International Journal of Analysis and Applications
ISSN 2291-8639
Volume 13, Number 1 (2017), 82-92
http://www.etamaths.com

ON THE BANACH SPACE TECHNIQUES IN THE EXISTENCE AND UNIQUENESS OF THE FUZZY FRACTIONAL KLEIN-GORDON EQUATION'S SOLUTION

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Abstract. In this paper, we study the existence and uniqueness of the solution of all fuzzy fractional differential equations, which are equivalent to the fuzzy integral equation. We use the Banach space techniques in this study. Also we will show that the fuzzy fractional Klein-Gordon equation (FFKGE) is equivalent to a fuzzy integral equation. We use parametric form of FFKGE with respect to definition and give new homotopy analysis method to obtain the approximate solution of this equation.

1. Introduction

In many cases of the modeling of real world phenomena, fuzzy initial value problems appear naturally, because information about the behavior of a dynamical system is uncertain. In order to obtain a more adequate model, we have to take into account these uncertainties. On the other hand, fractional calculus found many applications in various fields of physical sciences such as viscoelasticity, diffusion, control, relaxation, processes and so on [1].

The Klein-Gordon equation, which is denoted by KGE in this paper, is nowadays regarded as the relativistic form of the schrödinger equation. It affords appropriate description for spin zero particles. Since the solution of the KGE is often a complicated problem, use of pure mathematical methods is required.

Ebaid, applied Exp-function method for solving KGE [2]. Raicher *et.al* used a novel solution to the KGE in the presence of a strong rotating electric field [3]. But discussion on the fuzzy fractional Klein-Gordon equation (FFKGE) has not been done.

We consider the FFKGE with boundary conditions as follows

$$\frac{\partial^{2\gamma}\tilde{U}(x,t)}{\partial t^{2\gamma}} = \frac{\partial^{2}\tilde{U}(x,t)}{\partial x^{2}} + \tilde{U}(x,t) \quad 0<x<1\ ,\ 0<t<1\ ,\ 0<\gamma\leq 1$$

where γ is a parameter describing the order of fractional time derivative and , $\tilde{U}(x,0)=\tilde{K}(1+\sin x), \tilde{K}=(0.25\beta,-0.25\beta),\ 0\leq\beta\leq 1$ and $\tilde{U}_t(x,0)=0,\ 0<x<1,$
where $\tilde{U}(x,t):(0,1)\times[0,1)\longrightarrow\mathbb{R}_F$ is fuzzy number-valued function and $\mathbb{R}_F$ is the set of all fuzzy numbers.

In this paper we give a new homotopy analysis transform method for solving FFKGE. We use the parametric form of the above equation and find the approximate solution of this equation. The existence and uniqueness of the solution and convergence of the proposed method are proved in details. For this purpose we show that the FFKGE is equivalent to fuzzy integral equation. The concept of conformable fuzzy fractional derivative will define in this paper. Also, we define the fuzzy Banach space. Since the fixed point theorems in Banach spaces are powerful tools to prove existence and uniqueness of solution for integral equations, so in this study, we use fixed point theorem and introduce a contraction operator on a suitable Banach space.
The paper is organized as follows: in Sect. 2.2 we present some concepts and results about the fuzzy number. We explain the fractional transform and it is applied for FFKGE, then the equivalency to the

Received 13th August, 2016; accepted 7th October, 2016; published 3rd January, 2017.
2010 *Mathematics Subject Classification.* 34A12; 26A33.
Key words and phrases. fractional calculus; convergence; existence; homotopy analysis method; uniqueness; Klein-Gordon equation.

fuzzy integral equation is also proved in Sect. 3. The existence and uniqueness of the solution for fuzzy integral equation is discussed in Sect. 4, where we define the $C_F([0,1])$ and its properties and use the functional analysis methods. In Sect. 5 the Homotopy analysis transform method is applied to solve this equation.

2. Preliminaries

We now recall some definitions and symbols needed through the paper. We follow [4] in definitions and notations.

Definition 2.1. *A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ satisfying the following properties:*

a. *u is upper semicontinuous on $\mathbb{R}$,*
b. *$u(x) = 0$ outside of some interval $[c,d]$,*
c. *there are the real numbers a and b with $c \leq a \leq b \leq d$, such that u is increasing on $[c,a]$, decreasing on $[b,d]$ and $u(x) = 1$ for each $x \in [a,b]$,*
d. *u is fuzzy convex set (that is $u(\lambda x + (1-\lambda)y) \geq min\,\{u(x), u(y)\}$, $\forall x, y \in \mathbb{R}, \lambda \in [0,1]$)*

The set of all fuzzy numbers is denoted by $\mathbb{R}_F$.
In the following we introduce a concept that will be very efficient and useful to use and identification of fuzzy numbers.

