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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR A THIRD-ORDER DELAY DIFFERENTIAL EQUATION

FARID NOUIOUA[1,2], ABDELOUAHEB ARDJOUNI[2,3,*], ABDELKERIM MERZOUGUI[1] AND AHCENE DJOUDI[3]

Abstract. In this paper, the following third-order nonlinear delay differential equation with periodic coefficients

$$x'''(t)+p(t)x''(t)+q(t)x'(t)+r(t)x(t)=f\left(t,x\left(t\right),x(t-\tau(t))\right)+c(t)x'(t-\tau(t)),$$

is considered. By employing Green's function and Krasnoselskii's fixed point theorem, we state and prove the existence of positive periodic solutions to the third-order delay differential equation.

1. Introduction

Delay differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, see the monograph [8, 19] and the papers [1]- [18], [20]- [22], [24]- [27] and the references therein.

The second order nonlinear delay differential equation with periodic coefficients

$$x''\left(t\right)+p\left(t\right)x'\left(t\right)+q\left(t\right)x\left(t\right)=r\left(t\right)x'\left(t-\tau\left(t\right)\right)+f\left(t,x\left(t\right),x\left(t-\tau\left(t\right)\right)\right),$$

has been investigated in [25]. By using Krasnoselskii's fixed point theorem and the contraction mapping principle, Wang, Lian and Ge obtained existence and uniqueness of periodic solutions.

In [22], Ren, Siegmund and Chen discussed the existence of positive periodic solutions for the third-order differential equation

$$x'''\left(t\right)+p\left(t\right)x''\left(t\right)+q\left(t\right)x'\left(t\right)+c\left(t\right)x\left(t\right)=g\left(t,x\left(t\right)\right).$$

By employing the fixed point index, the authors obtained existence results for positive periodic solutions.

Inspired and motivated by the works mentioned above and the papers [1]- [18], [20]- [22], [24]- [27] and the references therein, we concentrate on the existence of positive periodic solutions for the third-order nonlinear delay differential equation

$$x'''(t)+p(t)x''(t)+q(t)x'(t)+r(t)x(t)=f\left(t,x\left(t\right),x(t-\tau(t))\right)+c(t)x'(t-\tau(t)). \tag{1.1}$$

where p, q, r are continuous real-valued functions. The function $c:\mathbb{R}\longrightarrow\mathbb{R}^{+}$ is continuously differentiable, $\tau:\mathbb{R}\longrightarrow\mathbb{R}^{+}$ is twice continuously differentiable and $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous in their respective arguments. To show the existence of positive periodic solutions, we transform (1.1) into an integral equation and then use Krasnoselskii's fixed point theorem. The obtained integral equation splits in the sum of two mappings, one is a contraction and the other is compact.

In this paper, we give the assumptions as follows that will be used in the main results.

(h1) There exist differentiable positive T-periodic functions a_1 and a_2 and a positive real constant ρ such that

$$\begin{cases} a_1(t)+\rho=p(t), \\ a_1'\left(t\right)+a_2\left(t\right)+\rho a_1(t)=q\left(t\right), \\ a_2'\left(t\right)+\rho a_2(t)=r\left(t\right). \end{cases}$$

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(h2) $p, q, r, c \in C(\mathbb{R}, \mathbb{R}^+)$ are T-periodic functions with $\tau(t) \geq \tau^* > 0$, $\tau'(t) \neq 1$ for all $t \in [0, T]$ and

$$\int_0^T p(s)ds > \rho, \quad \int_0^T q(s)ds > 0.$$

(h3) The function $f(t, x, y)$ is continuous T-periodic in t and continuous in x and y.

The organization of this paper is as follows. In section 2, we introduce some notations and lemmas, and state some preliminary results needed in later section, then we give the Green's function of (1.1), which plays an important role in this paper. In section 3, we present our main results on existence of positive periodic solutions of (1.1).

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1.1). For its proof we refer the reader to [23].

Theorem 1.1 (Krasnoselskii). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that H_1 and H_2 map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{M}$, implies $H_1 x + H_2 y \in \mathbb{M}$,

(ii) H_1 is compact and continuous,

(iii) H_2 is a contraction mapping.

Then there exists $z \in \mathbb{M}$ with $z = H_1 z + H_2 z$.

