International Journal of Analysis and Applications
ISSN 2291-8639
Volume 13, Number 2 (2017), 161-169
http://www.etamaths.com

ON THE COMPOSITION AND NEUTRIX COMPOSITION OF THE DELTA FUNCTION AND THE FUNCTION $\cosh^{-1}(|x|^{1/r}+1)$

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Abstract. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. The composition $F(f(x))$ of F and f is said to exist and be equal to the distribution $h(x)$ if the limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta function. It is proved that the neutrix composition $\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]$ exists and

$$\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)] = -\sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{k}{i}\frac{(-1)^{i+k}rc_{r,s,k}}{(kr+r)k!}\delta^{(k)}(x),$$

for $s = M-1, M, M+1, \ldots$ and $r = 1, 2, \ldots$, where

$$c_{r,s,k} = \sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{kr+r-i}(2j-i)^{s+1}}{2^{s+i+1}},$$

M is the smallest integer for which $s-2r+1 < 2Mr$ and $r \le s/(2M+2)$.

Further results are also proved.

1. Introduction

Let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

A sequence of functions $\{f_n\}$ is said to be regular if

(i) f_n is infinitely differentiable for all n,
(ii) the sequence $\{\langle f_n, \varphi\rangle\}$ converges to a limit $\langle f, \varphi\rangle$ for every $\varphi \in \mathcal{D}$,
(iii) $\langle f, \varphi\rangle$ is continuous in φ in the sense that $\lim_{n\to\infty}\langle f_n, \varphi\rangle = 0$ for each sequence $\varphi_n \to 0$ in $\mathcal{D}$, see [24].

There are many ways to construct a sequence of regular functions which converges to $\delta(x)$. For instance let ρ be a fixed infinitely differentiable function having the properties:

(i) $\rho(x) = 0$ for $|x| \ge 1$, (ii) $\rho(x) \ge 0$,
(iii) $\rho(x) = \rho(-x)$, (iv) $\int_{-1}^{1}\rho(x)\,dx = 1$,

putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.
Further, if F is a distribution in $\mathcal{D}'$ and $F_n(x) = \langle F(x-t), \delta_n(x)\rangle$, then $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to $F(x)$.

In the framework of the theory of distributions, no meaning can be generally given to expressions of the form $F(f(x))$ where F and f are arbitrary distributions. However, in elementary particle physics one finds the need to evaluate $\delta^2(x)$ when calculating the transition rates of certain particle interactions, [14]. In addition, there are terms proportional to powers of the δ functions at the origin

Received 7th September, 2016; accepted 4th November, 2016; published 1st March, 2017.
2010 *Mathematics Subject Classification.* 33B10, 46F30, 46F10, 41A30.
Key words and phrases. distribution; delta function; composition of distributions; neutrix composition of distributions.

coming from the measure of path integration [10]. The composition of a distribution and an infinitely differentiable function is extended to distributions by continuity provided the derivative of the infinitely differentiable function is different from zero, [2]. The composition of a distribution and an infinitely differentiable function is extended to distributions by continuity provided the derivative of the infinitely differentiable function is different from zero, [2]. Fisher [5] defined the composition of a distribution F and a summable function f which has a single simple root in the open interval (a,b), and it was recently generalized in [18] by allowing f to be a distribution. Antosik [1] defined the composition $g(f(x))$ as the limit of the sequence $\{g_n(f_n)\}$ providing the limit exists. By this definition he defined the compositions $\sqrt{\delta}=0$, $\sqrt{\delta^2+1}=1+\delta$, $\log(1+\delta)=0$, $\sin\delta=0$, $\cos\delta=1$ and $\frac{1}{1+\delta}=1$.

For many pairs of distributions, it is not possible to define their compositions by using the definition of Antosik. Using the neutrix calculus developed by van der Corput [3], Fisher gave a general principle for the discarding of unwanted infinite quantities from asymptotic expansions and this has been exploited in context of distributions, see [4,5]. The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from divergent integral is referred to as the Hadamard finite part, see [16]. In fact his method can be regarded as a particular applications of the neutrix calculus.

