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SOME GENERALIZED STEFFENSEN'S INEQUALITIES VIA A NEW IDENTITY FOR LOCAL FRACTIONAL INTEGRALS

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Abstract. In this study, we first give an identity for local fractional integrals. We then make use of this identity in order to derive several generalizations of the celebrated Steffensen's inequality associated with local fractional integrals. Relevant connections of the results presented in this paper with those that were proven in earlier works are also pointed out.

1. Introduction

As long ago as 1919, Steffensen [25] established the following result which is known in the literature as Steffensen's inequality.

Theorem 1.1. *Let a and b be real numbers such that $a < b$. Also let $f, g : [a,b] \to \mathbb{R}$ be integrable functions such that f is nonincreasing and, for every $x \in [a,b]$, $0 \leqq g(x) \leqq 1$. Then*

$$\int_{b-\lambda}^{b} f(x)dx \leqq \int_{a}^{b} f(x)g(x)dx \leqq \int_{a}^{a+\lambda} f(x)dx, \tag{1.1}$$

where

$$\lambda = \int_{a}^{b} g(x)dx.$$

Steffensen's inequality (1.1) happens to be the most basic inequality which deals with the comparison between integrals over a whole interval $[a,b]$ and integrals over a subset of $[a,b]$. In fact, the inequality (1.1) has attracted considerable attention and interest from mathematicians and researchers. In this connection, the interested reader is referred to a number of works (see, for example, [3], [4], [7]-[10], [13], [14], [17] and [26]) for various related integral inequalities.

Recently, Wu and Srivastava [27] proved the following inequality which is a weighted version of the inequality (1.1).

Theorem 1.2. *Let f, g and h be integrable functions defined on $[a,b]$ with f nonincreasing. Also let $0 \leqq g(x) \leqq h(x)$ for all $x \in [a,b]$. Then the following inequalities hold true:*

$$\begin{aligned}\int_{b-\lambda}^{b} f(x)h(x)dx &\leqq \int_{b-\lambda}^{b} \left(f(x)h(x) - [f(x)-f(b-\lambda)]\,[h(x)-g(x)]\right)dx\\ &\leqq \int_{a}^{b} f(x)g(x)dx\\ &\leqq \int_{a}^{a+\lambda} \big(f(x)h(x) - [f(x)-f(a+\lambda)]\,[h(x)-g(x)]\big)dx\\ &\leqq \int_{a}^{a+\lambda} f(x)h(x)dx,\end{aligned}$$

where λ is given by

$$\int_{a}^{a+\lambda} h(x)dx = \int_{a}^{b} g(x)dx = \int_{b-\lambda}^{b} h(x)dx.$$

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We recall the following identity given by Fink [1]:

$$\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right)-\frac{1}{b-a}\int_a^b f(t)dt$$
$$=\frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}\,\kappa(t.x)f^{(n)}(t)dt, \tag{1.2}$$

where

$$F_k(x)=\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a}\right) \tag{1.3}$$

and

$$\kappa(t,x)=\begin{cases} t-a & (a\leqq t\leqq x\leqq b)\\ t-b & (a\leqq x<t\leqq b)\end{cases}$$

Pečarić *et al.* [15] gave generalizations of Steffensen's inequality (1.1) via Fink's identity (1.2). Subsequently, Pečarić *et al.* [16] derived several new identities related to various generalizations of Steffensen's inequality (1.1).

2. Definitions, Notations and Preliminaries

The concepts of fractional calculus [8] and local fractional calculus (also called fractal calculus) (see, for details, [28] and [32]) are becoming increasingly useful in a wide variety of problems in mathematical, physical and engineering sciences (see, for example, the recent works [29] to [39]).

With a view to introducing the definition of the local fractional derivative and the local fractional integral, we need the following notations and preliminaries (see [28] and [32]).