2. Green's function of third-order differential equation

For $T > 0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0,T]} |x(t)|.$$

We consider

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = h(t), \tag{2.1}$$

where h is a continuous T-periodic function. Obviously, by the condition $(h1)$, (2.1) is transformed into

$$\begin{cases} y'(t) + \rho y(t) = h(t), \\ x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t). \end{cases}$$

Lemma 2.1 ([3]). *If $y, h \in P_T$, then y is a solution of equation*

$$y'(t) + \rho y(t) = h(t),$$

if only if

$$y(t) = \int_t^{t+T} G_1(t, s)h(s)ds, \tag{2.2}$$

where

$$G_1(t, s) = \frac{\exp(\rho(s - t))}{\exp(\rho T) - 1}. \tag{2.3}$$

Corollary 2.1. *Green function G_1 satisfies the following properties*

$$G_1(t + T, s + T) = G_1(t, s), \ G_1(t, t + T) = G_1(t, t)\exp(\rho T),$$
$$G_1(t + T, s) = G_1(t, s)\exp(-\rho T), \ G_1(t, s + T) = G_1(t, s)\exp(\rho T),$$
$$\frac{\partial}{\partial t}G_1(t, s) = -\rho G_1(t, s),$$
$$\frac{\partial}{\partial s}G_1(t, s) = \rho G_1(t, s),$$

and

$$m_1 \leq G_1(t, s) \leq M_1,$$

where

$$m_1 = \frac{1}{\exp(\rho T) - 1}, \ M_1 = \frac{\exp(\rho T)}{\exp(\rho T) - 1}.$$

Lemma 2.2 ([21]). *Suppose that* $(h1)$ *and* $(h2)$ *hold and*

$$\frac{R_1\left[\exp\left(\int_0^T a_1(v)dv\right)-1\right]}{Q_1 T}\geq 1, \tag{2.4}$$

where

$$R_1=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp\left(\int_0^T a_1(v)dv\right)}{\exp\left(\int_0^T a_1(v)dv\right)-1}a_2(s)\,ds\right|,$$

$$Q_1=\left(1+\exp\left(\int_0^T a_1(v)dv\right)\right)^2 R_1^2.$$

Then there are continuous T*-periodic functions* a *and* b *such that*

$$b(t)>0,\ \int_0^T a(v)dv>0,$$

and

$$a(t)+b(t)=a_1(t),\ b'(t)+a(t)b(t)=a_2(t),\ \textit{for } t\in\mathbb{R}.$$

Lemma 2.3 ([25]). *Suppose the conditions of Lemma 2.2 hold and* $y\in P_T$. *Then the equation*

$$x''(t)+a_1(t)x'(t)+a_2(t)x(t)=y(t),$$

has a T *periodic solution. Moreover, the periodic solution can be expressed by*

$$x(t)=\int_t^{t+T}G_2(t,s)y(s)ds, \tag{2.5}$$

where

$$G_2(t,s)=\frac{\int_t^s \exp\left[\int_t^v b(u)du+\int_v^s a(u)du\right]dv+\int_s^{t+T}\exp\left[\int_t^v b(u)du+\int_v^{s+T}a(u)du\right]dv}{\left[\exp\left(\int_0^T a(v)dv\right)-1\right]\left[\exp\left(\int_0^T b(v)dv\right)-1\right]}. \tag{2.6}$$

Corollary 2.2. *Green's function* G_2 *satisfies the following proprieties*

$$G_2(t+T,s+T)=G_2(t,s),\ G_2(t,t+T)=G_2(t,t),$$

$$G_2(t+T,s)=\exp\left(-\int_0^T b(v)dv\right)\left[G_2(t,s)+\int_t^{t+T}E(t,u)F(u,s)\,du\right],$$

$$\frac{\partial}{\partial t}G_2(t,s)=-b(t)G_2(t,s)+F(t,s),$$

$$\frac{\partial}{\partial s}G_2(t,s)=a(t)G_2(t,s)-E(t,s),$$

where

$$E(t,s)=\frac{\exp\left(\int_t^s b(v)dv\right)}{\exp\left(\int_0^T b(v)dv\right)-1},\ F(t,s)=\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right)-1}.$$

Lemma 2.4 ([21]). *Let* $A=\int_0^T a_1(v)dv$ *and* $B=T^2\exp\left(\frac{1}{T}\int_0^T \ln\left(a_2(v)\right)dv\right)$. *If*

$$A^2\geq 4B, \tag{2.7}$$

then

$$\min\left\{\int_0^T a(v)dv,\ \int_0^T b(v)dv\right\}\geq\frac{1}{2}\left(A-\sqrt{A^2-4B}\right)=l,$$

$$\max\left\{\int_0^T a(v)dv,\ \int_0^T b(v)dv\right\}\leq\frac{1}{2}\left(A+\sqrt{A^2-4B}\right)=L.$$

