International Journal of Analysis and Applications
ISSN 2291-8639
Volume 13, Number 2 (2017), 206-215
http://www.etamaths.com

DUNKL GENERALIZATION OF q-PARAMETRIC SZÁSZ-MIRAKJAN OPERATORS

M. MURSALEEN*, MD. NASIRUZZAMAN AND A.A.H. AL-ABIED

ABSTRACT. In this paper, we construct q-parametric Szász-Mirakjan operators generated by the q-Dunkl generalization of the exponential function. We obtain Korovkin's type approximation theorem and compute convergence of these operators by using the modulus of continuity. Furthermore, we obtain the rate of convergence of these operators for functions belonging to the Lipschitz class.

1. INTRODUCTION

In 1912, Bernstein [5] introduced the following sequence of operators $B_n : C[0,1] \to C[0,1]$ defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \ (n \in \mathbb{N}) \ x \in [0,1], \ f \in C[0,1]. \tag{1.1}$$

In 1950, Szász [27] introduced the operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \ x \geq 0, \ f \in C[0,\infty). \tag{1.2}$$

For the last two decades, the application of q-calculus emerged as a new area in the field of approximation theory. The first q-analogue of Bernstein polynomials was introduced by Lupaş [15] and later Phillips [23] considered another q-analogue of the Bernstein polynomials. Later on, many authors introduced q-generalization of various operators and investigated several approximation properties. For instance, [1], [2], [3], [8], [9], [10], [12], [16]– [22], [24].

The q-integer $[n]_q$, the q-factorial $[n]_q!$ and the q-binomial coefficient are defined by (see [13])

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\} \\ n, & \text{if } q = 1, \end{cases} \quad \text{for } n \in \mathbb{N} \text{ and } [0]_q = 0,$$

$$[n]_q! := \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & n \geq 1, \\ 1, & n = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

respectively. The q-analogue of $(1+x)^n$ is the polynomial

$$(1+x)_q^n := \begin{cases} (1+x)(1+qx)\cdots(1+q^{n-1}x) & n = 1,2,3,\cdots \\ 1 & n = 0. \end{cases}$$

A q-analogue of the common Pochhammer symbol also called a q-shifted factorial is defined as

$$(x;q)_0 = 1, \ (x;q)_n = \prod_{j=0}^{n-1}(1-q^j x), \ (x;q)_\infty = \prod_{j=0}^{\infty}(1-q^j x).$$

Received 1st September, 2016; accepted 8th November, 2016; published 1st March, 2017.

2010 *Mathematics Subject Classification.* 41A25, 41A36, 33C45.

Key words and phrases. q-integers; Szász operator; Szász-Mirakjan operator; Dunkl analogue; modulus of continuity; Lipschitz class.

The Gauss binomial formula is given by

$$(x+a)_q^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{k(k-1)/2} a^k x^{n-k}.$$

There are two q-analogue of the exponential function e^z, defined as (see also [14])

For $\mid z \mid< \frac{1}{1-q}$ and $\mid q \mid< 1$,

$$e(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \frac{1}{1-((1-q)z)_q^{\infty}}, \tag{1.3}$$

and for $\mid q \mid< 1$,

$$E(z) = \prod_{j=0}^{\infty} \left(1+(1-q)q^j z\right)_q^{\infty} = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{z^k}{k!} = (1+(1-q)z)_q^{\infty}, \tag{1.4}$$

where $(1-x)_q^{\infty} = \prod_{j=0}^{\infty}(1-q^j x)$.

The $q-$analogue of Bernstein operators [23] is defined as follows:

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k \prod_{s=0}^{n-k-1} (1-q^s x)\ f\left(\frac{[k]_q}{[m]_q}\right),\ x \in [0,1], n \in \mathbb{N}. \tag{1.5}$$

In [4] q-Szász-Mirakjan operators were defined as follows:

$$S_{n,q}(f;x) := E\left(-\frac{[n]_q x}{b_n}\right) \sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right) \frac{[n]_q^k x^k}{[k]_q! b_n^k}, \tag{1.6}$$

where $0 \leq x < \frac{b_n}{(1-q)[n]_q}$, $f \in C[0,\infty)$ and $\{b_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$.