For $0<\alpha\leqq 1$, we have the following α-type sets of elements:
$\mathbb{Z}^\alpha$: The α-type set of integers defined by

$$\mathbb{Z}^\alpha:=\{0^\alpha,\pm1^\alpha,\pm2^\alpha,\cdots,\pm n^\alpha,\cdots\}.$$

$\mathbb{Q}^\alpha$: The α-type set of the rational numbers defined by

$$\mathbb{Q}^\alpha:=\left\{m^\alpha : m^\alpha=\left(\frac{p}{q}\right)^\alpha \qquad (p,q\in\mathbb{Z};\ q\neq 0)\right\}.$$

$\mathbb{J}^\alpha$: The α-type set of the irrational numbers defined by

$$\mathbb{J}^\alpha:=\left\{m^\alpha : m^\alpha\neq\left(\frac{p}{q}\right)^\alpha \qquad (p,q\in\mathbb{Z};\ q\neq 0)\right\}.$$

$\mathbb{R}^\alpha$: The α-type set of the real line numbers defined by

$$\mathbb{R}^\alpha:=\mathbb{Q}^\alpha\cup\mathbb{J}^\alpha.$$

Proposition. *Let a^α, b^α and c^α belong to the set $\mathbb{R}^\alpha$ of real line numbers. Then*
(1) $a^\alpha+b^\alpha$ and $a^\alpha b^\alpha$ belong to the set $\mathbb{R}^\alpha$;
(2) $a^\alpha+b^\alpha=b^\alpha+a^\alpha=(a+b)^\alpha=(b+a)^\alpha$;
(3) $a^\alpha+(b^\alpha+c^\alpha)=(a+b)^\alpha+c^\alpha$;
(4) $a^\alpha b^\alpha=b^\alpha a^\alpha=(ab)^\alpha=(ba)^\alpha$;
(5) $a^\alpha(b^\alpha c^\alpha)=(a^\alpha b^\alpha)c^\alpha$;
(6) $a^\alpha(b^\alpha+c^\alpha)=a^\alpha b^\alpha+a^\alpha c^\alpha$;
(7) $a^\alpha+0^\alpha=0^\alpha+a^\alpha=a^\alpha$ and $a^\alpha 1^\alpha=1^\alpha a^\alpha=a^\alpha$.

The definitions of the local fractional derivative and the local fractional integral can now be given as follows.

Definition 2.1. *(see [28] and [32]) A non-differentiable function $f:\mathbb{R}\to\mathbb{R}^\alpha$ $\big(x\to f(x)\big)$ is said to be local fractional continuous at $x=x_0$ if, for any $\varepsilon>0$, there exists $\delta>0$ such that*

$$|f(x)-f(x_0)|<\varepsilon^\alpha$$

holds true for

$$|x-x_0|<\delta \qquad (\varepsilon,\delta\in\mathbb{R}).$$

If the function $f(x)$ is local continuous on the interval (a,b), we denote this property as follows:

$$f(x) \in C_\alpha(a,b).$$

Definition 2.2. *(see [28] and [32]) The local fractional derivative of $f(x)$ of order α $(0<\alpha \leqq 1)$ at $x=x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha\big(f(x)-f(x_0)\big)}{(x-x_0)^\alpha}, \tag{2.1}$$

where

$$\Delta^\alpha\big(f(x)-f(x_0)\big) \cong \Gamma(\alpha+1)\big(f(x)-f(x_0)\big).$$

If there exists

$$f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k+1 \text{ times}} f(x) \qquad (x \in I \subseteq \mathbb{R}),$$

then we write

$$f \in D_{(k+1)\alpha}(I) \qquad (k \in \mathbb{N}_0 := \{0,1,2,\cdots\} = \mathbb{N} \cup \{0\}).$$

Definition 2.3. *(see [28] and [32]) Let $f(x) \in C_\alpha[a,b]$. Then the local fractional integral of $f(x)$ of order α $(0<\alpha \leqq 1)$ is defined by*

$${}_aI_b^\alpha f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha \tag{2.2}$$

with

$$\Delta t_j = t_{j+1} - t_j \qquad \text{and} \qquad \Delta t = \max\{\Delta t_1, \Delta t_2, \cdots, \Delta t_{N-1}\},$$

where $[t_j, t_{j+1}]$ $(j=0,1,\cdots,N-1)$ and

$$a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$$

is a partition of the interval $[a,b]$.