Corollary 2.3. *Functions G_2, E and F satisfy*

$$m_2 \leq G_2(t,s) \leq M_2,$$
$$E(t,s) \leq \frac{e^L}{e^l - 1},$$
$$F(t,s) \leq e^L,$$

where

$$m_2 = \frac{T}{(\exp(L)-1)^2}, \; M_2 = \frac{T\exp\left(\int_0^T a_1(v)\,dv\right)}{(\exp(l)-1)^2}.$$

Lemma 2.5 ([11]). *Suppose the conditions of Lemma 2.2 hold and $h \in P_T$. Then the equation*

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = h(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)h(s)ds, \tag{2.8}$$

where

$$G(t,s) = \int_t^{t+T} G_2(t,\sigma)\,G_1(\sigma,s)\,d\sigma. \tag{2.9}$$

Corollary 2.4. *Green's function G satisfies the following properties*

$$G(t+T,s+T) = G(t,s), \; G(t,t+T) = G(t,t)\exp(\rho T),$$
$$\frac{\partial}{\partial t}G(t,s) = (\exp(-\rho T)-1)\,G_1(t,t)\,G_2(t,s) - b(t)\,G(t,s) + \int_t^{t+T} F(t,\sigma)G_1(\sigma,s)\,d\sigma,$$
$$\frac{\partial}{\partial s}G(t,s) = \rho G(t,s),$$

and

$$m \leq G(t,s) \leq M,$$

where

$$m = \frac{T^2}{(\exp(l)-1)^2(\exp(\rho T)-1)}, \; M = \frac{T^2\exp\left(\rho T + \int_0^T a(v)\,dv\right)}{(\exp(l)-1)^2(\exp(\rho T)-1)}.$$

3. Main Results

In this section we will study the existence of positive periodic solutions of (1.1).

Lemma 3.1. *Suppose* (h1)-(h3) *and (2.4) hold. The function $x \in P_T$ is a solution of (1.1) if and only if*

$$\begin{aligned} x(t) &= Z(t)(\exp(\rho T)-1)\,G(t,t)\,x(t-\tau(t)) \\ &\quad + \int_t^{t+T} G(t,s)\{f(s,x(s),x(s-\tau(s))) - R(s)\,x(s-\tau(s))\}\,ds, \end{aligned} \tag{3.1}$$

where

$$R(s) = \frac{(c'(s)+c(s)\rho)(1-\tau'(s)) + c(s)\tau''(s)}{(1-\tau'(s))^2}, \tag{3.2}$$

$$Z(t) = \frac{c(t)}{1-\tau'(t)}. \tag{3.3}$$

Proof. Let $x \in P_T$ be a solution of (1.1). From Lemma 2.5, we have

$$\begin{aligned} x(t) &= \int_t^{t+T} G(t,s)\,[f(s,x(s),x(s-\tau(s))) + c(s)\,x'(s-\tau(s))]\,ds \\ &= \int_t^{t+T} G(t,s)\,f(s,x(s),x(s-\tau(s)))\,ds + \int_t^{t+T} G(t,s)\,c(s)\,x'(s-\tau(s))\,ds. \end{aligned} \tag{3.4}$$

Performing an integration by parts, we get

$$\begin{aligned}
&\int_t^{t+T} G(t,s)\,c(s)\,x'(s-\tau(s))\,ds \\
&= \int_t^{t+T} \frac{c(s)\,(1-\tau'(s))\,x'(s-\tau(s))}{1-\tau'(s)} G(t,s)\,ds \\
&= \int_t^{t+T} \frac{c(s)}{1-\tau'(s)} G(t,s)\,dx(s-\tau(s)) \\
&= \frac{c(s)}{1-\tau'(s)} G(t,s)\,x(s-\tau(s))\Big|_t^{t+T} - \int_t^{t+T} \frac{\partial}{\partial s}\left[\frac{c(s)}{1-\tau'(s)} G(t,s)\right] x(s-\tau(s))\,ds \\
&= Z(t)\,(\exp(\rho T)-1)\,x(t-\tau(t))\,G(t,t) - \int_t^{t+T} R(s)\,G(t,s)\,x(s-\tau(s))\,ds, \qquad (3.5)
\end{aligned}$$

where R and Z are given by (3.2) and (3.3), respectively. We obtain (3.1) by substituting (3.5) in (3.4). Since each step is reversible, the converse follows easily. This completes the proof. $\square$

