International Journal of Analysis and Applications
ISSN 2291-8639
Volume 2, Number 2 (2013), 162-172
http://www.etamaths.com

PAIRWISE SC COMPACT SPACES

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Abstract. The main focus of this paper is to introduce the properties of SC compact spaces.

1. Introduction:

In 1963, J. C. Kelly introduced Bitopological space . Levine introduced the notion of semi open sets in topological spaces . Maheshwari and Prasad introduced semi open sets in bitopological spaces in 1977 and further properties of this notion were studied by Bose in 1981. Fukutake defined one kind of semi open sets in bitopological spaces and studied their properties in 1989.

G.L.Garg , D.Sivaraj introduced SC - compact space in 1984 . In 1969 , Viglino introduced the class of C - compact spaces as a subclass of QHC (almost compact spaces) . In 1976 , Thompson introduced the class of S closed spaces (using semi open covers) which is again a subclass of QHC spaces .

In [1] the authors, introduced the notion of pairwise extremally disconnected spaces and investigated its fundamental properties.

The main focus of this paper is to introduced the concept of pairwise SC compact spaces and to discuss the properties of SC compact spaces .

2010 Mathematics Subject Classification. 54E55.
Key words and phrases. Pairwise SC - compact space, pairwise C - compact space, $\tau_i \tau_j$ - S closed.

2. Preliminaries :

If A is a subset of X with a topology τ , then the closure of A is denoted by τ - cl (A) or cl (A) , the interior of A is denoted by τ - int (A) or int (A) and the complement of A in X is denoted by A^c . Now we shall require the following known definitions and prerequisites .

Definition 2.1 - A topological space (X , τ) is called **compact ,** if every open covering $\mathcal{C}$ of X contains a finite sub - collection that also covers X .

Definition 2.2 [7] - A topological space (X , τ) is said to be SC - Compact if for each closed subset A of X and τ - semi open over u of A $\exists$ a finite subfamily of elements of u , say V_1 , V_2 , . . . , V_n such that

$$A \subset \cup_{i=1}^{n} cl\ V_i \ .$$

Definition 2.3 : A topological space (X , τ) is said to be C - Compact if for each closed subset A of X and each open over $\mathcal{U}$ of A $\exists$ a finite subfamily of elements of $\mathcal{U}$, say V_1 , V_2 , . . . , V_n such that $A \subset \cup_{i=1}^{n} cl\ V_i$, where i , j = 1 , 2 and i $\neq$ j .

Definition 2.4 : A space (X , τ) is said to be extremally disconnected if the closure of every open set is open .

Definition 2.5 : A space (X , τ) is S - closed if every semi open cover of X has a finite subfamily whose closures cover X .

Definition 2.6 : A subset A of a topological space (X , τ) is said to be regular open if A = int [cl (A)].

Definition 2.7 : A subset A of a topological space (X , τ) is said to be semi open if A $\subseteq$ cl [int (A)].

Definition 2.8 [1] : A bitopological space (X , τ_1 , τ_2) is said to be

1. (τ_i , τ_j) - extremally disconnected [1] if τ_j - closure of every τ_i - open set is τ_j - open in X .
2. pairwise extremally disconnected if (X , τ_1 , τ_2) is (τ_i , τ_j) - extremally disconnected and (τ_j , τ_i) - extremally deisconnected.

3. Pairwise SC compact spaces :

In this section we define the pairwise SC compact spaces and Pairwise C compact spaces . Also we discussed the properties of pairwise SC compact spaces .

Definition 3.1 : A bitopological space (X , τ_1 , τ_2) is said to be Pairwise C - Compact if for each τ_i - closed subset A of X and each pairwise open over $\mathcal{U}$ of A $\exists$ a finite subfamily of elements of $\mathcal{U}$, say V_1 , V_2 , . . . , V_n such that A $\subset$ $\bigcup_{i=1}^{n} \tau_i - cl\ V_i$, where i , j = 1 , 2 and i $\neq$ j .