Definition 2.2. *For any $u \in \mathbb{R}_F$ the $\alpha-$cut set of u is denoted by $[u]^\alpha$ and defined by $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$, where $0 \leq \alpha \leq 1$. The notation,*

$$[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]; \qquad \alpha \in [0,1]$$

refers to the lower and upper branches on u, in other words

$$\underline{u}^\alpha = min\{x \mid x \in u^\alpha\},\ \overline{u}^\alpha = max\{x \mid x \in u^\alpha\}$$

An arbitrary fuzzy number u is represented, in parametric form, by an ordered pair of functions $u = (\underline{\mathrm{u}}, \overline{u})$, which define the end points of the $\alpha-$cuts, satisfying the three conditions:

a. $\underline{\mathrm{u}}$ is a bounded non-decreasing left continuous function on $(0,1]$, and right continuous at 0,
b. $\overline{u}$ is a bounded non-increasing left continuous function on $(0,1]$, and right continuous at 0,
c. $\underline{\mathrm{u}}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

For arbitrary $u = (\underline{\mathrm{u}}, \overline{u})$, $v = (\underline{v}, \bar{v})$ and $k \geq 0$, addition $(u+v)$ and multiplication by k as $(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r)$, $(\overline{u+v})(r) = \overline{u}(r) + \bar{v}(r)$, $\underline{ku}(r) = k\underline{u}(r)$, $\overline{ku}(r) = k\overline{u}(r), k \geq 0$, and $\underline{ku}(r) = k\bar{u}(r), \overline{ku}(r) = k\underline{u}(r), \quad k < 0$ are defined.
It is well-known that the addition and multiplication operations of real numbers can be extended to $\mathbb{R}_F$. In other words, for any $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^\alpha = [u]^\alpha \oplus [v]^\alpha,\ [\lambda \odot u]^\alpha = \lambda[u]^\alpha,\ \forall \alpha \in [0,1].$$

$\ominus$ is the Hukuhara difference (H-difference), it means that $w \ominus v = u$ if and only if $u \oplus v = w$ for all $u, v, w \in \mathbb{R}_F$.

Definition 2.3. *For arbitrary fuzzy number $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ the Hausdorff distance between these fuzzy numbers given by $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$,*

$$D(u,v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \bar{v}(r)|\},$$

where D is a metric on $\mathbb{R}_F$ and has the following properties (see [1]).

a. *$D(u \oplus w, v \oplus w) = D(u,v), \forall u, v, w \in \mathbb{R}_F$,*
b. *$D(k \odot u, k \odot v) = |k| D(u,v), \forall k \in \mathbb{R}, u, v \in \mathbb{R}_F$,*
c. *$D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \forall u, v, w, e \in \mathbb{R}_F$,*
d. *$(\mathbb{R}_F, D)$ is a complete metric space.*

Definition 2.4. *The function $f : T \longrightarrow \mathbb{R}_F$ is called a fuzzy function, and the $\alpha-$cut set of f is represented by*

$$f(t;\alpha) = [\underline{f}(t;\alpha), \overline{f}(t;\alpha)]; \ \forall\alpha \in [0,1],$$

where $\underline{f}(t;\alpha) = \underline{f(t)}^{\alpha}$, $\overline{f}(t;\alpha) = \overline{f(t)}^{\alpha}$.

A fuzzy function may have fuzzy domain and fuzzy range. So the function $f : \mathbb{R}_F \longrightarrow \mathbb{R}_F$ is also a fuzzy function.

Definition 2.5. *Let $f : \mathbb{R} \to \mathbb{R}_F$ be a fuzzy function. If for an arbitrary fixed number $t_0 \in \mathbb{R}$ and $\varepsilon > 0$, exists $\delta > 0$ such that*

$$|t - t_0| < \delta \Longrightarrow D(f(t), f(t_0)) < \varepsilon \qquad t \in \mathbb{R}_F,$$

then f is said to be continuous at t_0.

Definition 2.6. *The fuzzy function $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is called to be fuzzy bounded if there exists $M > 0$ such that $\| f \|_{F.u}$:=sup $D(f(u), \hat{0}) \leq M, (u \in \mathbb{R})$.*

Proposition 2.1. *Let $f : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}_F$ be a fuzzy continuous function. Then it is fuzzy bounded.*

In the following we consider the concept of integral of a fuzzy function. The integration on the α-cut of fuzzy function is also defined.

Definition 2.7. *let $f : [a,b] \longrightarrow \mathbb{R}_F$ be a fuzzy function. For each partition $p = \{x_1, x_2, \cdots, x_m\}$ of $[a,b]$ and for arbitrary $x_{i-1} \leq \xi_i \leq x_i$, $2 \leq i \leq m$, let*

$$R_p = \sum_{i=2}^{m} f(\xi_i)(x_i - x_{i-1}).$$

The define integral of $f(x)$ over $[a,b]$ is ,

$$\int_a^b f(x;y) = \lim R_p, \quad max\ |x_i - x_{i-1}| \longrightarrow 0$$

provided that this limit exists in metric D. If the function f is continuous in the metric D, its definite integral exists [5].