The following definition of the neutrix composition of distributions is a generalization of Gel'fand and Shilov's definition of the composition involving the delta function [15], and was given in [5].

Definition 1.1. *Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a,b), with $-\infty<a<b<\infty$, if*

$$\operatorname*{N-lim}_{n\to\infty}\int_{-\infty}^{\infty}F_n(f(x))\varphi(x)dx=\langle h(x),\varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$, where $F_n(x)=F(x)*\delta_n(x)$ for $n=1,2,\ldots$ and N is the neutrix, see [3], having domain N' the positive and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda\ln^{r-1}n,\quad \ln^r n:\qquad \lambda>0,\quad r=1,2,\ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a,b) if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}F_n(f(x))\varphi(x)dx=\langle h(x),\varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$.

Note that taking the neutrix limit of a function $f(n)$, is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$, see [4,6,7,16].

2. Main Results

By using Fisher's definition Koh and Li give meaning to δ^k and $(\delta')^k$ for $k=2,3,\ldots$, see [17], and the more general form $(\delta^{(r)})^k$ was considered by Kou and Fisher in [18]. The meaning has been given to the symbol δ_+^k in [22] and the k-th powers of δ for negative integers were defined in [21].

Recently, in [20] Chenkuan Li and Changpin Li used Caputo fractional derivatives and Definition 1.1 and chose the following $\delta-$sequence

$$\delta_n(x)=\Big(\frac{n}{\pi}\Big)e^{-nx^2}\quad(x\in\mathbb{R})$$

to redefine powers of the distributions $\delta^k(x)$ and $(\delta')^k(x)$ for some values of $k\in\mathbb{R}$.

The following two theorems were proved in [6] and [7] respectively.

Theorem 2.1. *The neutrix composition $\delta^{(s)}(\operatorname{sgn}x|x|^\lambda)$ exists and*

$$\delta^{(s)}(\operatorname{sgn}x|x|^\lambda)=0$$

for $s = 0, 1, 2, \dots$ *and* $(s+1)\lambda = 1, 3, \dots$ *and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^{\lambda}) = \frac{(-1)^{(s+1)(\lambda+1)}s!}{\lambda[(s+1)\lambda-1]!}\delta^{((s+1)\lambda-1)}(x)$$

for $s = 0, 1, 2, \dots$ *and* $(s+1)\lambda = 2, 4, \dots$.

Theorem 2.2. *The compositions* $\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s})$ *and* $\delta^{(s-1)}(|x|^{1/s})$ *exist and*

$$\begin{aligned} \delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s}) &= \tfrac{1}{2}(2s)!\delta'(x), \\ \delta^{(s-1)}(|x|^{1/s}) &= (-1)^{s-1}\delta(x) \end{aligned}$$

for $s = 1, 2, \dots$.

The next theorem was proved in [9].

Theorem 2.3. *The neutrix composition* $\delta^{(s)}(\sinh^{-1} x_+^{1/r})$ *exists and*

$$\delta^{(s)}(\sinh^{-1} x_+^{1/r}) = \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{i+k}ra_{s,k,i}}{2^{kr+r}k!}\delta^{(k)}(x),$$

for $s = 0, 1, 2, \dots$ *and* $r = 1, 2, \dots$, *where* M *is the smallest positive integer greater than* $(s-r+1)/r$ *and*

$$a_{r,s,k,i} = \frac{(-1)^s[(kr+r-2i)^s + (kr+r-2i-2)^s]}{2}.$$

In particular, the neutrix composition $\delta(\sinh^{-1} x_+^{1/r})$ *exists and*

$$\delta(\sinh^{-1} x_+^{1/r}) = 0,$$

for $r = 2, 3, \dots$.

In the following, we define the function $\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]$ by

$$\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)] = \begin{cases} \delta^{(s)}[\cosh^{-1}(|x|^{1/r}+1)], & x \geq 0, \\ 0, & x < 0 \end{cases}$$

and we define the function $\delta^{(s)}[\cosh^{-1}(x_-^{1/r}+1)]$ by

$$\delta^{(s)}[\cosh^{-1}(x_-^{1/r}+1)] = \begin{cases} \delta^{(s)}[\cosh^{-1}(|x|^{1/r}+1)], & x \leq 0, \\ 0, & x > 0 \end{cases}$$

for $r = 1, 2, \dots$ and $s = 0, 1, 2, \dots$.