Clearly, we find from Definition 2.3 that

$${}_aI_b^\alpha f(x) = \begin{cases} 0 & (a=b) \\ \\ -\,{}_bI_a^\alpha f(x) & (a<b). \end{cases} \tag{2.3}$$

If, for any $x \in [a,b]$, there exists ${}_aI_x^\alpha f(x)$, then we denote it simply as follows:

$$f(x) \in I_x^\alpha[a,b].$$

Lemma 2.1. *(see [28] and [32]) It is asserted that*

(i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}$;

(ii) $\dfrac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha}(dx)^\alpha = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right)$ $(k \in \mathbb{R})$.

Lemma 2.2. *(see [28] and [32]) Each of the following assertions holds true.*

(1) Local fractional integration is anti-differentiation: Suppose that

$$f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b].$$

Then

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) Local fractional integration by parts: Suppose that

$$f(x), g(x) \in D_\alpha[a,b] \qquad \text{and} \qquad f^{(\alpha)}(x),\ g^{(\alpha)}(x) \in C_\alpha[a,b].$$

Then

$${}_aI_b^\alpha f(x) g^{(\alpha)}(x) = f(x)g(x)\big|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x) g(x).$$

We next recall that Sarikaya *et al.* [23] proved the following generalized Steffensen inequality for local fractional integrals.

Theorem 2.1. *Let $f(x), g(x) \in I_x^\alpha [a,b]$ such that the function $f(x)$ is non-increasing and $0 \leqq g(x) \leqq 1$ on $[a,b]$ $(a<b)$. Then*

$$ {}_{b-\lambda}I_b^\alpha f(x) \leqq {}_aI_b^\alpha f(x)g(x) \leqq {}_aI_{a+\lambda}^\alpha f(x), \tag{2.4}$$

where

$$\lambda^\alpha = \Gamma(1+\alpha)\ {}_aI_b^\alpha g(x).$$

Sarikaya *et al.* [23] also stated the following identities which we shall use in order to prove our main results in this paper.

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)}\int_a^{a+\lambda} f(x)(dx)^\alpha - {}_aI_b^\alpha f(x)g(x) \\ &\quad= \frac{1}{\Gamma(\alpha+1)}\int_a^{a+\lambda} \left[f(x)-f\left(a+\lambda\right)\right]\left[1-g(x)\right](dx)^\alpha \\ &\qquad+ \frac{1}{\Gamma(\alpha+1)}\int_{a+\lambda}^{b} \left[f\left(a+\lambda\right)-f(x)\right]g(x)(dx)^\alpha \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &{}_aI_b^\alpha f(x)g(x) - \frac{1}{\Gamma(\alpha+1)}\int_{b-\lambda}^{b} f(x)(dx)^\alpha \\ &\quad= \frac{1}{\Gamma(\alpha+1)}\int_a^{b-\lambda} \left[f(x)-f\left(b-\lambda\right)\right]g(x)(dx)^\alpha \\ &\qquad+ \frac{1}{\Gamma(\alpha+1)}\int_{b-\lambda}^{b} \left[f\left(b-\lambda\right)-f(x)\right]\left[1-g(x)\right](dx)^\alpha. \end{aligned} \tag{2.6}$$

The interested reader is referred to several other related works including (for example) [2], [5], [6], [11], [12], [18] to [24] and [28] to [39] for the theory and applications of local fractional calculus.

In Section 3 of this paper, we give several inequalities which provide generalizations Steffensen's inequality (1.1) for local fractional integrals.