Furthermore:

$$\underline{\int_a^b f(x;\alpha)} = \int_a^b \underline{f}(x;\alpha),$$

and

$$\overline{\int_a^b f(x;\alpha)} = \int_a^b \bar{f}(x;\alpha)$$

More details about the properties of the fuzzy integral are given in [5].
Now, we want to introduce a new definition of fuzzy fractional derivative as [8]:

Definition 2.8. *Let $f : [a,b] \longrightarrow \mathbb{R}_F$. The fuzzy γ-fractional integral of fuzzy-valued function f is defined as follows:*

$$(I^{\gamma} f)(x) = \int_a^x \frac{f(t)}{t^{1-\gamma}} dt, \qquad x > a \ , \ 0 < \gamma < 1.$$

let us consider the $\alpha - cut$ representation of fuzzy-valued function f is $f(x;\alpha) = [\underline{f}(x;\alpha), \bar{f}(x;\alpha)]$, for $0 \leq \alpha \leq 1$, then we indicate the fuzzy $\gamma - fractional$ integral of fuzzy-valued function f based on its lower and upper functions as follows:

Theorem 2.1. *Let $f : [a,b] \longrightarrow \mathbb{R}_F$. The fuzzy $\gamma - fractional$ integral of fuzzy-valued function f can be expressed as follows:*

$$(I^{\gamma} f)(x;\alpha) = [(I^{\gamma} \underline{f})(x;\alpha), (I^{\gamma} \bar{f})(x;\alpha)], \quad 0 < \alpha < 1$$

where

$$(I^{\gamma} \underline{f})(x;\alpha) = \int_a^x \frac{\underline{f}(t;\alpha)}{t^{1-\gamma}} dt, (I^{\gamma} \bar{f})(x;\alpha) = \int_a^x \frac{\bar{f}(t;\alpha)}{t^{1-\gamma}} dt.$$

Theorem 2.2. *For $\gamma \in [0,1]$ and $f : [a,b] \longrightarrow \mathbb{R}_F$*

$$D_t^{\gamma} f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\gamma}) \ominus f(t)}{\varepsilon}.$$

For $t>0$, $\gamma \in (0,1)$; $D_t^{\gamma} f(t)$ is called the conformable fuzzy fractional derivative of f of order γ [9, 10].

Using this kind of fractional derivative and some useful formulas can convert differential equations into integer-order differential equations.
Some properties for the suggested conformable fuzzy fractional derivative given in [8] are as follows:

$$D_t^{\gamma}(t^{\eta}) = \eta t^{\eta-\gamma}, \eta \in \mathbb{R}, \tag{2.1}$$

$$D_t^{\gamma}(f(t)g(t)) = g(t) D_t^{\gamma} f(t) \oplus f(t) D_t^{\gamma} g(t). \tag{2.2}$$

$$D_t^{\gamma} f[g(t)] = f'_g[g(t)] D_t^{\gamma} g(t) = D_g^{\gamma} f[g(t)](g'(t))^{\gamma}. \tag{2.3}$$

3. The fractional transform

In this section, we reduce FFKGE to an ordinary differential equation, then we show that this equation is equivalent to fuzzy integral equation. We consider the FFKGE with boundary conditions as follows

$$\frac{\partial^{2\gamma}\tilde{U}(x,t)}{\partial t^{2\gamma}} = \frac{\partial^2 \tilde{U}(x,t)}{\partial x^2} + \tilde{U}(x,t) \quad 0<x<1 \ , \ 0<t<1 \ , \ 0<\gamma \leq 1. \tag{3.1}$$

Now, we introduce the following transformations :

$$\bar{U}(x,t) = \bar{U}(\xi), \quad \xi = ax + \frac{bt^{\gamma}}{\gamma} \qquad a,b>0, \quad a + \frac{b}{\gamma} < 1$$

So, we can say that 0< ξ <1 by using (1) and (3) and by substituting into equation (3.1) it is derived that

$$b^2 \overline{U}'' - a^2 \overline{U}'' - \overline{U} = 0. \tag{3.2}$$

Now, we show that this fuzzy equation is equivalent to fuzzy integral equation as form:

$$\begin{aligned} \overline{U}''(\xi) = \bar{f}(\xi) \quad &\Longrightarrow \quad \overline{U}'(\xi) = \overline{U}'(0) + \int_0^{\xi} \bar{f}(z)dz \\ &\Longrightarrow \quad \overline{U}(\xi) = \overline{U}(0) + \overline{U}'(0)\xi + \int_0^{\xi}\int_0^{\xi} \bar{f}(z)dzd\xi, \end{aligned}$$

on the other hand,

$$\int_0^{\xi} \cdots \int_0^{\xi} f(\xi)(d\xi)^n = \frac{1}{(n-1)!}\int_0^{\xi} (\xi - z)^{n-1} f(z)dz,$$

therefore,

$$\overline{U}(\xi) = \overline{U}(0) + \overline{U}'(0)\xi + \int_0^{\xi} (\xi - z)\bar{f}(z)dz.$$