We also use the following easily proved lemma.

Lemma 2.1.

$$\int_0^1 t^i \rho^{(s)}(t)\,dt = \begin{cases} 0, & 0 \leq i < s, \\ \frac{1}{2}(-1)^s s!, & i = s \end{cases}$$

for $s = 0, 1, 2, \dots$.

We now prove

Theorem 2.4. *The neutrix composition* $\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]$ *exists and*

$$\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)] = \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{k}{i}\frac{(-1)^k rc_{r,s,k}}{(kr+r)k!}\delta^{(k)}(x) \tag{2.1}$$

for $s = M-1, M, M+1, \dots$ *and* $r = 1, 2, \dots$, *where*

$$c_{r,s,k} = \sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{kr+r+s-i}(2j-i)^{s+1}}{2^{i+1}},$$

M *is the smallest integer for which* $s-2r+1 < 2Mr$ *and* $r \leq s/(2M+2)$.

In particular, the neutrix composition $\delta[\cosh^{-1}(x_+ + 1)]$ *exists and*

$$\delta[\cosh^{-1}(x_+ + 1)] = 0 \tag{2.2}$$

for $r = 1, 2, \ldots$ *and the neutrix composition* $\delta'[\cosh^{-1}(x_+ + 1)]$ *exists and*

$$\delta'[\cosh^{-1}(x_+ + 1)] = \frac{1}{4}\delta(x). \tag{2.3}$$

Proof. To prove equation (1), we first of all have to evaluate

$$\begin{aligned}
\int_{-1}^{1} \delta_n^{(s)}[\cosh^{-1}(x_+^{1/r} + 1)]x^k\,dx &= n^{s+1}\int_{-1}^{1} \rho^{(s)}[n\cosh^{-1}(x_+^{1/r} + 1)]x^k\,dx \\
&= n^{s+1}\int_{0}^{1} \rho^{(s)}[n\cosh^{-1}(x^{1/r} + 1)]x^k\,dx \\
&\quad + n^{s+1}\int_{-1}^{0} \rho^{(s)}(0)x^k\,dx \\
&= I_1 + I_2.
\end{aligned} \tag{2.4}$$

It is obvious that

$$\operatorname*{N-lim}_{n\to\infty} I_2 = \operatorname*{N-lim}_{n\to\infty} n^{s+1}\int_{-1}^{0} \rho^{(s)}(0)x^k\,dx = 0, \tag{2.5}$$

for $k = 0, 1, 2, \ldots$.

Making the substitution $t = n\cosh^{-1}(x^{1/r} + 1)$, we have for large enough n

$$\begin{aligned}
I_1 &= rn^s \int_0^1 [\cosh(t/n) - 1]^{kr+r-1}\sinh(t/n)\rho^{(s)}(t)\,dt \\
&= -\frac{rn^{s+1}}{kr+r}\int_0^1 [\cosh(t/n) - 1]^{kr+r}\rho^{(s+1)}(t)\,dt \\
&= -\frac{rn^{s+1}}{kr+r}\sum_{i=0}^{kr+r}\binom{kr+r}{i}(-1)^{kr+r-i}\int_0^1 \cosh^i(t/n)\rho^{(s+1)}(t)\,dt \\
&= -\frac{rn^{s+1}}{kr+r}\sum_{i=0}^{kr+r}\binom{kr+r}{i}\sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{kr+r-i}}{2^i}\int_0^1 \exp[(2j-i)t/n]\rho^{(s+1)}(t)\,dt \\
&= -\frac{rn^{s+1}}{kr+r}\sum_{i=0}^{kr+r}\binom{kr+r}{i}\sum_{j=0}^{i}\binom{i}{j}\sum_{m=0}^{\infty}\frac{(-1)^{kr+r-i}(2j-i)^m}{2^i m! n^m}\int_0^1 t^m\rho^{(s+1)}(t)\,dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} I_1 &= \frac{r}{kr+r}\sum_{i=0}^{kr+r}\binom{kr+r}{i}\sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{kr+r+s-i}(2j-i)^{s+1}}{2^i(s+1)!}\int_0^1 t^{s+1}\rho^{(s+1)}(t)\,dt \\
&= \frac{r}{kr+r}\sum_{i=0}^{kr+r}\binom{kr+r}{i}\sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{kr+r+s-i}(2j-i)^{s+1}}{2^{i+1}} \\
&= \frac{r}{kr+r}\sum_{i=0}^{kr+r}\binom{kr+r}{i}c_{r,s,k},
\end{aligned} \tag{2.6}$$