3. Main Results

We start with the following important identity for our work. Throughout this paper, $T_k(x)$ is defined by

$$T_k(x) = \frac{(n-1-k)^\alpha}{\Gamma(1+k\alpha)}\left(\frac{f^{(k\alpha)}(a)(x-a)^{k\alpha} - f^{(k\alpha)}(b)(x-b)^\alpha}{(b-a)^\alpha}\right). \tag{3.1}$$

Lemma 3.1. *Let $f^{(n-1)\alpha}(t)$ be absolutely continuous on $[a,b]$ with $f^{(n)\alpha} \in I_x^\alpha[a,b]$. Then*

$$\begin{aligned} &\frac{1}{n^\alpha}\left(\frac{f(x)}{\Gamma(1+\alpha)} + \sum_{k=1}^{n-1} F_k\right) - \frac{1}{(b-a)^\alpha\Gamma(1+\alpha)}\int_a^b f(y)(dy)^\alpha \\ &\quad= \frac{1}{n^\alpha\ (b-a)^\alpha\Gamma\big(1+(n-1)\alpha\big)[\Gamma(1+\alpha)]^2} \\ &\qquad\times \int_a^b (x-t)^{(n-1)\alpha}\ \kappa_\alpha(t,x) f^{(n\alpha)}(t)dt, \end{aligned}$$

where

$$F_k(x) = \frac{(n-k)^\alpha}{\Gamma(1+k\alpha)}\left(\frac{f^{(k-1)\alpha}(a)(x-a)^{k\alpha} - f^{(k-1)\alpha}(b)(x-b)^{k\alpha}}{(b-a)^\alpha}\right)$$

and

$$\kappa_\alpha(t,x) = \begin{cases} (t-a)^\alpha & (a \leqq t \leqq x \leqq b) \\ (t-b)^\alpha & (a \leqq x < t \leqq b). \end{cases}$$

Proof. We begin by recalling the following local fractional Taylor's formula:

$$f(x)=f(y)+\sum_{k=1}^{n-1}\frac{f^{(k\alpha)}(y)(x-y)^{k\alpha}}{\Gamma(1+k\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_a^b\frac{f^{(n\alpha)}(t)(x-t)^{(n-1)\alpha}}{\Gamma\big(1+(n-1)\alpha\big)}(dt)^\alpha,$$

which, upon integration with respect to y from $y=a$ to $y=b$, yields

$$\begin{aligned}\int_a^b(dy)^\alpha\int_y^x(dt)^\alpha&=\int_a^x(dy)^\alpha\int_y^x(dt)^\alpha+\int_x^b(dy)^\alpha\int_y^x(dt)^\alpha\\&=\int_a^x(dt)^\alpha\int_a^t(dy)^\alpha-\int_x^b(dy)^\alpha\int_x^y(dt)^\alpha\\&=\int_a^x(dt)^\alpha\int_a^t(dy)^\alpha-\int_x^b(dt)^\alpha\int_t^b(dy)^\alpha.\end{aligned}$$

Thus, by evaluating the last integral, we have

$$\begin{aligned}\frac{f(x)(b-a)^\alpha}{\Gamma(1+\alpha)}&=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(y)(dy)^\alpha+\sum_{k=1}^{n-1}I_k+\frac{1}{\Gamma\big(1+(n-1)\alpha\big)[\Gamma(1+\alpha)]^2}\\&\quad\times\int_a^b f^{(n\alpha)}(t)(x-t)^{(n-1)\alpha}\kappa_\alpha(t,x)(dt)^\alpha,\end{aligned}\tag{3.2}$$

where

$$I_k(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b\frac{f^{(k\alpha)}(y)(x-y)^{k\alpha}}{\Gamma(1+k\alpha)}(dy)^\alpha.$$

By local fractional integration by parts in the above expression for $I_k(x)$, we get

$$I_k(x)=I_{k-1}(x)-(b-a)^\alpha F_k(x)(n-k)^{-\alpha}\qquad(1\leqq k\leqq n-1).$$