Sucu [26] defined a Dunkl analogue of Szász operators via a generalization of the exponential function given by [25] as follows:

$$S_n^*(f;x) := \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu\theta_k}{n}\right) \quad (n \in \mathbb{N}), \tag{1.7}$$

where $x \geq 0, f \in C[0,\infty), \mu \geq 0$ and

$$e_\mu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_\mu(n)}.$$

Also here

$$\gamma_\mu(2k) = \frac{2^{2k} k! \Gamma\left(k+\mu+\frac{1}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)},$$

and

$$\gamma_\mu(2k+1) = \frac{2^{2k+1} k! \Gamma\left(k+\mu+\frac{3}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)}.$$

A recursion formula for γ_μ is given by

$$\gamma_\mu(k+1) = (k+1+2\mu\theta_{k+1})\gamma_\mu(k),\ \ k=0,1,2,\cdots,$$

where

$$\theta_k = \begin{cases} 0 & \text{if } k \in 2\mathbb{N} \\ 1 & \text{if } k \in 2\mathbb{N}+1 \end{cases}$$

In [6], Cheikh et al. studied the q-Dunkl classical q-Hermite type polynomials and presented the definitions of q-Dunkl analogues of exponential functions, recursion relations and notaions for $\mu > -\frac{1}{2}$ and $0 < q < 1$, respectively.

$$e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)},\ x \in \mathbb{R}, \tag{1.8}$$

$$E_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{\gamma_{\mu,q}(n)}, \; x \in \mathbb{R}, \tag{1.9}$$

$$\gamma_{\mu,q}(n+1) = \left(\frac{1 - q^{2\mu\theta_{n+1}+n+1}}{1-q} \right) \gamma_{\mu,q}(n), \; n \in \mathbb{N}, \tag{1.10}$$

$$\theta_n = \begin{cases} 0 & \text{if } n \in 2\mathbb{N}, \\ 1 & \text{if } n \in 2\mathbb{N}+1. \end{cases}$$

An explicit formula for $\gamma_{\mu,q}(n)$ is given by

$$\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1}, q^2)_{[\frac{n+1}{2}]} (q^2, q^2)_{[\frac{n}{2}]}}{(1-q)^n} \gamma_{\mu,q}(n), \; n \in \mathbb{N}.$$

Some of the special cases of $\gamma_{\mu,q}(n)$ are listed as:

$$\gamma_{\mu,q}(0) = 1, \; \gamma_{\mu,q}(1) = \frac{1-q^{2\mu+1}}{1-q}, \; \gamma_{\mu,q}(2) = \left(\frac{1-q^{2\mu+1}}{1-q} \right) \left(\frac{1-q^2}{1-q} \right),$$

$$\gamma_{\mu,q}(3) = \left(\frac{1-q^{2\mu+1}}{1-q} \right) \left(\frac{1-q^2}{1-q} \right) \left(\frac{1-q^{2\mu+3}}{1-q} \right),$$

$$\gamma_{\mu,q}(4) = \left(\frac{1-q^{2\mu+1}}{1-q} \right) \left(\frac{1-q^2}{1-q} \right) \left(\frac{1-q^{2\mu+3}}{1-q} \right) \left(\frac{1-q^4}{1-q} \right).$$

In [11], Içöz and Çekim gave a Dunkl generalization of Szász operators via q-calculs as:

$$D_{n,q}(f;x) = \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n} \right) \quad (n \in \mathbb{N}), \tag{1.11}$$

where $\mu > \frac{1}{2}$, $x \geq 0$, $0 < q < 1$ and $f \in C[0,\infty)$.

In this paper, we define a Dunkl generalization of q-parametric Szász-Mirakjan operators:

For any $x \in [0,\infty)$, $f \in C[0,\infty)$, $0 < q < 1$, and $\mu > \frac{1}{2}$, we define

$$D^*_{n,q}(f;x) = \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} q^{\frac{k(k-1)}{2}} f\left(\frac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)} \right) \quad (n \in \mathbb{N}). \tag{1.12}$$

2. Main results

Lemma 2.1. *Let $D^*_{n,q}(.\,;\,.)$ be the operators given by (1.12). Then we have the following identities:*

1. $D^*_{n,q}(e_0;x) = 1$
2. $D^*_{n,q}(e_1;x) = qx$
3. $qx^2 + \frac{q^{2(1+\mu)}}{[n]_q}[1-2\mu]_q x \leq D^*_{n,q}(e_2;x) \leq qx^2 + \frac{q^{2(1+\mu)}}{[n]_q}[1+2\mu]_q x,$

where $e_j(t) = t^j$, $j = 0,1,2,\cdots$.