By substituting into equation (8) we have

$$(b^2 - a^2)\bar{f}(\xi) - \overline{U}(0) - \overline{U}'(0)\xi - \int_0^{\xi} (\xi - z)\bar{f}(z)dz = 0,$$

$$\bar{f}(\xi) = \underbrace{\left[\frac{-(\overline{U}(0) + \overline{U}'(0)\xi)}{b^2 - a^2}\right]}_{\bar{g}(\xi)} + \int_0^{\xi} \underbrace{\frac{(\xi - z)}{b^2 - a^2}}_{K(\xi,z)} \bar{f}(z)dz = 0,$$

$$\bar{f}(\xi) = \bar{g}(\xi) + \int_0^{\xi} K(\xi,z)\bar{f}(z)dz.$$

similarly,

$$\underline{f}(\xi) = \underline{g}(\xi) + \int_0^{\xi} K(\xi,z)\underline{f}(z)dz.$$

4. Existence And convergence Analysis

In this section, we prove the existence and uniqueness of the solution and convergence of the method by using the following assumptions. We consider fuzzy integral equation as follow:

$$f(\xi) = g(\xi) + \int_0^{\xi} K(\xi, z) f(z) dz,$$

where k is an arbitrary positive kernel on $[0,1] \times [0,\xi]$ and functions $f, g : [0,1] \longrightarrow \mathbb{R}_F$ are continuous fuzzy number-valued functions. We assume that K is continuous and therefore it is uniformly bounded so there exists $M_1 > 0$ such that

$$|K(\xi, z)| \leq M_1 \quad 0 \leq \xi \leq 1, \quad 0 \leq z \leq \xi.$$

Now consider the set,

$$C_F([0,1]) = \left\{ f : [0,1] \longrightarrow \mathbb{R}_F; \ f \ is \ continuous \right\},$$

which is the space of fuzzy continuous function because for $f, g \in C_F([0,1])$ and $\alpha \in \mathbb{R}, \alpha f + g$ is continuous. Regarding to Def. 2.6 we define the fuzzy uniform norm as form

$$\|f\|_{F.u} := \sup_{\xi \in [0,1]} D(f(\xi), \hat{0}).$$

In the next theorem we show that $C_F([0,1])$ is a Banach space.

Theorem 4.1. *$(C_F([0,1]), \|.\|_{F.u})$ is a Banach space.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a cauchy sequence in $C_F([0,1])$. Then for each $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $D(f_n(\xi), f_m(\xi)) < \varepsilon$ for all $n, m \geq M$, and for all $\xi \in [0,1]$.
That is,

$$\sup \ D(f_n(\xi), f_m(\xi)) = \| f_n - f_m \|_{F.u} < \varepsilon \ \ for \ all \ \ n, m \geq M.$$

This implies that for each $\xi \in [0,1]$, $\{f_n(\xi)\}$ is a cauchy sequence in the complete metric space $\mathbb{R}_F$. So there exists a function f such that $f_n(\xi) \longrightarrow f(\xi)$ for all $\xi \in [0,1]$. It means that the pointwise limit function $f(\xi) = \lim_{n\to\infty} f_n(\xi)$ exists. At first, we want to prove that $\{f_n\}$ also converges uniformly to f, that is $\|f_n - f\|_{F.u} \longrightarrow 0$ $(n \to \infty)$.
In other words for each $\varepsilon > 0$ we need to find M such that $\|f_n - f\|_{F.u} \leq \varepsilon$ for $n > M$.
For this, let $\varepsilon > 0$ and then fix M such that $\|f_n - f_m\|_{F.u} < \frac{\varepsilon}{2}$ for all $n, m \geq M$. We can do this since $\{f_n\}$ is a cauchy sequence. Using the triangle inequality, we have

$$\|f_n - f\|_{F.u} \leq \|f_n - f_M\|_{F.u} + \|f - f_M\|_{F.u}.$$

As we know that for $n \geq M$, we have $\|f_n - f_M\|_{F.u} < \frac{\varepsilon}{2}$. Therefore

$$\|f - f_M\| = \lim_{n\to\infty} \|f_n - f_M\|_{F.u} < \frac{\varepsilon}{2}.$$

So $\|f_n - f\|_{F.u} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $n > M$ which means $\|f_n - f\|_{F.u} \longrightarrow 0$ $(n \to \infty)$.
Now, we have to show that f is continuous.
So let $\varepsilon > 0$ and $\xi_1 \in [0,1]$, we want to find $\delta > 0$ such that for an arbitrary fixed number ξ_2; $D(f(\xi_1), f(\xi_2)) < \varepsilon$ when $|\xi_1 - \xi_2|$, with using the triangle inequality

$$D(f(\xi_1), f(\xi_2)) \leq D(f(\xi_1), f_n(\xi_1)) + D(f_n(\xi_1), f_n(\xi_2)) + D(f_n(\xi_2), f(\xi_2))$$

for some n.
Now, we pick n such that $D(f(\xi_1), f_n(\xi_1)) < \frac{\varepsilon}{3}$ and $D(f_n(\xi_2), f(\xi_2)) < \frac{\varepsilon}{3}$, on the other hand f_n is continuous. So, $D(f_n(\xi_1), f_n(\xi_2)) < \frac{\varepsilon}{3}$ and we have $D(f(\xi_1), f(\xi_2)) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.
Hence $f \in C_F([0,1])$, so $C_F([0,1])$ is a Banach space. □

Now we define the operator T as

$$T(f)(\xi) = g(\xi) + \int_0^{\xi} K(\xi, z) f(z) \ dz, \qquad \forall \xi \in [0,1], \quad \forall f \in C_F([0,1]), \ g : [0,1] \to \mathbb{R}_F.$$

$T(f)(\xi)$ can be represented as form $T(f)(\xi) = (\underline{T(f)}(\xi), \overline{T(f)}(\xi))$ where,

$$\underline{T(f)}(\xi) = \underline{g}(\xi) + \int_0^\xi K(\xi, z)\underline{f}(z)\ dz, \qquad \overline{T(f)}(\xi) = \overline{g}(\xi) + \int_0^\xi K(\xi, z)\overline{f}(z)\ dz.$$

Sufficient conditions for the existence of a unique solution for the above integral equation will be given in the following.