Hence

$$(n-k)^\alpha\left[I_k(x)-I_{k-1}(x)\right]=-(b-a)^\alpha F_k(x).\tag{3.3}$$

The sum from $k=1$ to $k=n-1$ in (3.3) is given by

$$\sum_{k=1}^{n-1}I_k=-(b-a)^\alpha\sum_{k=1}^{n-1}F_k(x)+(n-1)^\alpha I_0.\tag{3.4}$$

Substituting from (3.4) into (3.2) and rearranging the resulting equation, we have desired inequality asserted by Lemma 3.1. □

Theorem 3.1. *Let $f:[a,b]\mapsto\mathbb{R}^\alpha$ be such that $f^{(n-1)\alpha}$ is absolutely continuous for some $n\in\mathbb{N}\setminus\{1\}$. Suppose that the functions $g,h\in I_x^\alpha[a,b]$ are such that h is positive and $0^\alpha\leqq g\leqq 1^\alpha$ on $[a,b]$. Also let*

$${}_aI_{a+\lambda}h(t)={}_aI_bg(t)h(t)$$

and the function S_1 be defined by

$$S_1(x)=\begin{cases}\dfrac{1}{\Gamma(1+\alpha)}\int_a^x\left[1-g(t)\right]h(t)(dt)^\alpha & (x\in[a,a+\lambda])\\[2ex]\dfrac{1}{\Gamma(1+\alpha)}\int_x^b g(t)h(t)(dt)^\alpha & (x\in[a+\lambda,b]).\end{cases}\tag{3.5}$$

Then

$$\begin{aligned}\frac{1}{\Gamma(1+\alpha)}\int_a^{a+\lambda} f(t)h(t)(dt)^\alpha &- \frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)g(t)h(t)(dt)^\alpha \\ &- \Gamma(1+\alpha)\sum_{k=0}^{n-2}\frac{1}{\Gamma(1+\alpha)}\int_a^b S_1(x)T_k(x)(dx)^\alpha \\ = -&\frac{1}{(b-a)^\alpha\Gamma\big(1+(n-2)\alpha\big)[\Gamma(1+\alpha)]^2} \\ &\times\int_a^b\left(\int_a^b S_1(x)(x-t)^{(n-2)\alpha}\,\kappa_\alpha(t,x)(dx)^\alpha\right)f^{(n\alpha)}(t)(dt)^\alpha. \end{aligned} \tag{3.6}$$

Proof. It is easily seen that

$$\begin{aligned}\frac{1}{\Gamma(1+\alpha)}\int_a^{a+\lambda} f(t)h(t)(dt)^\alpha &- \frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)g(t)h(t)(dt)^\alpha \\ =&\frac{1}{\Gamma(1+\alpha)}\int_a^{a+\lambda}\left[f(t)-f\left(a+\lambda\right)\right]\left[1-g(t)\right]h(t)(dt)^\alpha \\ &+\frac{1}{\Gamma(1+\alpha)}\int_{a+\lambda}^{b}\left[f\left(a+\lambda\right)-f(t)\right]g(t)h(t)(dt)^\alpha \\ =&\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t\left[1-g(x)\right]h(x)(dx)^\alpha\right)\left[f(t)-f\left(a+\lambda\right)\right]\Big|_a^{a+\lambda} \\ &-\frac{1}{\Gamma(1+\alpha)}\int_a^{a+\lambda}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t\left[1-g(x)\right]h(x)(dx)^\alpha\right)(df(t))^\alpha \\ &+\left(\frac{1}{\Gamma(1+\alpha)}\int_b^t g(x)h(x)(dx)^\alpha\right)\left[f\left(a+\lambda\right)-f(t)\right]\Big|_{a+\lambda}^{b} \\ &-\frac{1}{\Gamma(1+\alpha)}\int_{a+\lambda}^{b}\left(\frac{1}{\Gamma(1+\alpha)}\int_t^b g(x)h(x)(dx)^\alpha\right)(df(t))^\alpha \\ =&-\frac{1}{\Gamma(1+\alpha)}\int_a^{a+\lambda}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t\left[1-g(x)\right]h(x)(dx)^\alpha\right)(df(t))^\alpha \\ &-\frac{1}{\Gamma(1+\alpha)}\int_{a+\lambda}^{b}\left(\frac{1}{\Gamma(1+\alpha)}\int_t^b g(x)h(x)(dx)^\alpha\right)(df(t))^\alpha \\ =&-\frac{1}{\Gamma(1+\alpha)}\int_a^b S_1(t)d(f(t))^\alpha \\ =&-\frac{1}{\Gamma(1+\alpha)}\int_a^b S_1(x)f^{(\alpha)}(x)(dx)^\alpha. \end{aligned}$$