Proof. (1) $D^*_{n,q}(1;x) = \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} q^{\frac{k(k-1)}{2}} = 1$ (from (1.9)).

(2)

$$\begin{aligned} D^*_{n,q}(e_1;x) &= \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} q^{\frac{k(k-1)}{2}} \left(\frac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)} \right) \\ &= \frac{q}{[n]_q} \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=1}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k-1)} q^{\frac{(k-1)(k-2)}{2}} \\ &= \frac{q}{[n]_q} \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^{k+1}}{\gamma_{\mu,q}(k)} q^{\frac{k(k-1)}{2}} \\ &= \frac{qx}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} q^{\frac{k(k-1)}{2}} \\ &= qx. \end{aligned}$$

(3)

$$\begin{aligned} D^*_{n,q}(e_2;x) &= \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} q^{\frac{k(k-1)}{2}} \left(\frac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)} \right)^2 \\ &= \frac{1}{[n]_q^2} \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=1}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k-1)} q^{\frac{k^2-5k+8}{2}} \left(\frac{1-q^{2\mu\theta_k+k}}{(1-q)} \right) \\ &= \frac{q^2}{[n]_q^2} \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=1}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k-1)} q^{\frac{(k-1)(k-4)}{2}} \left(\frac{1-q^{2\mu\theta_k+k}}{(1-q)} \right), \end{aligned}$$

hence

$$D^*_{n,q}(e_2;x) = \frac{q^2}{[n]_q^2} \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^{k+1}}{\gamma_{\mu,q}(k)} q^{\frac{k(k-3)}{2}} \left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{(1-q)} \right). \tag{2.1}$$

A simple calculation yields

$$[2\mu\theta_{k+1}+k+1]_q = [2\mu\theta_k+k]_q + q^{2\mu\theta_k+k}[2\mu(-1)^k+1]_q. \tag{2.2}$$

Replacing k by $2k$, then (2.2) implies that

$$[2\mu\theta_{2k+1}+2k+1]_q = \left(\frac{1-q^{2\mu\theta_{2k}+2k}}{1-q} \right) + q^{2\mu\theta_{2k}+2k}[1+2\mu]_q, \tag{2.3}$$

and by replacing k by $2k+1$, we have

$$[2\mu\theta_{2k+2}+2k+2]_q = \left(\frac{1-q^{2\mu\theta_{2k+1}+2k+1}}{1-q} \right) + q^{2\mu\theta_{2k+1}+2k+1}[1-2\mu]_q. \tag{2.4}$$

Now by separating (2.1), to even and odd terms and using (2.3) and (2.4)

$$\begin{aligned} D^*_{n,q}(e_2;x) &= \frac{q^2}{[n]_q^2} \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^{k+1}}{\gamma_{\mu,q}(k)} q^{\frac{k(k-3)}{2}} \left(\frac{1-q^{2\mu\theta_k+k}}{(1-q)} \right) \Bigg|_{\text{for } k=2k,2k+1} \\ &+ \frac{q^2}{[n]_q^2} \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^{2k+1}}{\gamma_{\mu,q}(2k)} q^{k(2k-3)} q^{2\mu\theta_{2k}+2k}[1+2\mu]_q \\ &+ \frac{q^2}{[n]_q^2} \frac{1}{E_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^{2k+2}}{\gamma_{\mu,q}(2k+1)} q^{(k-1)(2k+1)} q^{2\mu\theta_{2k+1}+2k+1}[1-2\mu]_q. \end{aligned}$$

Since

$$[1-2\mu]_q \le [1+2\mu]_q, \tag{2.5}$$

using the inequality (2.5), we have

$$\begin{aligned} D^*_{n,q}(e_2;x) &\leq qx^2+\frac{q^2}{[n]_q}\frac{x}{E_{\mu,q}([n]_qx)}[1+2\mu]_q\sum_{k=0}^{\infty}\frac{(q[n]_qx)^{2k}}{\gamma_{\mu,q}(2k)}q^{k(2k-3)} \\ &+ \frac{q^{2(\mu+1)}}{[n]_q}\frac{x}{E_{\mu,q}([n]_qx)}[1+2\mu]_q\sum_{k=0}^{\infty}\frac{(q[n]_qx)^{2k+1}}{\gamma_{\mu,q}(2k+1)}q^{(k-1)(2k+1)} \\ &\leq qx^2+\frac{q^{2(\mu+1)}}{[n]_q}\frac{x}{E_{\mu,q}([n]_qx)}[1+2\mu]_q\sum_{k=0}^{\infty}\frac{([n]_qx)^{k}}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}} \\ &\leq qx^2+\frac{q^{2(\mu+1)}}{[n]_q}[1+2\mu]_qx. \end{aligned}$$