Theorem 4.2. *let $K = K(\xi, z)$ be continuous and positive for $0 \le \xi \le 1$, $0 \le z \le \xi$ and $f, g : [0,1] \longrightarrow \mathbb{R}_F$ be fuzzy continuous functions on $[0,1]$. If $M_1\ \xi < 1$, than the Homotopy analysis method*

$$f_0(\xi) = g(\xi),$$

$$f_m(\xi) = g(\xi) + \int_0^\xi K(\xi, z) f_{m-1}(z) dz \qquad m \ge 1.$$

convergence to the unique solution f.

Proof. First we show that $T(C_F([0,1])) \subseteq C_F([0,1]))$. Since g is continuous on the compact set $[0,1]$, so it is uniformly continuous. Therefore

$$\forall \varepsilon_1 > 0 \qquad \exists \rho_1 > 0 \quad s.t. \quad |\xi_1 - \xi_2| < \rho_1 \Longrightarrow D(g(\xi_1), g(\xi_2)) < \varepsilon_1.$$

It means; $\sup \max\{|\underline{g}(\xi_1) - \underline{g}(\xi_2)|, |\overline{g}(\xi_1) - \overline{g}(\xi_2)|\} < \varepsilon_1 \qquad 0 \le \xi_1, \xi_2 \le 1,$
consequently,

$$|\underline{g}(\xi_1) - \underline{g}(\xi_2)| < \varepsilon_1\ ,\ |\overline{g}(\xi_1) - \overline{g}(\xi_2)| < \varepsilon_1.$$

As mentioned above f is continuous thus f is bounded. It means

$$\exists\ M_2 > 0 \quad s.t. \quad |f| \le M_2,$$

we must show that,

$$\forall \varepsilon > 0 \quad \exists\ \rho > 0 \quad s.t. \quad |\xi_1 - \xi_2| < \rho \Longrightarrow \|T(f)(\xi_1) - T(f)(\xi_2)\|_{F.u} < \varepsilon.$$

Since

$$\|T(f)(\xi_1) - T(f)(\xi_2)\|_{F.u} = \sup_{\xi_1, \xi_2 \in [0,1]} \max\ \{|\underline{T(f)}(\xi_1) - \underline{T(f)}(\xi_2)|, |\overline{T(f)}(\xi_1) - \overline{T(f)}(\xi_2)|\}$$

It is enough to show that,

$$|\underline{T(f)}(\xi_1) - \underline{T(f)}(\xi_2)| < \varepsilon\ ,\ |\overline{T(f)}(\xi_1) - \overline{T(f)}(\xi_2)| < \varepsilon$$

$$\begin{aligned}
|\underline{T(f)}(\xi_1) - \underline{T(f)}(\xi_2)| &\le |\underline{g}(\xi_1) - \underline{g}(\xi_2)| + |\int_0^{\xi_1} K(\xi_1, z)\underline{f}(z) dz - \int_0^{\xi_2} K(\xi_2, z)\underline{f}(z) dz| \\
&\le \varepsilon_1 + |\int_0^{\xi_1} K(\xi_1, z)\underline{f}(z) dz + \int_{\xi_2}^0 K(\xi_2, z)\underline{f}(z) dz| \\
&\le \varepsilon_1 + \int_0^{\xi_1} |K(\xi_1, z)||\underline{f}(z)| dz + \int_{\xi_2}^0 |K(\xi_2, z)||\underline{f}(z)| dz \\
&\le \varepsilon_1 + \int_0^{\xi_1} M_1|\underline{f}(z)| dz + \int_{\xi_2}^0 M_1|\underline{f}(z)| dz \\
&= \varepsilon_1 + \int_{\xi_2}^{\xi_1} M_1|\underline{f}(z)| dz \\
&\le \varepsilon_1 + M_1.M_2 \int_{\xi_2}^{\xi_1} dz \\
&= \varepsilon_1 + M_1 M_2(\xi_1 - \xi_2).
\end{aligned}$$

