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# NEWTON'S METHOD FOR CONVEX OPERATORS AND APPLICATIONS

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Abstract. This review work presents the general statement of a variant of Newton's method for convex monotone operators and its applications. We consider the estimation of the absolute error too. One makes the connection to the contraction principle. One of the applications is approximating $A^{1/p}$, $p > 1$, where $A$ a positive selfadjoint operator is acting on a Hilbert space. One works with "global" convex monotone operators. For the local approach, we mention appropriate references.

## 1. Introduction

There are several version of the Newton's method for operators, some of them "working locally", around the solution of the given equation ([1], [4]). The evaluation of the "absolute error" is of main interest. We prove a version for "global" convex monotone operators. This statement and its proof are important due to the generality of the obtained result. One applies this method to equations involving functions of matrices or of operators, as well as for "scalar equations". One makes the connection to the successive approximation method from the contraction principle. In particular, one

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approximates $A^{1/p}$, $p>1$, $p\in R$, $A$ being a selfadjoint operator with the spectrum contained in the positive semi axis. This work is related to the papers [2], [5], [6]. For some other aspects on the same subject (involving analyticity), see [1]. The background is contained in [3] and [4]. The rest of the paper is organized as follows. Section 2 contains the detailed proof of the "abstract" version of the Newton's method for convex monotone operators. In Section 3, some direct applications to concrete scalar and operatorial equations are considered. Section 4 makes the connection to the contraction principle, by approximating $A^{1/p}$, $p>1$, $A$ being a positive selfadjoint operator acting on a Hilbert space.

## 2. General statements

Let X be a $\sigma-$ complete vector lattice, endowed with a solid $(|x|\le|y|\Rightarrow\|x\|\le\|y\|)$ and $o-$ continuous norm $(x_n\to_{in\ order} x\Rightarrow\|x_n-x\|\to 0)$. Let $Y$ be a normed vector space, endowed with an order relation defined by a closed convex cone. For $a,b\in X$, $a<b$, we denote $[a,b]=\{x\in X,\ a\le x\le b\}$. Pet $P\in C^1([a,b],Y)$. In most of our applications, we have $X=Y$, where $Y$ is an order complete Banach lattice of selfadjoint operators, that is also a commutative algebra (see [3], p. 303-305).

Theorem 2.1 *Assume additionally that for each* $x\in[a,b]$, $\exists[P'(x)]^{-1}\in L_+(Y,X)$ *and that*

$$a\le x\le b\Rightarrow P'(a)\le P'(x)\le P'(b).$$

*If* $P(a)<0$, $P(b)>0$, *then there exists a unique solution* $x^*$ *of the equation* $P(x)=0$, *where*

$$x^*:=\inf x_k=\lim x_k,\ x_0:=b,\ x_{k+1}=x_k-[P'(x_k)]^{-1}[P(x_k)],\ k\in N. \tag{1}$$

*Moreover, we have*

$$a < x^* < b, \quad \left\|x_k - x^*\right\| \le \left\|[P'(a)]^{-1}\right\| \cdot \left\|P(x_k)\right\| \to 0. \tag{2}$$

Proof. Using induction upon $k$, we prove that

$$P(x_k) \ge 0,\ x_{k+1} \le x_k, \quad k \in N. \tag{3}$$

The last relations (1) and the convexity of $P$, yield

$$P(x_0) = P(b) > 0,\ P(x_{k+1}) \ge P(x_k) + [P'(x_k)](x_{k+1} - x_k) = P(x_k) - P(x_k) = 0$$

Hence $P(x_k) \ge 0\ \forall k \in N$. This relations lead to

$$x_{k+1} - x_k = -[P'(x_k)]^{-1}[P(x_k)] \le 0 \Rightarrow x_{k+1} \le x_k \quad \forall k \in N.$$

We derive the following useful relations

$$P(a) < 0,\ -P(x_k) \le 0 \Rightarrow 0 \ge [P'(x_k)]^{-1}(P(a) - P(x_k)) \ge$$
$$[P'(x_k)]^{-1}([P'(x_k)](a - x_k)) = a - x_k \Rightarrow x_k \ge a, \quad \forall k \in N.$$

Using the hypothesis on the space $X$, there exists $x^*$ defined by the first relations (1), and from (3) we infer that the sequence $(x_k)_k$ is decreasing. Passing through the limit in the recurrence relations (1) one obtains

$$\left[P'\left(x^*\right)\right]^{-1}\left[P\left(x^*\right)\right] = 0 \Rightarrow P\left(x^*\right) = 0.$$