Similarly, we can show that

$$D^*_{n,q}(e_2;x)\geq qx^2+\frac{q^{2(\mu+1)}}{[n]_q}[1-2\mu]_qx.$$

□

Lemma 2.2. *Let the operators $D^*_{n,q}(.\ ;\ .)$ be given by (1.12). Then we have the following identities:*

(1) $D^*_{n,q}(e_1-1;x)=qx-1$
(2) $D^*_{n,q}(e_1-x;x)=(q-1)x$
(3) $(1-q)x^2+\frac{q^{2(1+\mu)}}{[n]_q}[1-2\mu]_qx\leq D^*_{n,q}((e_1-x)^2;x)\leq(1-q)x^2+\frac{q^{2(1+\mu)}}{[n]_q}[1+2\mu]_qx.$

Next, we obtain the Korovkin's type approximation properties for our operators defined by (1.12).

In order to obtain the convergence results for the operators $D^*_{n,q}(.,.)$, we write $q=q_n$ where $q_n\in(0,1)$ such that,

$$\lim_n q_n\to 1 \tag{2.6}$$

Theorem 2.1. *Let $q=q_n$ satisfy (2.6), for $0<q_n<1$ and if $D^*_{n,q_n}(.\ ;\ .)$ be the operators given by (1.12). Then for any function $f\in X[0,\infty)\cap H$,*

$$D^*_{n,q_n}(f;x)=f(x)$$

uniformly on each compact subset of $[0,\infty)$.

Proof. The proof is based on the well known Korovkin's theorem regarding the convergence of a sequence of linear positive operators, so it is enough to prove the conditions

$$D^*_{n,q_n}((e_j;x)=x^j,\ j=0,1,2,\ \{\text{as } n\to\infty\}$$

uniformly on $[0,1]$.
Clearly from (2.6) and $\frac{1}{[n]_{q_n}}\to 0,(n\to\infty)$, we have

$$\lim_{n\to\infty}D^*_{n,q_n}(e_1;x)=x,\ \lim_{n\to\infty}D^*_{n,q_n}(e_2;x)=x^2.$$

Which completes the proof. □

Let $C_B(\mathbb{R}^+)$ be the set of all bounded and continuous functions on $\mathbb{R}^+=[0,\infty)$, which is linear normed space with

$$\parallel f\parallel_{C_B}=\sup_{x\geq 0}\mid f(x)\mid.$$

We write

$$H:=\{f:x\in[0,\infty),\frac{f(x)}{1+x^2}\text{ is convergent as } x\to\infty\}.$$

We recall the weighted spaces defined as follows:

$$\begin{aligned} P_\rho(\mathbb{R}^+) &= \{f : \mid f(x) \mid \leq M_f \rho(x)\}, \\ Q_\rho(\mathbb{R}^+) &= \left\{f : f \in P_\rho(\mathbb{R}^+) \cap C[0,\infty)\right\}, \\ Q_\rho^k(\mathbb{R}^+) &= \left\{f : f \in Q_\rho(\mathbb{R}^+) \text{ and } \lim_{x\to\infty} \frac{f(x)}{\rho(x)} = k (k \text{ is a constant})\right\}, \end{aligned}$$

where $\rho(x) = 1 + x^2$ is a weight function and M_f is a constant depending only on f. $Q_\rho(\mathbb{R}^+)$ is a normed space with the norm $\parallel f \parallel_\rho = \sup_{x\geq 0} \frac{|f(x)|}{\rho(x)}$.