Define the mapping $H : P_T \to P_T$ by

$$\begin{aligned}
(H\varphi)(t) = &\int_t^{t+T} G(t,s)\,\{f(s,\varphi(s),\varphi(s-\tau(s))) - R(s)\,\varphi(s-\tau(s))\}\,ds \\
&+ Z(t)\,(\exp(\rho T)-1)\,G(t,t)\,\varphi(t-\tau(t)). \qquad (3.6)
\end{aligned}$$

Note that to apply Krasnoselskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is compact. Therefore, we express (3.6) as

$$(H\varphi)(t) = (H_1\varphi)(t) + (H_2\varphi)(t).$$

where $H_1, H_2 : P_T \to P_T$ are given by

$$(H_1\varphi)(t) = \int_t^{t+T} G(t,s)\,\{f(s,\varphi(s),\varphi(s-\tau(s))) - R(s)\,\varphi(s-\tau(s))\}\,ds, \qquad (3.7)$$

and

$$(H_2\varphi)(t) = Z(t)\,(\exp(\rho T)-1)\,G(t,t)\,\varphi(t-\tau(t)). \qquad (3.8)$$

To simplify notations, we introduce the constants

$$\alpha = \max_{t\in[0,T]} |Z(t)|,\ \beta = \max_{t\in[0,T]} \{b(t)\},\ \delta = \frac{\exp(L)}{\exp(l)-1},\ \gamma = \exp(\rho T)-1. \qquad (3.9)$$

In this section we obtain the existence of a positive periodic solution of (1.1) by considering the two cases; $c(t) \geq 0$ and $c(t) \leq 0$ for all $t \in \mathbb{R}$. For a non-negative constant $\mathbf{K}$ and a positive constant $\mathbf{L}$ we define the set

$$\mathbb{D} = \{\varphi \in P_T : \mathbf{K} \leq \varphi \leq \mathbf{L}\},$$

which is a closed convex and bounded subset of the Banach space P_T.

In case $c(t) \geq 0$, we assume that there exist a positive constant η such that

$$\eta \leq Z(t),\ \text{for all } t \in [0,T], \qquad (3.10)$$

$$\alpha M \gamma < 1, \qquad (3.11)$$

and for all $s \in [0,T],\ x, y \in \mathbb{D}$

$$\frac{\mathbf{K}(1-\eta m\gamma)}{mT} \leq f(s,x,y) - R(s)\,y \leq \frac{\mathbf{L}(1-\alpha M\gamma)}{MT}. \qquad (3.12)$$

Lemma 3.2. *Suppose* (h1)-(h3), *(2.4), (2.7) and (3.10)-(3.12) hold. Then $H_1 : \mathbb{D} \to P_T$ is compact.*

Proof. Let H_1 be defined by (3.7). Obviously, $H_1\varphi$ is continuous and it is easy to show that $(H_1\varphi)(t+T) = (H_1\varphi)(t)$. For $t \in [0,T]$ and for $\varphi \in \mathbb{D}$, we have

$$\begin{aligned}\left|(H_1\varphi)(t)\right| &= \left|\int_t^{t+T} G(t,s)\left\{f(s,\varphi(s),\varphi(s-\tau(s))) - R(s)\varphi(s-\tau(s))\right\} ds\right| \\ &\le MT\frac{\mathbf{L}(1-\alpha M\gamma)}{MT} = \mathbf{L}(1-\alpha M\gamma).\end{aligned}$$

Thus from the estimation of $|(H_1\varphi)(t)|$ we have

$$\|H_1\varphi\| \le \mathbf{L}(1-\alpha M\gamma).$$

This shows that $H_1(\mathbb{D})$ is uniformly bounded.

To show that $H_1(\mathbb{D})$ is equicontinuous, let $\varphi_n \in \mathbb{D}$, where n is a positive integer. Next we calculate $\frac{d}{dt}(H_1\varphi_n)(t)$ and show that it is uniformly bounded. By using $(h1)$, $(h2)$ and $(h3)$ we obtain by taking the derivative in (3.7) that

$$\begin{aligned}&\frac{d}{dt}(H_1\varphi_n)(t) \\ &= \int_t^{t+T}\left[(\exp(-\rho T)-1)G_1(t,t)G_2(t,s) - b(t)G(t,s) + \int_t^{t+T} F(t,\sigma)G_1(\sigma,s)\,d\sigma\right] \\ &\times \left[f(s,\varphi_n(s),\varphi_n(s-\tau(s))) - R(s)\varphi_n(s-\tau(s))\right] ds.\end{aligned}$$