Proposition 3.1 : If A and B are two pairwise C - compact subsets of a bitopological space (X , τ_1 , τ_2) then A $\cup$ B is pairwise C - compact subset of X .

Proof : Given A and B are two pairwise C - compact subsets of a bitopological space (X , τ_1 , τ_2) .

To prove A $\cup$ B is pairwise C compact subset of X ,

ie) To prove that for any pairwise open cover of A $\cup$ B has a finite sub cover .

Let $\{U_i \ / \ i \in \Lambda\}$ be any pairwise open cover of A $\cup$ B , then A $\cup$ B $\subseteq$ { $\cup$ U_i / i$\in$ Λ } and therefore $A \subseteq \bigcup U_i$ and $B \subseteq \bigcup U_i$, which implies that $\{\bigcup U_i \ / \ i \in \Lambda\}$ is an pairwise open cover of A and B , where i , j = 1 , 2 and i $\neq$ j .

But A and B are pairwise SC - compact subsets , therefore there exists i_1 , i_2 , . . . , $i_n \in \Lambda$ and t_1 , t_2 , . . . , $t_n \in \Lambda$ such that $\{ U_{i_1}, U_{i_2}, \ldots, U_{i_n} \}$ and $\{ U_{t_1}, U_{t_2}, \ldots, U_{t_n} \}$ is a finite sub cover of A and B respectively , then $\{ U_{i_1}, U_{i_2}, \ldots, U_{i_n} \} \cup \{ U_{t_1}, U_{t_2}, \ldots, U_{t_n} \}$ is a finite sub cover of A $\cup$ B , therefore is an pairwise C compact subset of X . ■

Definition 3.2 : A bitopological space (X , τ_1 , τ_2) is said to be pairwise S - closed if every τ_j - open over $\mathcal{U}$ of X has a finite subfamily whose τ_i - closure covers X , where i , j = 1 , 2 and i $\neq$ j .

Definition 3.3 : A bitopological space (X , τ_1 , τ_2) is said to be Pairwise SC - compact if for each τ_i - closed subset A of X and τ_j - semi open over $\mathcal{U}$ of A $\exists$ a finite subfamily of elements of $\mathcal{U}$, say V_1 , V_2 , . . . , V_n such that

$\tau_i - cl\ (A) = A \subset \cup_{i=1}^{n} \tau_i - cl\ V_i$, where i , j = 1 , 2 and i $\neq$ j .

Example 3.1 : Consider ($\mathbb{R}$, τ_1) , where τ_1 is the co - countable topology on $\mathbb{R}$.

Then ($\mathbb{R}$, τ_1) is SC - compact … … (1)

Let τ_2 be the cofinite topology on $\mathbb{R}$.

Hence ($\mathbb{R}$, τ_2) is also SC - compact … … (2)

Thus , ($\mathbb{R}$, τ_1 , τ_2) is pairwise SC - compact from (1) and (2) .

Proposition 3.2 : If A and B are two pairwise SC - compact subsets of a bitopological space (X , τ_1 , τ_2) then A $\cup$ B is pairwise SC - compact subset of X .

Proof : Given A and B are two pairwise SC - compact subsets of a bitopological space (X , τ_1 , τ_2) .

We shall prove that A $\cup$ B is pairwise SC compact subset of X ,

We have to prove that for any τ_j - semi open cover of A $\cup$ B has a finite sub cover .

Let {U_i / i$\in \Lambda$ } be any τ_j - semi open cover of A $\cup$ B .

Then A $\cup$ B $\subseteq$ { $\cup$ U_i / i$\in \Lambda$ } and therefore $A \subseteq \cup U_i$ and B $\subseteq \cup U_i$, which implies that $\{\cup U_i\ /\ i \in \Lambda\}$ is an τ_j - semi open cover of A and B , where i , j = 1 , 2 and i $\neq$ j .

But A and B are pairwise SC - compact subsets .