for $k = 0, 1, 2, \ldots$.

When $k = M$, we have

$$\begin{aligned}
|I_1| &\le \frac{rn^{s+1}}{Mr+r}\int_0^1 \Big|[\cosh(t/n) - 1]^{Mr+r}\rho^{(s+1)}(t)\Big|\,dt \\
&\le rn^{s+1}\int_0^1 [(t/n)^2 + O(n^{-4})]^{Mr+r}|\rho^{(s+1)}(t)|\,dt \\
&\le rn^{s-2Mr-2r+1}\int_0^1 [1 + O(n^{-4Mr-4r})]|\rho^{(s+1)}(t)|\,dt \\
&= O(n^{s-2Mr-2r+1}).
\end{aligned}$$

Thus, if ψ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 \delta_n^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]x^M\psi(x)\,dx = 0, \tag{2.7}$$

since $s-2Mr-2r+1<0$.

We also have

$$\int_{-1}^0 \delta_n^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]\psi(x)\,dx \quad = \quad n^{s+1}\int_{-1}^0 \rho^{(s)}(0)\psi(x)\,dx$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^0 \delta_n^{(s)}[(\sinh^{-1}x_+)^{1/r}]\psi(x)\,dx = 0. \tag{2.8}$$

If now φ is an arbitrary function in $\mathcal{D}[-1,1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{M-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^M}{M!}\varphi^{(M)}(\xi x),$$

where $0<\xi<1$, and so

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(s)}[\cosh^{-1}(x_+^{1/r}+1)],\varphi(x)\rangle \;&=\; \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{M-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1 \delta_n^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]x^k\,dx \\
&\quad + \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{M-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^0 \delta_n^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]x^k\,dx \\
&\quad + \lim_{n\to\infty}\frac{1}{M!}\int_0^1 \delta_n^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]x^M\varphi^{(M)}(\xi x)\,dx \\
&\quad + \lim_{n\to\infty}\frac{1}{M!}\int_{-1}^0 \delta_n^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]x^M\varphi^{(M)}(\xi x)\,dx \\
&= \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{k}{i}\frac{rc_{r,s,k}\varphi^{(k)}(0)}{(kr+r)k!} + 0 \\
&= \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{k}{i}\frac{(-1)^k rc_{r,s,k}}{(kr+r)k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,
\end{aligned} \tag{2.9}$$

on using equations (4) to (9). This proves equation (1) on the interval $(-1,1)$.

It is clear that $\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)] = 0$ for $x>0$ and so equation (1) holds for $x>0$.

Now suppose that φ is an arbitrary function in $\mathcal{D}[a,b]$, where $a<b<0$. Then

$$\int_a^b \delta_n^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]\varphi(x)\,dx = n^{s+1}\int_a^b \rho^{(s)}(0)\varphi(x)\,dx$$

and so

$$\operatorname*{N-lim}_{n\to\infty}\int_a^b \delta_n^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]\varphi(x)\,dx = 0.$$

It follows that $\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)] = 0$ on the interval (a,b). Since a and b are arbitrary, we see that equation (1) holds on the real line.

To prove equation (2), we note that in this case $s=0$ and so $M=0$ for $r=1,2,\ldots$. The sum in equation (1) is therefore empty and equation (2) follows.