By applying Lemma 3.1 with $f^{(\alpha)}$ and replacing n by $n-1$ ($n\in\mathbb{N}\setminus\{1\}$), we obtain

$$\begin{aligned} f^{(\alpha)}(x) = -\Gamma(1+\alpha)\sum_{k=0}^{n-2}T_k(x) &+ \frac{1}{(b-a)^\alpha\Gamma\big(1+(n-2)\alpha\big)\Gamma(1+\alpha)} \\ &\times\int_a^b (x-t)^{(n-2)\alpha}\,\kappa_\alpha(t,x)f^{(n\alpha)}(t)(dt)^\alpha. \end{aligned} \tag{3.7}$$

Moreover, by using the equation (3.7), we get

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_a^b S_1(x) f^{(\alpha)}(x)(dx)^\alpha \\
&\quad = -\Gamma(1+\alpha) \sum_{k=0}^{n-2} \frac{1}{\Gamma(1+\alpha)} \int_a^b S_1(x) T_k(x)(dx)^\alpha \\
&\qquad + \frac{1}{(b-a)^\alpha \Gamma\big(1+(n-2)\alpha\big)[\Gamma(1+\alpha)]^2} \\
&\qquad \times \int_a^b S_1(x) \left(\int_a^b (x-t)^{(n-2)\alpha}\ \kappa_\alpha(t,x) f^{(n\alpha)}(t)(dt)^\alpha \right)(dx)^\alpha.
\end{aligned} \tag{3.8}$$

Finally, by applying Fubini's theorem for local fractional double integrals in the last term in (3.8), we arrive at the assertion (3.6) of Theorem 3.1. □

Theorem 3.2. *Let $f : [a,b] \mapsto \mathbb{R}^\alpha$ be such that $f^{(n-1)\alpha}$ is absolutely continuous for some $n \in \mathbb{N} \setminus \{1\}$. Suppose that the functions $g, h \in I_x^\alpha[a,b]$ are such that h is positive and $0^\alpha \leqq g \leqq 1^\alpha$ on $[a,b]$. Also let*

$$_{b-\lambda}I_b h(t) = {}_aI_b g(t)h(t)$$

and the function S_2 be defined by

$$S_2(x) = \begin{cases} \dfrac{1}{\Gamma(1+\alpha)} \int_a^x g(t)h(t)\ (dt)^\alpha & (x \in [a, b-\lambda]) \\ \\ \dfrac{1}{\Gamma(1+\alpha)} \int_x^b [1-g(t)]\, h(t)(dt)^\alpha & (x \in [b-\lambda, b]). \end{cases} \tag{3.9}$$

Then

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)g(t)h(t)(dt)^\alpha - \frac{1}{\Gamma(1+\alpha)} \int_{b-\lambda}^b f(t)h(t)(dt)^\alpha \\
&\qquad - \Gamma(1+\alpha) \sum_{k=0}^{n-2} \frac{1}{\Gamma(1+\alpha)} \int_a^b S_2(x) T_k(x)(dx)^\alpha \\
&\quad = -\frac{1}{(b-a)^\alpha \Gamma\big(1+(n-2)\alpha\big)[\Gamma(1+\alpha)]^2} \\
&\qquad \times \int_a^b \left(\int_a^b S_2(x)(x-t)^{(n-2)\alpha}\ \kappa_\alpha(t,x)(dx)^\alpha \right) f^{(n\alpha)}(t)(dt)^\alpha.
\end{aligned} \tag{3.10}$$