Theorem 2.2. *Let $q = q_n$ satisfy (2.6), for $0 < q_n < 1$ and if $D^*_{n,q_n}(. \ ; \ .)$ be the operators given by (1.12). Then for any function $f \in Q_\rho^k(\mathbb{R}^+)$ we have*

$$\lim_{n\to\infty} \parallel D^*_{n,q_n}(f;x) - f \parallel_\rho = 0.$$

Proof. From Lemma 2.1, the first condition of (1) is fulfilled for $\tau = 0$. Now for $\tau = 1, 2$ it is easy to see from (2), (3) of Lemma 2.1 by using (2.6) that

$$\parallel D^*_{n,q_n}\,(e_1)^\tau; x) - x^\tau \parallel_\rho = 0.$$

This complete the proof. □

3. Rate of convergence

Next, we calculate the rate of convergence of operators (1.12) by means of modulus of continuity and Lipschitz type maximal functions.

Let $f \in C[0,\infty]$. The modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by the relation

$$\omega(f,\delta) = \sup_{|y-x|\leq\delta} \mid f(y) - f(x) \mid, \ \ x, y \in [0,\infty). \tag{3.1}$$

It is known that $\lim_{\delta\to 0+} \omega(f,\delta) = 0$ for $f \in C[0,\infty)$ and for any $\delta > 0$ one has

$$\mid f(y) - f(x) \mid \leq \left(\frac{\mid y - x \mid}{\delta} + 1\right) \omega(f,\delta). \tag{3.2}$$

Theorem 3.1. *Let $q = q_n$ satisfy (2.6) for $x \geq 0, \ 0 < q_n < 1$ and if $D^*_{n,q_n}(. \ ; \ .)$ be the operators defined by (1.12). Then for any function $f \in \tilde{C}[0,\infty)$, we have*

$$\mid D^*_{n,q_n}(f;x) - f(x) \mid \leq \left\{1 + \sqrt{(1-q_n)[n]_{q_n} x^2 + q_n^{2(1+\mu)}[1+2\mu]_{q_n} x}\right\} \omega\left(f; \frac{1}{\sqrt{[n]_{q_n}}}\right),$$

where $\tilde{C}[0,\infty)$ is the space of uniformly continuous functions on $\mathbb{R}^+$ and $\omega(f,\delta)$ is the modulus of continuity of the function $f \in \tilde{C}[0,\infty)$ defined in (3.2).

Proof. We prove it by using the result (3.2),(3.3) and Cauchy-Schwarz inequality:
$\mid D^*_{n,q}(f;x)-f(x)\mid$

$$\begin{aligned}
&\leq \frac{1}{E_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\left|f\left(\frac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)}\right)-f(x)\right|\\
&\leq \frac{1}{E_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\left\{1+\frac{1}{\delta}\left|\left(\frac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)}\right)-x\right|\right\}\omega(f;\delta)\\
&= \left\{1+\frac{1}{\delta}\left(\frac{1}{E_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\left|\frac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)}-x\right|\right)\right\}\omega(f;\delta)\\
&\leq \left\{1+\frac{1}{\delta}\left(\frac{1}{E_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\left(\frac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)}-x\right)^2\right)^{\frac{1}{2}}\left(D^*_{n,q}(e_0;x)\right)^{\frac{1}{2}}\right\}\omega(f;\delta)\\
&= \left\{1+\frac{1}{\delta}\left(D^*_{n,q}(e_1-x)^2;x\right)^{\frac{1}{2}}\right\}\omega(f;\delta)\\
&\leq \left\{1+\frac{1}{\delta}\sqrt{(1-q)x^2+\frac{q^{2(1+\mu)}}{[n]_q}[1+2\mu]_qx}\right\}\omega(f;\delta),
\end{aligned}$$

if we choose $\delta=\delta_n=\sqrt{\frac{1}{[n]_q}}$, then we get our result. ∎

Now we give the rate of convergence of the operators $D^*_{n,q}(f;x)$ defined in (1.12) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$.

Let $f\in C[0,\infty)$, $M>0$ and $0<\nu\leq 1$. We recall that f belongs to the class $Lip_M(\nu)$ if

$$Lip_M(\nu)=\{f:\mid f(\zeta_1)-f(\zeta_2)\mid\leq M\mid \zeta_1-\zeta_2\mid^{\nu}\quad(\zeta_1,\zeta_2\in[0,\infty))\} \tag{3.3}$$

is satisfied.

Theorem 3.2. *Let $D^*_{n,q}(.\ ;\ .)$ be the operator defined in (1.12). Then for each $f\in Lip_M(\nu),(M>0,\ 0<\nu\leq 1)$ satisfying (3.3), we have*

$$\mid D^*_{n,q}(f;x)-f(x)\mid\leq M\left(\lambda_n(x)\right)^{\frac{\nu}{2}}$$

where $\lambda_n(x)=D^*_{n,q}\left((e_1-x)^2;x\right)$.