Therefore there exist i_1 , i_2 , . . . , $i_n \in \Lambda$ and t_1 , t_2 , . . . , $t_n \in \Lambda$ such that $\{ U_{i_1}, U_{i_2}, \ldots, U_{i_n} \}$ and $\{ U_{t_1}, U_{t_2}, \ldots, U_{t_n} \}$ is a finite sub cover of A and B respectively , then $\{ U_{i_1}, U_{i_2}, \ldots, U_{i_n} \} \cup \{ U_{t_1}, U_{t_2}, \ldots, U_{t_n} \}$ is a finite sub cover of A $\cup$ B .

Therefore is an pairwise SC compact subset of X . ■

Theorem 3.3 : The pairwise semi *continuous* image of a pairwise SC - compact space is a pairwise SC - compact space.

Proof : Let (X , τ_1 , τ_2) be a pairwise semi compact space, and let f : (X , τ_1 , τ_2) $\rightarrow$ (Y, τ_1', τ_2') be a semi continuous , surjection .

We shall show that (Y, τ_1', τ_2') is a pairwise SC - compact space.

Let {U_i / i$\in \Lambda$ } be any τ_j - semi open cover of Y , then $\{f^{-1}(U_i)/$ i$\in \Lambda$ } is a τ_j - semi open cover of X , which is compact space.

So there exists , i_1 , i_2 , . . . , $i_n \in \Lambda$ such that the family $\left\{f^{-1}\left(U_{i_j}\right),\ j = 1,2,\dots,n\right\}$ covers X and since f is onto, then $\left\{\ U_{i_j}\ ,\ j = 1,2,\dots,n\ \right\}$ is a finite sub cover of Y.

Hence Y is a pairwise SC - compact space. ■

Theorem 3.4 : Every $\tau_i \tau_j$ - extremally disconnected , pairwise C - compact space (X , τ_1 , τ_2) is pairwise SC - compact and hence pairwise S - closed .

Proof : Let A be a τ_i - closed subset of X and let $\{A_\alpha \ / \ \alpha \in \Lambda\}$ be any τ_j - semi open cover of A , where i , j = 1 , 2 and i ≠ j .

Since the τ_i -closure of a τ_j -semi open set is τ_i τ_j - regular closed and τ_i τ_j - regular closed sets in an τ_i τ_j - extremally disconnected spaces are τ_i - open and τ_j - closed , $\{\overline{A_\alpha} \ / \ \alpha \in \Lambda\}$ is a τ_i - open τ_j - closed cover of A .

Since X is pairwise C - Compact , we have

$$A \subset \cup_{i=1}^{n} \overline{A_{\alpha_i}} \text{ for some } n \in \Lambda .$$

Hence X is pairwise SC compact . ■

Theorem 3.5 : For a bitopological space (X , τ_1 , τ_2) the following assertions are equivalent .

i) X is pairwise SC - Compact .

ii) Every τ_i - regular closed cover of each τ_j - closed subset of X has a finite sub cover .

Proof : Since every τ_i τ_j - regular closed set is τ_i τ_j - semi open , trivially i) ⇒ ii) .

Conversely , let $\{A_\alpha \ / \ \alpha \in \Lambda\}$ be any τ_j - semi open cover of a τ_i - closed subset A of X , where i , j = 1 , 2 and i ≠ j .

Since $\{\overline{A_\alpha} \ / \ \alpha \in \Lambda\}$ is a τ_i τ_j - regular closed cover of A , by (ii) , $A \subset \cup_{i=1}^{n} \overline{A_{\alpha_i}}$ for some n ∈ Λ .

Hence X is pairwise SC compact .

Hence i) ⇒ ii) . ■

Theorem 3.6 : If A is τ_i - closed subset of a pairwise SC compact space (X , τ_1 , τ_2) then A is pairwise SC compact .

Proof : Let A be a τ_i - closed subset of a pairwise SC compact space (X , τ_1 , τ_2) .