From the assumptions on the positivity of $P(b), -P(a)$, and from the definition of $x^*$, we infer that $a < x^* < b$. In order to prove (2), one uses the convexity once more:

$$P(x_k) = P(x_k) - P\left(x^*\right) \ge P'\left(x^*\right)\left(x_k - x^*\right) \ge P'(a)\left(x_k - x^*\right) \Rightarrow$$
$$[P'(a)]^{-1}(P(x_k)) \ge x_k - x^* \ge 0 \Rightarrow \left\|x_k - x^*\right\| \le \left\|[P'(a)]^{-1}\right\| \cdot \left\|(P(x_k))\right\|,\ k \in N.$$

The uniqueness of the solution follows quite easily:

$$0 = P\left(x_1^*\right) - P\left(x_2^*\right) \ge P'\left(x_2^*\right)\left(x_1^* - x_2^*\right) \Rightarrow 0 \ge \left[P'\left(x_2^*\right)\right]^{-1}\left(P'\left(x_2^*\right)\left(x_1^* - x_2^*\right)\right) = x_1^* - x_2^*.$$

Similarly, we can write: $0 \ge x_2^* - x_1^*$, hence $x_1^* = x_2^*$.

□

The corresponding statement for convex decreasing operators holds.

Theorem 2.2 Assume that for any $x \in [a,b]$ there exists $[P'(x)]^{-1}$ such that $-[P'(x)]^{-1}(Y_+) \subset X_+$, $a \le x \le b \Rightarrow [P'(x)]^{-1} \ge [P'(b)]^{-1}$. *If* $P(a) > 0, \quad P(b) < 0$, *Then there exists an unique solution* $x^* \in ]a,b[$ *of the equation* $P(x) = 0, \; x^*$ *being given by:*

$$x^* = \sup x_k = \lim x_k, \; x_0 = a, \; x_{k+1} = x_k - [P'(x_k)]^{-1}(P(x_k)), \quad k \in N.$$

*Moreover, the sequence* $(x_k)_k$ *is increasing and the convergence rate is given by the inequalities*

$$\left\| x_k - x^* \right\| \le \left\| [P'(b)]^{-1} \right\| \cdot \left\| P(x_k) \right\|, \; k \in N.$$

3. Direct applications

During this Section we mention some applications of the general theorems of Section 2. The difficulties consists only in technical details concerning verifying conditions from general theorems. That is why we do not prove all the statements.

Theorem 3.1 (see Theorem 2.4 [6]). *Let* $H$ *be a Hilbert space and* $X$ *the commutative algebra defined in* [3], *p. 303-305. Let*
$B_j \in X_+, \; j \in \{0,1,\ldots, n\}, \; B_0 > 0, \; B_n > 0$ *be such that*

$$B_0 < \sum_{j=1}^{n} B_j$$

*and* $B_1 + 2B_2U + \cdots + nB_nU^{n-1}$ *is invertible for any* $U \in [0, I]$. *Then there exists a unique* $\overline{U}, \; 0 < \overline{U} < I$, *such that*

$$B_n\overline{U}^n + B_{n-1}\overline{U}^{n-1} + \cdots + B_1\overline{U} - B_0 = 0,$$

*and this solution verifies in particular the relation*

$$\left\| I - \overline{U} \right\| \le \left\| \sum_{j=1}^{n} B_j - B_0 \right\|$$

Proof. The space $X$ is an order-complete Banach lattice and a commutative algebra of selfadjoint operators. Let

$$P : [0, I] \subset X_+ \to X$$

be defined by

$$P(U) = B_n U^n + \cdots + B_1 U - B_0 .$$

One can show that the operator $P_n(U) = U^n$ is convex on $X_+$ (see [6] for details). Now it follows easily that $P$ is also convex on $X_+$, since all the coefficients $B_k \in X_+$ and all the operators in $X$ are permutable. On the other hand, we have:

$$P'(U)(V) = \left(nB_n U^{n-1} + \cdots + 2B_2 U + B_1\right) \cdot V \Rightarrow$$
$$[P'(U)]^{-1}(\tilde{V}) = \left(nB_n U^{n-1} + \cdots + 2B_2 U + B_1\right)^{-1} \cdot \tilde{V}, \quad \tilde{V} \in X, U \in [0, I].$$

We also have

$$[P'(U)]^{-1} \geq 0,\ 0 \leq U \leq I \Rightarrow P'(0)V = B_1 V$$
$$\leq \left(nB_n U^{n-1} + \cdots + B_1\right) \cdot V \leq \left(nB_n + \cdots + B_1\right) \cdot V \Rightarrow$$
$$P'(0) \leq P'(U) \leq P'(I) \quad \forall U \in [0, I].$$