When $r=s=1$ it follows that $M=1$ and equation (3) then follows from equation (1). This completes the proof of the theorem.

Corollary 2.1. *The neutrix composition* $\delta^{(s)}[\cosh^{-1}(x_-^{1/r}+1)]$ *exists and*

$$\delta^{(s)}[\cosh^{-1}(x_-^{1/r}+1)] \quad = \quad \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{k}{i}\frac{rc_{r,s,k}}{(kr+r)k!}\delta^{(k)}(x) \tag{2.10}$$

for $s = M-1, M, M+1, \ldots$ *and* $r = 1, 2, \ldots,$

In particular, the neutrix composition $\delta[\cosh^{-1}(x_-^{1/r}+1)]$ *exists and*

$$\delta[\cosh^{-1}(x_-^{1/r}+1)] = 0 \tag{2.11}$$

for $r = 1, 2, \ldots$ *and the neutrix composition* $\delta'[\cosh^{-1}(x_-^{1/r}+1)]$ *exists and*

$$\delta'[\cosh^{-1}(x_-^{1/r}+1)] = \frac{1}{4}\delta(x). \tag{2.12}$$

Proof. Equations (10) to (12) follow immediately on replacing x by $-x$ in equations (1) to (3) respectively.

Corollary 2.2. *The neutrix composition* $\delta^{(s)}[\cosh^{-1}(|x|^{1/r}+1)]$ *exists and*

$$\delta^{(s)}[\cosh^{-1}(|x|^{1/r}+1)] \quad = \quad \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{k}{i}\frac{[1+(-1)^k]rc_{r,s,k}}{(kr+r)k!}\delta^{(k)}(x) \tag{2.13}$$

for $s = M-1, M, M+1, \ldots$ *and* $r = 1, 2, \ldots,$

In particular, the neutrix composition $\delta[\cosh^{-1}(|x|^{1/r}+1)]$ *exists and*

$$\delta[\cosh^{-1}(|x|^{1/r}+1)] = 0 \tag{2.14}$$

for $r = 1, 2, \ldots$ *and the neutrix composition* $\delta'[\cosh^{-1}(|x|^{1/r}+1)]$ *exists and*

$$\delta'[\cosh^{-1}(|x|^{1/r}+1)] = \frac{1}{2}\delta(x). \tag{2.15}$$

Proof. Equation (13) follows from equations (1) and (10) on noting that

$$\delta^{(s)}[\cosh^{-1}(|x|^{1/r}+1)] = \delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)] + \delta^{(s)}[\cosh^{-1}(x_-^{1/r}+1)].$$

Equations (14) to (15) follow similarly.

Theorem 2.5. *The neutrix composition* $\delta^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}]$ *exists and*

$$\delta^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}] \quad = \quad \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{kr+r}{i}\frac{(-1)^k b_{r,s,k}}{k!}\delta^{(k)}(x) \tag{2.16}$$

for $s = M-1, M, M+1, \ldots$ *and* $r = 1, 2, \ldots,$ *where*

$$b_{r,s,k} = \sum_{j=0}^{ri+r}\binom{ri+r}{j}\frac{(-1)^{s+k-i}r(2j-ri-r)^{s+1}}{2^{ri+r+1}(ri+r)},$$

M *is the smallest integer for which* $s+1 < 2Mr$ *and* $r \le (s+1)/(2M)$.

Proof. To prove equation (16), we first of all have to evaluate

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}]x^k\,dx &= n^{s+1}\int_{-1}^{1}\rho^{(s)}[n\cosh^{-1}(x_+ + 1)^{1/r}]x^k\,dx \\ &= n^{s+1}\int_{0}^{1}\rho^{(s)}[n\cosh^{-1}(x_+ + 1)^{1/r}]x^k\,dx \\ &\quad + n^{s+1}\int_{-1}^{0}\rho^{(s)}(0)x^k\,dx \\ &= J_1 + J_2. \end{aligned} \tag{2.17}$$

It is obvious that

$$\operatorname*{N-lim}_{n\to\infty} J_2 = \operatorname*{N-lim}_{n\to\infty} n^{s+1}\int_{-1}^{0}\rho^{(s)}(0)x^k\,dx = 0, \tag{2.18}$$

for $k = 0, 1, 2, \ldots$.