Let $\mathcal{U} = \{V_\alpha : \alpha \in \Lambda\}$ be a τ_j - semi open cover of a τ_i - closed subset B of A .

Since A is τ_i - closed , $\mathcal{U}$ is a τ_j - semi open of a τ_i - closed subset B of A .

Therefore ,

$$B \subset \cup_{i=1}^{n} \tau_i - cl\ V_i \ .$$

Hence A is a pairwise SC compact . ■

Theorem 3.7 : Every pairwise SC compact subset of a pairwise Hausdorff space is pairwise closed .

Proof : Suppose that A be a pairwise SC compact subset of a pairwise Hausdorff space X .

Since X is pairwise Hausdorff , the subspace A is pairwise Hausdorff .

By hypothesis , A is a pairwise SC compact .

Hence A is pairwise compact .

Let $x \in X - A$.

For every $a \in A$ we have $a \neq x$.

But X is pairwise Hausdorff .

Hence there exist τ_i - open nbds U_α of a and a τ_j - open nbds V_α of x such that

$U_\alpha \cap V_\alpha = \phi$ ... (1) , where i , j = 1 , 2 and $i \neq j$.

But then the collection $\varsigma = \{U_a : a \in A\}$ is an pairwise open cover of A .

But A is pairwise compact .

Hence ς has a finite sub collection $\{U_{a_1}, U_{a_2}, \dots, U_{a_n}\}$ covering A .

Put $U = U_{a_1} \cup U_{a_2} \cup \dots \cup U_{a_n}$.

Then U is an τ_i- open set with $A \subset U$.

Consider the corresponding τ_j- open sets $V_{a_1}, V_{a_2}, \dots, V_{a_n}$.

Write $V = V_{a_1} \cap V_{a_2} \cap \dots \cap V_{a_n}$.

Then V is an τ_j- open set with $x \in V$.

By virtue of (1)

$$U \cap V = \phi .$$

$$\Rightarrow x \in U \subset X - V \subset X - A$$

$$\Rightarrow x \in U \subset X - A$$

$$\Rightarrow X - A \text{ is } \tau_i\text{- open}$$

$$\Rightarrow A \text{ is } \tau_i\text{- closed .}$$

Similarly , A is τ_j - closed .

Hence A is pairwise closed . ■

Theorem 3.8 : Let X be a pairwise SC - compact space and let Y be a pairwise Hausdorff space . If $f : X \to Y$ is a pairwise continuous , pairwise irresolute bijection , then f is a pairwise semi homeomorphism .

Proof : Let A be a semi closed subset of the pairwise SC - compact space X .

Then A is pairwise SC - compact .

$\Rightarrow$ f (A) is pairwise SC – compact inY because f is pairwise irresolute .

$\Rightarrow$ f (A) is pairwise semi closed in Y .

$\Rightarrow$ $(f^{-1})^{-1}$(A) is pairwise semi closed inY .

$\Rightarrow$ f^{-1} is pairwise irresolute .

$\Rightarrow$ pairwise semi homeomorphism .

■

Theorem 3.9 : Let X be a pairwise SC compact space and Y be a pairwise Hausdorff space . If h : X $\rightarrow$ Y is a pairwise semi continuous bijection , then h is a pairwise homeomorphism .

Proof : It is given h is a pairwise semi continuous and bijection .

To P.T h is a pairwise homeomorphism .

It is enough to show that h is an pairwise open map .

Let G be any τ_i - open set in X .

Let A = X – G .

Then X – G is a τ_i - closed set in X .

Since X is pairwise SC compact , by theorem 3.6 , A is also pairwise SC compact .

Using theorem 3.3 , since h is pairwise semi continuous , we find that h (A) is pairwise SC compact in Y .

But Y is pairwise Hausdorff and by theorem it follows that h (A) is a τ_i - closed in Y.

But then Y – h (A) = h (X) – h (A) = h (X – A) = h (G) .