Because of the hypothesis, one has

$$P(0) = -B_0 < 0, \quad P(I) = \sum_{k=1}^{n} B_k - B_0 > 0,$$

so that all requirements of Theorem 2.1 are accomplished. It follows that there exists a unique solution $\overline{U} \in ]0, I[$ of the equation $P(U) = 0$, that verifies relation (9) for $k = 0$:

$$\left\| I - U \right\| \leq \left\| [P'(0)]^{-1} \right\| \cdot \left\| P(I) \right\| = \left\| B_1^{-1} \right\| \cdot \left\| B_n + \cdots + B_1 - B_0 \right\|.$$

Now the proof is complete.

□

Theorem 3.2 *Let* $H, X$ *be as in the preceding theorem,* $\alpha > 1,\ B \in X$ *such that the spectrum* $S(B) \subset ]\ln(\alpha), \infty[$. *There is a unique solution* $\overline{U} \in ]0, I[ \subset X$ *of the equation*

$$\exp(BU) - \alpha I = 0$$

*and this solution verifies in particular the relation*

$$\left\| I - \overline{U} \right\| \leq \left\| B^{-1} \right\| \cdot \left\| \exp B - \alpha I \right\|.$$

The next result is an application of the scalar version of Theorem 2.2, when $X = Y = R$.

Proposition 3.1 *Let* $\alpha, \beta, \gamma > 0$ *be such that*

$$1 - \alpha \cdot \exp(-\alpha) - \beta < \gamma < 1.$$

*Then there exists a unique solution* $x^* \in ]0,1[$ *of the equation*

$$\exp(-\alpha x) - \beta x - \gamma = 0$$

*and we have*

$$0 < x^* \leq \frac{1 - \gamma}{\alpha \exp(-\alpha) + \beta} \to 0, \quad \gamma \uparrow 1.$$

Theorem 3.3 *Let* $H$ *be a finite dimensional Hilbert space, and* $X$ *the space defined in* [3], *p. 303-305. Let* $\tilde{A}, B, C \in X$ *be such the spectrums of* $\tilde{A}$ *and* $B$ *are contained in* $]0, \infty[$. *Assume also that*

$$\exp\left(-\tilde{A}\right) - B < C < I.$$

*Then there exists a unique solution* $\overline{U} \in ]0, I[$ *of the equation*

$$\exp\left(-\tilde{A}U\right) - BU - C = 0$$

*and the following estimation holds:*

$$\left\| \tilde{U} \right\| \leq \left\| \left[ \tilde{A} \exp\left(-\tilde{A}\right) + B \right]^{-1} \right\| \cdot \left\| I - C \right\| \to 0,\ C \to I$$

Proposition 3.2 *There is a unique solution* $x^* \in ]3/2, 2[$ *of the equation*

$$2x^3 - 4x^2 + 1 = 0,$$

*and this solution verifies*

$$2 - \frac{27}{152} < x^* < 2.$$

Theorem 3.4 *Let* $A$ *be a selfadjoint operator acting on a Hilbert space, with the spectrum* $S(A) \subset [3/2, 2]$. *Let* $X = Y = X(A)$ *defined in* [3], *p. 303-305. Then there exists a unique operator* $\overline{U} \in X$ *such that:*

$$2\overline{U}^3 - 4\overline{U}^2 + I = 0$$

*and this operator verifies*

$$S(\overline{U}) \subset ]2 - \frac{27}{152}, 2[$$

## 4. Approximating $A^{1/p}$, $p > 1$; connection to contraction principle

Let H be a Hilbert space, $A$ a selfadjoint operator acting on $H$, with the spectrum
$S(A) \subset ]0, \infty[$, $X = X(A)$ the associated commutative algebra according to [3], p. 303-305. We denote

$$\omega_A = \inf_{\|h\|=1} \langle Ah, h \rangle, \quad \Omega_A = \sup_{\|h\|=1} \langle Ah, h \rangle$$

Theorem 4.1 (see Theorem 2.1 [2]). *Let* $A$ *be as above,* $A \notin Sp\{I\}$, $p > 1$, $p \in R$. *There exists a unique operator* $U_p \in ]\omega_A^{1/p} I, \Omega_A^{1/p} I[$ *such that*

$$U_p^p - A = 0,$$

*and this solution verifies the relations*

$$\left\|\Omega_A^{1/p} I - U_p\right\| \leq \frac{1}{p\,\omega_A^{(p-1)/p}} \left\|\Omega_A I - A\right\|, \left\|U_p - \omega_A^{1/p} I\right\| \leq \frac{\Omega_A^{(p+1)/p}}{p} \left\|\omega_A^{-1} I - A^{-1}\right\|.$$