Making the substitution $t = n\cosh^{-1}(x_+ + 1)^{1/r}$, we have for large enough n

$$\begin{aligned} J_1 &= rn^s \int_0^1 [\cosh^r(t/n) - 1]^k \cosh^{r-1}(t/n)\sinh(t/n)\rho^{(s)}(t)\,dt \\ &= rn^s \sum_{i=0}^k \binom{k}{i}(-1)^{k-i} \int_0^1 \cosh^{ri+r-1}(t/n)\sinh(t/n)\rho^{(s)}(t)\,dt \\ &= -rn^{s+1} \sum_{i=0}^k \binom{k}{i}\frac{(-1)^{k-i}}{ri+r} \int_0^1 \cosh^{ri+r}(t/n)\rho^{(s+1)}(t)\,dt \\ &= -rn^{s+1} \sum_{i=0}^k \binom{k}{i}\sum_{j=0}^{ri+r}\binom{ri+r}{j}\frac{(-1)^{k-i}}{2^{ri+r}(ri+r)} \int_0^1 \exp[(2j-ri-r)t/n]\rho^{(s+1)}(t)\,dt \\ &= -rn^{s+1} \sum_{i=0}^{kr+r} \binom{kr+r}{i}\sum_{j=0}^{ri+r}\binom{ri+r}{j}\sum_{m=0}^\infty\frac{(-1)^{k-i}(2j-ri-r)^m}{2^{ri+r}(ri+r)m!n^m} \int_0^1 t^m\rho^{(s+1)}(t)\,dt. \end{aligned}$$

It follows that

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty} J_1 &= -\sum_{i=0}^{kr+r}\binom{kr+r}{i}\sum_{j=0}^{ri+r}\binom{ri+r}{j}\frac{(-1)^{k-i}r(2j-ri-r)^{s+1}}{2^{ri+r}(ri+r)(s+1)!}\int_0^1 t^{s+1}\rho^{(s+1)}(t)\,dt \\ &= \sum_{i=0}^{kr+r}\binom{kr+r}{i}\sum_{j=0}^{ri+r}\binom{ri+r}{j}\frac{(-1)^{s+k-i}r(2j-ri-r)^{s+1}}{2^{ri+r+1}(ri+r)} \\ &= \sum_{i=0}^{kr+r}\binom{kr+r}{i}b_{r,s,k}, \end{aligned} \tag{2.19}$$

for $k = 0, 1, 2, \ldots$.

When $k = M$, we have

$$\begin{aligned} |J_1| &\le rn^s \int_0^1 \Big|[\cosh^r(t/n) - 1]^M \cosh^{r-1}(t/n)\sinh(t/n)\rho^{(s)}(t)\Big|\,dt \\ &\le rn^s \int_0^1 \Big|[(t/n)^{2r} + O(n^{-4r})]^M \cosh^{r-1}(t/n)\sinh(t/n)\rho^{(s)}(t)\Big|\,dt \\ &= O(n^{s-2Mr-1}). \end{aligned}$$

Thus, if ψ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 \delta_n^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}]x^M\psi(x)\,dx = 0, \tag{2.20}$$

since $s - 2Mr - 1 < 0$.

We also have

$$\int_{-1}^0 \delta_n^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}]\psi(x)\,dx = n^{s+1}\int_{-1}^0 \rho^{(s)}(0)\psi(x)\,dx$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^0 \delta_n^{(s)}[(\sinh^{-1}x_+)^{1/r}]\psi(x)\,dx = 0. \tag{2.21}$$