Proof. We prove it by using the result (3.3) and Hölder inequality:

$$\mid D^*_{n,q}(f;x)-f(x)\mid\leq\mid D^*_{n,q}(f(e_1)-f(x);x)\mid\leq D^*_{n,q}\left(\mid f(e_1)-f(x)\mid;x\right)\leq\mid MD^*_{n,q}\left(\mid e_1-x\mid^{\nu};x\right).$$

Therefore

$$\begin{aligned}
&\mid D^*_{n,q}(f;x)-f(x)\mid\\
&\leq M\tfrac{1}{E_{\mu,q}([n]_qx)}\textstyle\sum_{k=0}^{\infty}\tfrac{([n]_qx)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\left|\tfrac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)}-x\right|^{\nu}\\
&\leq M\tfrac{1}{E_{\mu,q}([n]_qx)}\textstyle\sum_{k=0}^{\infty}\left(\tfrac{([n]_qx)^kq^{\frac{k(k-1)}{2}}}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\nu}{2}}\left(\tfrac{([n]_qx)^kq^{\frac{k(k-1)}{2}}}{\gamma_{\mu,q}(k)}\right)^{\frac{\nu}{2}}\left|\tfrac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)}-x\right|^{\nu}\\
&\leq M\left(\tfrac{1}{(E_{\mu,q}([n]_qx))}\textstyle\sum_{k=0}^{\infty}\tfrac{([n]_qx)^kq^{\frac{k(k-1)}{2}}}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\nu}{2}}\left(\tfrac{1}{(E_{\mu,q}([n]_qx))}\textstyle\sum_{k=0}^{\infty}\tfrac{([n]_qx)^kq^{\frac{k(k-1)}{2}}}{\gamma_{\mu,q}(k)}\left|\tfrac{1-q^{2\mu\theta_k+k}}{q^{k-2}(1-q^n)}-x\right|^2\right)^{\frac{\nu}{2}}\\
&\leq M\left(D^*_{n,q}(e_1-x)^2;x\right)^{\frac{\nu}{2}}.
\end{aligned}$$

This complete the proof. ∎

We denote $C_B[0,\infty)$ for the space of all bounded and continuous functions on $\mathbb{R}^+=[0,\infty)$, and

$$C_B^2(\mathbb{R}^+)=\{g\in C_B(\mathbb{R}^+):g',g''\in C_B(\mathbb{R}^+)\}, \tag{3.4}$$

with the norm

$$\| g \|_{C_B^2(\mathbb{R}^+)}=\| g \|_{C_B(\mathbb{R}^+)} + \| g' \|_{C_B(\mathbb{R}^+)} + \| g'' \|_{C_B(\mathbb{R}^+)}, \tag{3.5}$$

also

$$\| g \|_{C_B(\mathbb{R}^+)}= \sup_{x\in\mathbb{R}^+} \mid g(x) \mid . \tag{3.6}$$

Theorem 3.3. *Let $D^*_{n,q}(.\ ;\ .)$ be the operators defined by (1.12). Then for any $g \in C_B^2(\mathbb{R}^+)$, we have*

$$\mid D^*_{n,q}(f;x) - f(x) \mid \leq \left((1-q)x + \frac{\lambda_n(x)}{2} \right) \| g \|_{C_B^2(\mathbb{R}^+)}$$

where $\lambda_n(x)$ is given as in Theorem 3.2.

Proof. Let $g \in C_B^2(\mathbb{R}^+)$. Then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(e_1) = g(x) + g'(x)(e_1 - x) + g''(\psi)\frac{(e_1-x)^2}{2},\ \psi \in (x, e_1).$$

By applying linearity property on $D^*_{n,q}$, we have

$$D^*_{n,q}(g,x) - g(x) = g'(x)D^*_{n,q}\left((e_1 - x);x\right) + \frac{g''(\psi)}{2}D^*_{n,q}\left((e_1-x)^2;x\right),$$

which implies that,

$$\mid D^*_{n,q}(g;x) - g(x) \mid \leq (1-q)x \| g' \|_{C_B(\mathbb{R}^+)} + \left((1-q)x^2 + \frac{q^{2(1+\mu)}}{[n]_q}[1+2\mu]_q x \right) \frac{\| g'' \|_{C_B(\mathbb{R}^+)}}{2}$$