Choosing $\varepsilon_1 = \dfrac{\varepsilon}{2}$, $\xi_1 - \xi_2 = \dfrac{\xi}{2M_1M_2}$, we have,

$$|\underline{T(f)}(\xi_1) - \underline{T(f)}(\xi_2)| < \varepsilon.$$

Similarly;

$$|\overline{T(f)}(\xi_1) - \overline{T(f)}(\xi_2)| < \varepsilon.$$

So $T(C_F([0,1])) \subseteq C_F([0,1])$.
Now, we show that the operator T is a contraction. So for $f, h \in C_F([0,1])$ and $\xi \in [0,1]$

$$\begin{aligned} D(T(f)(\xi), T(h)(\xi)) &\leq D(g(\xi), g(\xi)) + D(\int_0^\xi K(\xi,z)f(z)dz, \int_0^\xi K(\xi,z)h(z)dz) \\ &= \int_0^\xi |K(\xi,z)| D(f(z), h(z))dz \\ &\leq M_1 \int_0^\xi D(f(z), h(z))dz \\ &= M_1 \xi D(f,h), \end{aligned}$$

therefore,

$$D(T(f)(\xi), T(h)(\xi)) \leq M_1 \xi \ D(f,h).$$

After taking supremum we have

$$\|T(f)(\xi) - T(h)(\xi)\|_{F.u} \leq M_1 \xi \ \|f-h\|_{F.u},$$

since $M_1\xi < 1$ the operator T is a contraction on Banach space $(C_F([0,1]), \| \ . \ \|_{F.u})$ consequently, the Banach's fixed point theorem implies that this integral equation has a unique solution f in $C_F([0,1])$. $\square$

The existence and uniqueness of solution for integral equation was proved. We conclude that the FFKGE also has a unique solution.

Corollary 4.1. *The FFKGE has a unique solution.*

Proof. Combine the Sect. 3 (equivalency the FFKGE and fuzzy integral equation) and Theorem. 4.2. $\square$

5. The homotopy analysis transform method

The application of Homotopy analysis method in linear and nonlinear problems has been devoted by scientists and engineers. The fundamental work was done by liao and He [6]. He's technique in particular, eliminated some of the traditional limitations of methods and was successfully applied to solve many problems in various, fields including fluid mechanics, heat transfer and so on [7]. This method has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied science. To illustrate the basic idea of this method to solve the FFKGE. We consider the parametric from of this equation as follows :

$$\frac{\partial^{2\gamma}\overline{U}(x,t)}{\partial t^{2\gamma}} = \frac{\partial^2 \overline{U}(x,t)}{\partial x^2} + \overline{U}(x,t), \tag{5.1}$$

$$\frac{\partial^{2\gamma}\underline{U}(x,t)}{\partial t^{2\gamma}} = \frac{\partial^2 \underline{U}(x,t)}{\partial x^2} + \underline{U}(x,t).$$

Equation (5.1) can be written as

$$\frac{\partial \overline{U}(x,t)}{\partial t} = \frac{\partial^{1-2\gamma}\partial^2 \overline{U}(x,t)}{\partial t^{1-2\gamma}\partial x^2} + \frac{\partial^{1-2\gamma}}{\partial t^{1-2\gamma}}\overline{U}(x,t).$$

Now the methodology consists of applying the laplace transform first on both sides of above equation, we get

$$L[\overline{U}(x,t)] - \frac{\overline{U}(x,0)}{s} - \frac{\overline{U}_t(x,0)}{s^2} = \frac{1}{s^{2\gamma}} L\left[\frac{\partial^{1-2\gamma}}{\partial t^{1-2\gamma}} \frac{\partial^2 \overline{U}(x,t)}{\partial x^2}\right] + \frac{1}{s^{2\gamma}} L\left[\frac{\partial^{1-2\gamma}}{\partial t^{1-2\gamma}} \overline{U}(x,t)\right]. \tag{5.2}$$

Equation (5.2) can be written as a nonlinear operator form as follow:

$$N[\overline{U}(x,t)] = 0,$$

where N is nonlinear operator, $\overline{U}(x,t)$ is unknown function and x,t are independent variables, $\overline{U}(x,0)$ is auxiliary parameter.

Using $q \in [0,1]$ as an embedding parameter, then we have

$$\begin{aligned} N[\bar{\phi}(x,t;q)] &= L[\bar{\phi}(x,t;q)] - \frac{\overline{U}(x,0)}{s} - \frac{\overline{U}_t(x,0)}{s^2} = \frac{1}{s^{2\gamma}} L\left[\frac{\partial^{1-2\gamma}}{\partial t^{1-2\gamma}} \frac{\partial^2 \bar{\phi}(x,t;q)}{\partial x^2}\right] \\ &+ \frac{1}{s^{2\gamma}} L\left[\frac{\partial^{1-2\gamma}}{\partial t^{1-2\gamma}} \bar{\phi}(x,t;q)\right], \end{aligned}$$

where $\bar{\phi}(x,t;q)$ is the real function of x,t and q. By means of generalizing the traditional homotopy methods construct the zero-order deformation equation.

$$(1-q)L[\bar{\phi}(x,t;q) - U_0(x,t)] = qhN[\phi(x,t;q)], \tag{5.3}$$

where h is a nonzero auxiliary parameter. When $q=0$ and $q=1$, it holds

$$\bar{\phi}(x,t;0) = \overline{U}_0(x,t), \qquad \bar{\phi}(x,t;1) = \overline{U}(x,t).$$

