Bounded Variation,* RGMIA Research Report Collection, 17(2014), Article ID 1.

[13] W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, (2012).

[14] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[15] M. Z. Sarikaya, H. Budak, T. Tunc, S. Erden and H. Yaldiz, *Perturbed companion of Ostrowski type inequality for twice differentiable functions*, RGMIA Research Report Collection, 19 (2016), Article ID 59.

[16] E. Set and M. Z. Sarikaya, *On a new Ostrowski-type inequality and related results*, Kyungpook Mathematical Journal, 54(2014), 545-554.

[17] J. Park, *Some Companions of an Ostrowski-like Type Inequality for Twice Differentiable Functions,* Applied Mathematical Sciences, 8 (47) (2014), 2339 - 2351.

[18] M. Liu, Y. Zhu and J. Park, *Some companions of perturbed Ostrowski-type inequalities based on the quadratic kernel function with three sections and applications,* J. of Ineq. and Applications, 2013 (2013), Article ID 226.

[19] A. Qayyum, M. Shoaib and I. Faye, *Companion of Ostrowski-type inequality based on 5-step quadratic kernel and applications*, Journal of Nonlinear Science and Applications, 9 (2016), 537–552.

[20] A. Qayyum, I. Faye and M. Shoaib, A companion of Ostrowski Type Integral Inequality using a 5-step kernel With Some Applications, Filomat, in Press.

[21] A. Qayyum, M. Shoaib and I. Faye, *Derivation and applications of inequalities of Ostrowski type for n-times differentiable mappings for cumulative distribution function and some quadrature rules*, Journal of Nonlinear Sciences and Applications, 9 (2016), 1844-1857.

[1]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.

[2]Department of Mathematics, Faculty of Science, Bartin University, Bartin,Turkey.

[3]Department of Mathematics, University of Hail, P. O. Box 2440, Saudi Arabia.

*Corresponding author: hsyn.budak@gmail.com