From (3.5) we have $\| g' \|_{C_B[0,\infty)} \leq \| g \|_{C_B^2[0,\infty)}$.

$$\mid D^*_{n,q}(g;x) - g(x) \mid \leq (1-q)x \| g \|_{C_B^2(\mathbb{R}^+)} + \left((1-q)x^2 + \frac{q^{2(1+\mu)}}{[n]_q}[1+2\mu]_q x \right) \frac{\| g \|_{C_B^2(\mathbb{R}^+)}}{2}.$$

From 3 of Lemma 2.2, we get the required result. ☐

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{C_B^2(\mathbb{R}^+)} \left\{ \left(\| f - g \|_{C_B(\mathbb{R}^+)} + \delta \| g'' \|_{C_B^2(\mathbb{R}^+)} \right) : g \in \mathcal{W}^2 \right\}, \tag{3.7}$$

where

$$\mathcal{W}^2 = \left\{ g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+) \right\}. \tag{3.8}$$

Then there exits a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2(f, \delta^{\frac{1}{2}})$, $\delta > 0$, where the second order modulus of continuity is given by

$$\omega_2(f,\delta^{\frac{1}{2}}) = \sup_{0<h<\delta^{\frac{1}{2}}} \sup_{x\in\mathbb{R}^+} \mid f(x+2h) - 2f(x+h) + f(x) \mid . \tag{3.9}$$

Theorem 3.4. *Let $D^*_{n,q}(.\ ;\ .)$ be the operators defined by (1.12) and $C_B[0,\infty)$ be the space of all bounded and continuous functions on $\mathbb{R}^+$. Then for $x \in \mathbb{R}^+$, $f \in C_B(\mathbb{R}^+)$, we have*

$$\mid D^*_{n,q}(f;x) - f(x) \mid \leq 2M \left\{ \omega_2\left(f; \sqrt{\frac{2x(1-q)+\lambda_n(x)}{4}}\right) + \min\left(1, \frac{2x(1-q)+\lambda_n(x)}{4}\right) \| f \|_{C_B(\mathbb{R}^+)} \right\},$$

where M is a positive constant, $\lambda_n(x)$ is given in Theorem 3.2 and $\omega_2(f;\delta)$ is the second order modulus of continuity of the function f defined in (3.9).

Proof. We prove this by using the Theorem (3.3)

$$\begin{aligned} \mid D^*_{n,q}(f;x) - f(x) \mid &\leq \mid D^*_{n,q}(f-g;x) \mid + \mid D^*_{n,q}(g;x) - g(x) \mid + \mid f(x) - g(x) \mid \\ &\leq 2 \| f - g \|_{C_B(\mathbb{R}^+)} + (1-q)x \| g \|_{C_B(\mathbb{R}^+)} + \frac{\lambda_n(x)}{2} \| g \|_{C_B^2(\mathbb{R}^+)} \end{aligned}$$

From (3.5) clearly we have $\| g \|_{C_B[0,\infty)} \leq \| g \|_{C_B^2[0,\infty)}$.
Therefore,

$$| D_{n,q}^*(f;x) - f(x) | \leq 2 \left(\| f - g \|_{C_B(\mathbb{R}^+)} + \frac{2x(1-q)+\lambda_n(x)}{4} \| g \|_{C_B^2(\mathbb{R}^+)} \right),$$

where $\lambda_n(x)$ is given in Theorem 3.2.

By taking infimum over all $g \in C_B^2(\mathbb{R}^+)$ and by using (3.7), we get

$$| D_{n,q}^*(f;x) - f(x) | \leq 2K_2 \left(f; \frac{2x(1-q)+\lambda_n(x)}{4} \right)$$

Now for an absolute constant $C > 0$ in [7] we use the relation

$$K_2(f;\delta) \leq C\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta) \| f \|\}.$$

This complete the proof. □

Acknowledgement. Research of the first author was supported by the Department of Science and Technology, New Delhi, under grant No.SR/S4/MS:792/12. The second author acknowledges the financial support of University Grants Commission for awarding BSR Fellowship.