Expanding $\bar{\phi}_i(x,t;q)$ in Taylor's series with respect to q we have

$$\bar{\phi}(x,t;q) = \overline{U}_0(x,t) + \Sigma_{m=1}^{\infty} \overline{U}_m(x,t) q^m,$$

where

$$\overline{U}_m(x,t) = \frac{1}{m!} \frac{\partial^m \bar{\phi}(x,t;q)}{\partial q^m}\bigg|_{q=0}.$$

Differentiating deformation (5.2), m times with respect to q, dividing by $m_!$ and setting $q=1$, we have the mth-order deformation

$$L[\overline{U}_m(x,t) - X_m \overline{U}_{m-1}(x,t)] = hR_m(\overline{U}_{m-1}(x,t)),$$

where

$$R_m(\overline{U}_{m-1}(x,t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\bar{\phi}(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0},$$

and

$$X_m = \begin{cases} 1 & m>1, \\ 0 & m\leqslant 1. \end{cases}$$

Therefore for this equation,

$$\begin{aligned} R_m(\overline{U}_{m-1}(x,t)) &= L[\overline{U}_{m-1}(x,t)] - (1-X_m)\frac{\overline{K}(\beta)(1+\sin x)}{s} = \frac{1}{s^{2\gamma}} L[\frac{\partial^{1-2\gamma}}{\partial t^{1-2\gamma}} \frac{\partial^2 \overline{U}_{m-1}(x,t)}{\partial x^2}] \\ &+ \frac{1}{s^{2\gamma}} L[\frac{\partial^{1-2\gamma}}{\partial t^{1-2\gamma}} \overline{U}_{m-1}(x,t)], \end{aligned}$$

$$\overline{U}_0(x,t;\beta) = \overline{K}(\beta)(1+\sin x),$$

$$m = 1 \implies R_1(\overline{U}_0(x,t;\beta) = \frac{-\overline{K}(\beta)}{s^{2\gamma+1}},$$

$$if \quad h=-1 \quad then \quad \overline{U}_1(x,t;\beta) = \frac{\overline{K}(\beta)t^{2\gamma}}{\Gamma(2\gamma+1)}.$$

By using this method,

$$\overline{U}_2(x,t;\beta) = \frac{\overline{K}(\beta)t^{4\gamma}}{\Gamma(4\gamma+1)},$$

$$\overline{U}_3(x,t;\beta) = \frac{\overline{K}(\beta)t^{6\gamma}}{\Gamma(6\gamma+1)},$$

$$\overline{U}_4(x,t;\beta) = \frac{\overline{K}(\beta)t^{8\gamma}}{\Gamma(8\gamma+1)},$$

$$\overline{U}_5(x,t;\beta) = \frac{\overline{K}(\beta)t^{10\gamma}}{\Gamma(10\gamma+1)}.$$

Proceeding in this manner, the rest of the components $\bar{U}_n(x,t;\beta)$ for $n \geq 5$ can be completely obtained. We get the approximated solution of fuzzy fractional differential equation as follow

$$\overline{U}(x,t;\beta) = [(1+\sin x) + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} + \frac{t^{6\gamma}}{\Gamma(6\gamma+1)} + \frac{t^{8\gamma}}{\Gamma(8\gamma+1)} + \frac{t^{10\gamma}}{\Gamma(10\gamma+1)}]\overline{K}(\beta),$$

$$\underline{U}(x,t;\beta) = [(1+\sin x) + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} + \frac{t^{6\gamma}}{\Gamma(6\gamma+1)} + \frac{t^{8\gamma}}{\Gamma(8\gamma+1)} + \frac{t^{10\gamma}}{\Gamma(10\gamma+1)}]\underline{K}(\beta).$$

Let $\overline{K}(\beta) = 0.25\beta$ and $\underline{K}(\beta) = -0.25\beta$ for $0 \le \beta \le 1$
so the exact solution is given by

$$\overline{U}(x,t;\beta) = (1+\sin x)\overline{K}(\beta) + \Sigma_{r=1}^{\infty}\frac{t^{2r\gamma}}{\Gamma(2r\gamma+1)}\overline{K}(\beta),$$

$$\underline{U}(x,t;\beta) = (1+\sin x)\underline{K}(\beta) + \Sigma_{r=1}^{\infty}\frac{t^{2r\gamma}}{\Gamma(2r\gamma+1)}\underline{K}(\beta),$$

and

$$U(x,t;\beta) = (\underline{U}(x,t;\beta), \overline{U}(x,t;\beta)).$$

In the following tables and figures comparison between the exact solution and the different terms of approximation solution for $\overline{U}, \underline{U}$ is given by the homotopy analysis transform method at $\gamma = \frac{1}{8}$.
If we increase the computational process, the approximation solution will be closer to the exact solution.

Table. 1.