Corollary 4.1 *With the above notations and assumptions, we have*

$$\ln \Omega_A - \ln \omega_A \leq \frac{1}{\omega_A} \left\|\Omega_A I - A\right\| + \Omega_A \left\|\omega_A^{-1} I - A^{-1}\right\|.$$

Remark 4.1 If in the recurrence relation of Newton's method:

$$x_{k+1} = \varphi(x_k),\ \varphi(x) := x - [P'(x)]^{-1}(P(x))$$

the mapping $\varphi$ is a contraction, the rate of convergence of the sequence $(x_k)_k$ is given by contraction principle. Next, we show that this is the case of the operator $P(U) = U^p - A,$ which leads to the positive solution $U_p = A^{1/p}$.

Theorem 4.2 (see [2]). *Let* $p, A, X$ *be as above. Then the Newton recurrence for the equation*

$$P(U) = U^p - A = 0$$

*is*

$$U_0 = \Omega_A^{1/p} I, \quad U_{k+1} = \varphi(U_k) = \frac{p-1}{p} U_k + \frac{1}{p} U_k^{-p+1} A, \quad k \in N.$$

*The convergence rate for* $U_k \to A^{1/p}$ *is given by*

$$\left\|U_k - A^{1/p}\right\| \leq \left(\frac{p-1}{p}\right)^k \Omega_A^{1/p} \left\|I - \Omega_A^{-1} A\right\|, \quad k \in N.$$

Proof. Newton's sequence for the convex operator $P$ is:

$$U_0 = b = \Omega_A^{1/p} I, U_{k+1} = U_k - \left(pU_k^{p-1}\right)^{-1}\left(U_k^p - A\right) =$$
$$U_k - \frac{1}{p} U_k^{-p+1+p} + \frac{1}{p} U_k^{-p+1} A = \frac{p-1}{p} U_k + \frac{1}{p} U_k^{-p+1} A = \varphi(U_k),\ k \in N.$$

Let

$$M := \{U \in X ; U \geq A^{1/p}\},$$

where the root is obtained by the aid of functional calculus for $A$. Clearly, $M$ is closed in $X$, hence it is complete. Let $\varphi : M \to X$ be defined by

$$\varphi(U) = \frac{p-1}{p} U + \frac{1}{p} U^{-p+1} A, \quad U \in M.$$

A straightforward computation shows that $\varphi(A^{1/p}) = A^{1/p}$. First we show that

$$S(U) \subset ]0, \infty[ \Rightarrow \varphi(U) \in M;$$

(in particular this proves that $\varphi(M) \subset M$). One can show that $\varphi$ is convex on the subset of all operators in $X$ having the spectrum contained in the positive semiaxis. In particular, $\varphi$ is convex on $M$. Direct computations yield

$$\varphi(U) \geq \varphi(A^{1/p}) + \varphi'(A^{1/p})(U - A^{1/p}) = A^{1/p}.$$

Thus $\varphi(U) \in M$ for all $U$ with spectrum $S(U) \subset ]0, \infty[$. Now we prove that $\varphi : M \to M$ is a contraction, with contraction constant $q = \frac{p-1}{p}$. Precisely we prove that

$$\|\varphi'(U)\| \leq \frac{p-1}{p}, \quad \forall U \in M.$$

We have:

$$\|\varphi'(U)\| = \frac{p-1}{p} \|I - U^{-p} A\|; U \in M \Rightarrow U^{p} \geq A \Rightarrow I - U^{-p} A \geq 0 \Rightarrow$$

$$\|I - U^{-p} A\| \leq \|I\| = 1 \Rightarrow \|\varphi'(U)\| \leq \frac{p-1}{p}, \quad U \in M.$$

Now the conclusion on $\varphi$ being a contraction follows by a standard differential calculus argument. Now application of contraction theorem and an elementary computation shows that

$$\left\|U_k - A^{1/p}\right\| \le \frac{q^k}{1-q}\left\|U_0 - \varphi(U_0)\right\| = \left(\frac{p-1}{p}\right)^k \Omega_A^{1/p}\left\|I - \Omega_A^{-1}A\right\|, \quad k \in N.$$

The proof is complete.
□

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