References

[1] M. Altınok, M. Küçükaslan, A-statistical supremum-infimum and A-statistical convergence, Azerbaijan Journal of Mathematics, 4 (2) (2014), 31-42.
[2] A. Aral, V. Gupta, On q-analogue of Stancu-beta operators, Appl. Math. Letters, 25 (2012), 67–71.
[3] A. Aral, O. Doğru, Bleimann Butzer and Hahn operators based on q-integers, J. Ineq. Appl., 2007 (2007), Art. ID 79410.
[4] A. Aral, V. Gupta: The q-derivative and applications to q-Szász Mirakyan operators, Calcolo 43 (3) (2006), 151–170.
[5] S.N. Bernstein, Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités, Commun. Soc. Math. Kharkow, 2 (13) (1912), 1–2.
[6] B. Cheikh, Y. Gaied, M. Zaghouani, A q-Dunkl-classical q-Hermite type polynomials. Georgian Math. J., 21 (2) (2014), 125–137.
[7] A. Ciupa, A class of integral Favard-Szász type operators. Stud. Univ. Babeş-Bolyai, Math., 40 (1) (1995), 39–47.
[8] S. Ersan, O. Doğru, Statistical approximation properties of q-Bleimann, Butzer and Hahn operators, Math. Comput. Modell., 49 (2009), 1595–1606.
[9] A.D. Gadjiev, Simultaneous statistical approximation of analytic functions and their derivatives by k-positive linear operators, Azerbaijan Journal of Mathematics, 1 (1) (2011), 57-66.
[10] V. Gupta, C. Radu, Statistical approximation properties of q-Baskokov-Kantorovich operators, Cent. Eur. J. Math., 7 (4) (2009), 809–818.
[11] G. İçöz, Bayram Çekim, Dunkl generalization of Szász operators via q-calculus, Jour. Ineq. Appl., 2015 (2015), Art. ID 284.
[12] N. Ispir, Approximation by modified complex Szasz-Mirakjan operators, Azerbaijan Journal of Mathematics, 3 (2) (2013), 95-107.
[13] V. Kac, P. Cheung, Quantum Calculus. Springer-Verlag New York, 2002.
[14] V. Kac, P. Cheung, Quantum Calculu., Universitext, Springer-Verlag, New York, 2002.
[15] A. Lupaş, A q-analogue of the Bernstein operator, University of Cluj-Napoca, Seminar on numerical and statistical calculus, No. 9, 1987.
[16] N.I. Mahmudov, V. Gupta, On certain q-analogue of Szász Kantorovich operators, J. Appl. Math. Comput., 37 (2011) 407–419.
[17] M. Mursaleen, Asif Khan, H.M. Srivastava, K.S. Nisar, Operators constructed by means of q-Lagrange polynomials and A-statistical approximation, Appl. Math. Comput., 219 (2013), 6911–6918.
[18] M. Mursaleen, Asif Khan, Generalized q-Bernstein-Schurer operators and some approximation theorems, Jour. Function Spaces Appl., 2013 (2013), Article ID 719834, 7 pages.
[19] M. Mursaleen, Asif Khan, Statistical approximation properties of modified q-Stancu-Beta operators, Bull. Malaysian Math. Sci.Soc.(2), 36 (3) (2013), 683–690.
[20] M. Mursaleen, Faisal Khan and Asif Khan, Approximation properties for modified q-Bernstein-Kantorovich operators, Numer. Funct. Anal. Optim., 36(9) (2015), 1178–1197.
[21] M. Mursaleen, Faisal Khan and Asif Khan, Approximation properties for King's type modified q-Bernstein-Kantorovich operators, Math. Meth. Appl. Sci., 38 (2015), 5242–5252.
[22] M. Örkcü, O. Doğru, Weighted statistical approximation by Kantorovich type q-Szász Mirakjan operators, Appl. Math. Comput., 217 (2011), 7913–7919.
[23] G.M. Phillips, Bernstein polynomials based on the q- integers, The heritage of P.L. Chebyshev, A Festschrift in honor of the 70th-birthday of Professor T. J. Rivlin. Ann. Numer. Math., 4 (1997), 511–518.

[24] C. Radu, Statistical approximation properties of Kantorovich operators based on q-integers, Creat. Math. Inform., 17 (2) (2008), 75–84.

[25] M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, Oper. Theory, Adv. Appl., 73 (1994), 369-396.

[26] S. Sucu, Dunkl analogue of Szász operators, Appl. Math. Comput., 244, (2014), 42–48.

[27] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Natl. Bur. Stand., 45 (1950), 239–245.

Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India

*Corresponding author: mursaleenm@gmail.com