(x,t,β)	$\overline{U}_{approx[5]}$	$\overline{U}_{approx[15]}$	$\overline{U}_{approx[25]}$	$\overline{U}_{exact}$
(0.2,0.2,0.1)	0.07711367095	0.08075077885	0.08075381528	0.08075381592
(0.2,0.4,0.1)	0.1004436229	0.1133482422	0.1134055766	0.1134055435
(0.2,0.6,0.1)	0.1498635015	0.1495343794	0.1498624662	0.1498635015
(0.2,0.8,0.1)	0.1409872942	0.1914962138	0.1926482315	0.1926555602
(0.4,0.2,0.1)	0.08188239622	0.08551950412	0.08552254055	0.08552254120
(0.6,0.2,0.1)	0.08626299950	0.08990010740	0.08990314382	0.08990314448
(0.8,0.2,0.1)	0.09008083995	0.09371794785	0.09372098428	0.09372098492
(0.2,0.2,0.2)	0.1542273419	0.1615015577	0.1615076306	0.1615076318
(0.2,0.4,0.2)	0.2008872458	0.2266964845	0.2268109531	0.2268110870
(0.2,0.6,0.2)	0.2425774142	0.2990687587	0.2997249324	0.2997270030
(0.2,0.8,0.2)	0.2819745885	0.3829924276	0.3852964630	0.3853111204
(0.4,0.2,0.2)	0.1637647924	0.1710390082	0.1710450811	0.1710450824
(0.6,0.2,0.2)	0.1725259990	0.1798002148	0.1798062876	0.1798062890
(0.8,0.2,0.2)	0.1801616799	0.1874358957	0.1874419686	0.1874419698

Table. 2.

(x,t,β)	$\underline{U}_{approx[5]}$	$\underline{U}_{approx[15]}$	$\underline{U}_{approx[25]}$	$\underline{U}_{exact}$
(0.2,0.2,-0.1)	-0.07711367095	-0.08075077885	-0.08075381528	-0.08075381592
(0.2,0.4,-0.1)	-0.1004436229	-0.1133482422	-0.1134055766	-0.1134055435
(0.2,0.6,-0.1)	-0.1498635015	-0.1495343794	-0.1498624662	-0.1498635015
(0.2,0.8,-0.1)	-0.1409872942	-0.1914962138	-0.1926482315	-0.1926555602
(0.4,0.2,-0.1)	-0.08188239622	-0.08551950412	-0.08552254055	-0.08552254120
(0.6,0.2,-0.1)	-0.08626299950	-0.08990010740	-0.08990314382	-0.08990314448
(0.8,0.2,-0.1)	-0.09008083995	-0.09371794785	-0.09372098428	-0.09372098492
(0.2,0.2,-0.2)	-0.1542273419	-0.1615015577	-0.1615076306	-0.1615076318
(0.2,0.4,-0.2)	-0.2008872458	-0.2266964845	-0.2268109531	-0.2268110870
(0.2,0.6,-0.2)	-0.2425774142	-0.2990687587	-0.2997249324	-0.2997270030
(0.2,0.8,-0.2)	-0.2819745885	-0.3829924276	-0.3852964630	-0.3853111204
(0.4,0.2,-0.2)	-0.1637647924	-0.1710390082	-0.1710450811	-0.1710450824
(0.6,0.2,-0.2)	-0.1725259990	-0.1798002148	-0.1798062876	-0.1798062890
(0.8,0.2,-0.2)	-0.1801616799	-0.1874358957	-0.1874419686	-0.1874419698

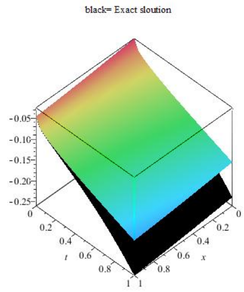

FIGURE 1. Comparison between the exact solution and the 5th-order of approximation solution given by the homotopy analysis transform method($\underline{U}_{approx[5]}(x,t,\beta)$)

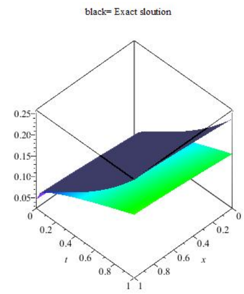

FIGURE 2. Comparison between the exact solution and the 5th-order of approximation solution given by the homotopy analysis transform method($\overline{U}_{approx[5]}(x,t,\beta)$)

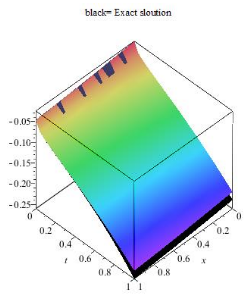

FIGURE 3. Comparison between the exact solution and the 10th-order of approximation solution given by the homotopy analysis transform method($\underline{U}_{approx[10]}(x,t,\beta)$)

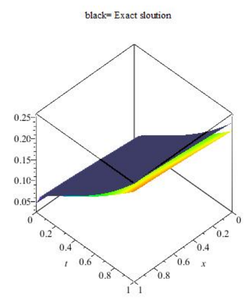

FIGURE 4. Comparison between the exact solution and the 10th-order of approximation solution given by the homotopy analysis transform method($\overline{U}_{approx[10]}(x,t,\beta)$)

6. CONCLUSION

The homotopy analysis method is applied for solving the FFKGE. This equation is equivalent to fuzzy integral equation. The existence, uniqueness of the solution and convergence of this method are proved.

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