Therefore , h (G) is τ_i - open in Y.

Consequently , h is an τ_i - open map .

Similarly , h is an τ_j - open map .

Therefore , h is pairwise open map .

Already h is pairwise continuous and bijection .

Hence h is a pairwise homeomorphism . ∎

Theorem 3.10 : If (X , τ_1 , τ_2) is pairwise SC compact space , then both (X , τ_1) and (X , τ_2) are SC compact.

Proof : To prove (X , τ_1) is SC compact space, we must prove for any semi open cover of
X , has a finite sub cover .
Let { U_i } , i ∈ Λ be any semi open cover of X , implies { U_i }, i ∈ Λ is a semi open cover of X and since X is pairwise semi compact space, implies there exists a finite sub cover of X , so (X , τ_2) is SC compact.
And by the same way we prove (X , τ_2) is semi compact.

Theorem 3.11 : A pairwise continuous , pairwise irresolute image of an pairwise SC compact space is pairwise SC compact .

Proof :

Let f : X → Y be a pairwise continuous , pairwise irresolute function from an pairwise SC compact space X on to a space Y .

Let A be τ_i - closed in Y and { V_α / α ∈ Λ } be a τ_j - semi open cover of the τ_i – closed set f^{-1}(A) in X .

Since A is pairwise SC compact , $f^{-1}(A) \subset \cup_{i=1}^{n} \tau_i - cl\ \left(f^{-1}(V_{\alpha_i})\right)$ for some i ∈ I .

This implies that $A \subset \cup_{i=1}^{n} \tau_i - cl\ (V_{\alpha_i})$, where i = 1 , 2 and i ≠ j .

Hence Y is pairwise SC compact .

4. Comparison :

Theorem 4.1 : Pairwise compactness $\Rightarrow$ pairwise C - compactness .

Proof : Let A be a τ_i - closed subset of X .

Hence X – A is τ_i - open in X .

Let $\mathcal{U}$ be an pairwise open cover of A .

Then $\mathcal{C} = \mathcal{U} \cup (X - A)$ is an pairwise open cover of X .

But X is pairwise compact ,

$\Rightarrow \exists$ a finite subfamily $V_1 , V_2 , \ldots \ldots , V_n$ of $\mathcal{C}$ such that X = $V_1 \cup V_2 \cup \ldots \ldots \cup V_n$.

$\Rightarrow A \subset V_1 \cup V_2 \cup \ldots \ldots \cup V_n \cup (X - A)$

$\Rightarrow A \subset V_1 \cup V_2 \cup \ldots \ldots \cup V_n$, where $V_1 , V_2 , \ldots \ldots , V_n \in \mathcal{U}$.

$\Rightarrow$ X is pairwise C - compact .

Remark 4.1 : The converse of the above theorem need not be true .

Theorem 4.2 : Pairwise SC - compactness $\Rightarrow$ pairwise C - compactness .

Proof : Let A be a τ_i - closed subset of X .

Then X – A is τ_i - open in X .

We know every τ_i - open set is τ_i - semi open set , we have X – A is τ_i - semi open in X .

Let $\mathcal{U}$ be an τ_j - open cover of A .

Then $\mathcal{U}$ be an τ_j - semi open cover of X .

But X is pairwise SC compact

$\Rightarrow \exists$ a finite subfamily $V_1 , V_2 , \ldots \ldots , V_n$ of $\mathcal{C}$ such that X = $V_1 \cup V_2 \cup \ldots \ldots \cup V_n$.

$\Rightarrow A \subset V_1 \cup V_2 \cup \ldots \ldots \cup V_n \cup (X - A)$

$\Rightarrow A \subset V_1 \cup V_2 \cup \ldots \ldots \cup V_n$, where $V_1 , V_2 , \ldots \ldots , V_n \in \mathcal{U}$.

$\Rightarrow$ X is pairwise C compact .

Remark 4.2 : The converse of the above theorem need not be true .

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