Consequently, by invoking (3.9) and (3.12), we obtain

$$\left|\frac{d}{dt}(H_1\varphi_n)(t)\right| \le \left[(1-\exp(-\rho T))M_1M_2 + M\beta + M_1\delta T\right]\frac{\mathbf{L}(1-\alpha M\gamma)}{M} \le D,$$

for some positive constant D. Hence the sequence $(H_1\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(H_1\varphi_{n_k})$ of $(H_1\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus H_1 is continuous and $H_1(\mathbb{D})$ is contained in a compact subset of $\mathbb{D}$. □

Lemma 3.3. *Suppose that (3.11) holds. If H_2 is given by (3.8), then $H_2 : \mathbb{D} \to P_T$ is a contraction.*

Proof. Let H_2 be defined by (3.8). It is easy to show that $(H_2\varphi)(t+T) = (H_2\varphi)(t)$. Let $\varphi, \psi \in \mathbb{D}$, we have

$$\|H_2\varphi - H_2\psi\| = \sup_{t\in[0,T]}\left|(H_2\varphi)(t) - (H_2\psi)(t)\right| \le \alpha\gamma M\|\varphi-\psi\|.$$

Hence $H_2 : \mathbb{D} \to P_T$ is a contraction by (3.11). □

Theorem 3.1. *Suppose that conditions $(h1)$-$(h3)$, (2.4), (2.7) and (3.10)-(3.12) hold. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

Proof. By Lemma 3.2, the operator $H_1 : \mathbb{D} \to P_T$ is compact and continuous. Also, from Lemma 3.3, the operator $H_2 : \mathbb{D} \to P_T$ is a contraction. Moreover, if $\varphi, \psi \in \mathbb{D}$, we see that

$$\begin{aligned}&(H_2\psi)(t) + (H_1\varphi)(t) \\ &= \gamma Z(t)G(t,t)\varphi(t-\tau(t)) + \int_t^{t+T} G(t,s)\left\{f(s,\varphi(s),\varphi(s-\tau(s))) - R(s)\varphi(s-\tau(s))\right\} ds \\ &\le \gamma\alpha M\mathbf{L} + \mathbf{L}(1-\alpha M\gamma) = \mathbf{L}.\end{aligned}$$

On the other hand

$$\begin{aligned}&(H_2\psi)(t) + (H_1\varphi)(t) \\ &= \gamma Z(t)G(t,t)\varphi(t-\tau(t)) + \int_t^{t+T} G(t,s)\left\{f(s,\varphi(s),\varphi(s-\tau(s))) - R(s)\varphi(s-\tau(s))\right\} ds \\ &\ge \gamma\alpha m\mathbf{K} + \mathbf{K}(1-\alpha m\gamma) = \mathbf{K}.\end{aligned}$$

This shows that $H_2\psi + H_1\varphi \in \mathbb{D}$. Clearly, all the Hypotheses of Theorem 1.1, are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = H_1\psi + H_2\varphi$. By Lemma 3.1 this fixed point is a solution of (1.1) and the proof is complete. □

In the case $c(t) \leq 0$, we substitute conditions (3.10)-(3.12) with the following conditions respectively. We assume that there exist a negative constant z_1 and a non-positive constant z_2 such that

$$z_1 \leq Z(t) \leq z_2, \text{ for all } t \in [0,T], \tag{3.13}$$

$$-z_1 M\gamma < 1, \tag{3.14}$$

and for all $s \in [0,T]$, $x, y \in \mathbb{D}$

$$\frac{\mathbf{K} - z_1 M\gamma \mathbf{L}}{mT} \leq f(s,x,y) - R(s)y \leq \frac{\mathbf{L} - z_2 m\gamma \mathbf{K}}{MT}. \tag{3.15}$$

Theorem 3.2. *Suppose that conditions* $(h1)$-$(h3)$*, (2.4), (2.7) and (3.13)-(3.15) hold. Then equation (1.1) has a positive* T*-periodic solution* x *in the subset* $\mathbb{D}$.

The proof follows along the lines of Theorem 3.1, and hence we omit it.

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[1]Faculty Mathematics and Informatics, Department of Mathematics, Univ M'sila, P.O. Box 166, 28000 M'sila, Algeria

[2]Faculty of Sciences and Technology, Department of Mathematics and Informatics, Univ Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria

[3]Applied Mathematics Lab, Faculty of Sciences, Department of Mathematics, Univ Annaba, P.O. Box 12, Annaba 23000, Algeria

*Corresponding author: abd_ardjouni@yahoo.fr