Proof. We observe that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)g(t)h(t)(dt)^\alpha - \frac{1}{\Gamma(1+\alpha)}\int_{b-\lambda}^b f(t)h(t)(dt)^\alpha \\
&\quad= \frac{1}{\Gamma(1+\alpha)}\int_a^{b-\lambda}\left[f(t)-f(b-\lambda)\right]g(t)h(t)(dt)^\alpha \\
&\qquad+ \frac{1}{\Gamma(1+\alpha)}\int_{b-\lambda}^b\left[f(b-\lambda)-f(t)\right]\left[1-g(t)\right]h(t)(dt)^\alpha \\
&\quad= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^t g(x)h(x)(dx)^\alpha\right)\left[f(t)-f(b-\lambda)\right]\Big|_a^{b-\lambda} \\
&\qquad- \frac{1}{\Gamma(1+\alpha)}\int_a^{b-\lambda}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t g(x)h(x)(dx)^\alpha\right)(df(t))^\alpha \\
&\qquad+ \left(\frac{1}{\Gamma(1+\alpha)}\int_b^t\left[1-g(x)\right]h(x)(dx)^\alpha\right)\left[f(b-\lambda)-f(t)\right]\Big|_{b-\lambda}^b \\
&\qquad- \frac{1}{\Gamma(1+\alpha)}\int_{b-\lambda}^b\left(\frac{1}{\Gamma(1+\alpha)}\int_t^b\left[1-g(x)\right]h(x)(dx)^\alpha\right)\big(df(t)\big)^\alpha \\
&\quad= -\frac{1}{\Gamma(1+\alpha)}\int_a^{b-\lambda}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^t g(x)h(x)(dx)^\alpha\right)\big(df(t)\big)^\alpha \\
&\qquad- \frac{1}{\Gamma(1+\alpha)}\int_{b-\lambda}^b\left(\frac{1}{\Gamma(1+\alpha)}\int_t^b\left[1-g(x)\right]h(x)(dx)^\alpha\right)\big(df(t)\big)^\alpha \\
&\quad= -\frac{1}{\Gamma(1+\alpha)}\int_a^b S_2(t)d\big(f(t)\big)^\alpha \\
&\quad= -\frac{1}{\Gamma(1+\alpha)}\int_a^b S_2(x)f^{(\alpha)}(x)(dx)^\alpha.
\end{aligned}$$

By making use of Lemma 3.1 with $f^{(\alpha)}$ and replacing n by $n-1$ ($n\in\mathbb{N}\setminus\{1\}$), we obtain

$$\begin{aligned}
f^{(\alpha)}(x) = -\Gamma(1+\alpha)\sum_{k=0}^{n-2}T_k(x) &+ \frac{1}{(b-a)^\alpha\Gamma\big(1+(n-2)\alpha\big)\Gamma(1+\alpha)} \\
&\times\int_a^b (x-t)^{(n-2)\alpha}\,\kappa_\alpha(t,x)f^{(n\alpha)}(t)(dt)^\alpha. \qquad (3.11)
\end{aligned}$$

Thus, by using this last equation (3.11), we get

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_a^b S_2(x)f^{(\alpha)}(x)(dx)^\alpha \\
&\quad= -\Gamma(1+\alpha)\sum_{k=0}^{n-2}\frac{1}{\Gamma(1+\alpha)}\int_a^b S_2(x)T_k(x)(dx)^\alpha \\
&\qquad+ \frac{1}{(b-a)^\alpha\Gamma\big(1+(n-2)\alpha\big)[\Gamma(1+\alpha)]^2} \\
&\qquad\times\int_a^b S_2(x)\left(\int_a^b (x-t)^{(n-2)\alpha}\,\kappa_\alpha(t,x)f^{(n\alpha)}(t)(dt)^\alpha\right)(dx)^\alpha. \qquad (3.12)
\end{aligned}$$

Finally, by applying Fubini's theorem of local fractional double integrals in the last term in (3.12), we deduce the result asserted by Theorem 3.2. □

Theorem 3.3. *Let $f:[a,b]\mapsto\mathbb{R}^\alpha$ be such that $f^{(n-1)\alpha}$ is absolutely continuous for some $n\in\mathbb{N}\setminus\{1\}$. Suppose that the functions $g,h\in I_x^\alpha[a,b]$ are such that h is positive and $0^\alpha\leqq g\leqq 1^\alpha$ on $[a,b]$. Also let*

$${}_aI_{a+\lambda}h(t) = {}_aI_b g(t)h(t)$$

and the function S_1 be defined by (3.5). If the function f is n-convex for local fractional calculus and

$$\int_a^b S_1(x)(x-t)^{(n-2)\alpha}\,\kappa_\alpha(t,x)(dx)^\alpha \leqq 0 \qquad (t\in[a,b]),$$

then the following inequality holds true:

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)g(t)h(t)(dt)^\alpha \leqq \frac{1}{\Gamma(1+\alpha)}\int_a^{a+\lambda} f(t)h(t)(dt)^\alpha - \Gamma(1+\alpha)\sum_{k=0}^{n-2}\frac{1}{\Gamma(1+\alpha)}\int_a^b S_1(x)T_k(x)(dx)^\alpha.$$

Proof. Since the function f is n-convex, we can suppose that f is n times differentiable and $f^{(n\alpha)} \geqq 0$. Using this property and the assumption (3.6) of Theorem 3.1, we get the required inequality asserted by Theorem 3.3. □

Theorem 3.4. *Let $f:[a,b]\mapsto \mathbb{R}^\alpha$ be such that $f^{(n-1)\alpha}$ is absolutely continuous for some $n\in\mathbb{N}\setminus\{1\}$. Suppose that the functions $g,h\in I_x^\alpha[a,b]$ are such that h is positive and $0^\alpha \leqq g \leqq 1^\alpha$ on $[a,b]$. Also let*

$${}_{b-\lambda}I_b h(t) = {}_aI_b g(t)h(t)$$

and the function S_2 be defined by (3.9)*. If the function f is n-convex for local fractional calculus and*

$$\int_a^b S_2(x)(x-t)^{(n-2)\alpha}\,\kappa_\alpha(t,x)(dx)^\alpha \leqq 0 \qquad (t\in[a,b]),$$

then the following inequality holds true:

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)g(t)h(t)(dt)^\alpha \geqq \frac{1}{\Gamma(1+\alpha)}\int_{b-\lambda}^b f(t)h(t)(dt)^\alpha + \Gamma(1+\alpha)\sum_{k=0}^{n-2}\frac{1}{\Gamma(1+\alpha)}\int_a^b S_2(x)T_k(x)(dx)^\alpha.$$

Proof. Since the function f is n-convex, we can suppose that f is n times differentiable and $f^{(n\alpha)} \geqq 0$. Using this property and the assumption (3.10) of Theorem 3.2, we obtain the required inequality asserted by Theorem 3.4. □

4. Concluding Remarks and Observations

The present investigation is motivated essentially by widespread applications of fractional calculus and local fractional calculus (also called fractal calculus) in a large variety of problems in mathematical, physical and engineering sciences. Here, in this paper, we have first derived an identity for local fractional integrals. We then make use of this identity in order to derive several generalizations of the celebrated Steffensen's inequality associated with local fractional integrals. We have also briefly considered relevant connections of the results presented in this paper with the results which were proven in earlier works.

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