If now φ is an arbitrary function in $\mathcal{D}[-1,1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{M-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^M}{M!}\varphi^{(M)}(\xi x),$$

where $0 < \xi < 1$, and so

$$\begin{aligned}
\mathop{\mathrm{N-lim}}_{n\to\infty}\langle\delta_n^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}],\varphi(x)\rangle = \\
&= \mathop{\mathrm{N-lim}}_{n\to\infty}\sum_{k=0}^{M-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1 \delta_n^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}]x^k\,dx \\
&\quad + \mathop{\mathrm{N-lim}}_{n\to\infty}\sum_{k=0}^{M-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^0 \delta_n^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}]x^k\,dx \\
&\quad + \lim_{n\to\infty}\frac{1}{M!}\int_0^1 \delta_n^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}]x^M\varphi^{(M)}(\xi x)\,dx \\
&\quad + \lim_{n\to\infty}\frac{1}{M!}\int_{-1}^0 \delta_n^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}]x^M\varphi^{(M)}(\xi x)\,dx \\
&= \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{kr+r}{i}\frac{b_{r,s,k}\varphi^{(k)}(0)}{k!} + 0 \\
&= \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{kr+r}{i}\frac{(-1)^k b_{r,s,k}}{k!}\langle\delta^{(k)}(x),\varphi(x)\rangle, \qquad (2.22)
\end{aligned}$$

on using equations (17) to (22). This proves equation (16) on the interval $(-1,1)$.

Replacing x by $-x$ in equation (16), we get

Corollary 2.3. *The neutrix composition $\delta^{(s)}[\cosh^{-1}(x_- + 1)^{1/r}]$ exists and*

$$\delta^{(s)}[\cosh^{-1}(x_- + 1)^{1/r}] = \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{kr+r}{i}\frac{b_{r,s,k}}{k!}\delta^{(k)}(x) \qquad (2.23)$$

for $s = M-1, M, M+1, \dots$ and $r = 1, 2, \dots$,

Corollary 2.4. *The neutrix composition $\delta^{(s)}[\cosh^{-1}(|x| + 1)^{1/r}]$ exists and*

$$\delta^{(s)}[\cosh^{-1}(|x| + 1)^{1/r}] = \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{kr+r}{i}\frac{[1+(-1)]^k b_{r,s,k}}{k!}\delta^{(k)}(x) \qquad (2.24)$$

for $s = M-1, M, M+1, \dots$ and $r = 1, 2, \dots$,

Proof. Equation (24) follows from equations (16) and (23) on noting that

$$\delta^{(s)}[\cosh^{-1}(|x|+1)^{1/r}] = \delta^{(s)}[\cosh^{-1}(x_+ + 1)^{1/r}] + \delta^{(s)}[\cosh^{-1}(x_- + 1)^{1/r}].$$

For further related results on the neutrix composition of distributions, see [11], [12], [13], [19] and [23].

References

[1] P. Antosik, Composition of Distributions, Technical Report no.9 University of Wisconsin, Milwaukee, (1988-89), pp.1-30.

[2] P. Antosik, J. Mikusinski and R. Sikorski, Theory of Distributions, The sequential Approach, PWN-Elsevier, Warszawa-Amsterdam (1973).

[3] J.G. van der Corput, Introduction to the neutrix calculus, J. Analyse Math., 7(1959), 291-398.

[4] B. Fisher, On defining the distribution $\delta^{(r)}(f(x))$, Rostock. Math. Kolloq., 23(1983), 73-80.

[5] B. Fisher, On defining the change of variable in distributions, Rostock. Math. Kolloq., 28(1985), 33-40.

[6] B. Fisher, The delta function and the composition of distributions, Dem. Math., 35(1)(2002), 117-123.

[7] B. Fisher, The composition and neutrix composition of distributions, in: Kenan Taş et al. (Eds.), Mathematical Methods in Engineering, Springer, Dordrecht, 2007, pp. 59-69.

[8] B. Fisher and B. Jolevska-Tuneska, Two results on the composition of distributions, Thai. J. Math., 3(1)(2005), 17-26.

[9] B. Fisher and A. Kılıçman, On the composition and neutrix composition of the delta function and powers of the inverse hyperbolic sine function, Integral Transforms Spec. Funct. 21(12)(2010), 935-944.

[10] H. Kleinert, A. Chervyakov, Rules for integrals over products of distributions from coordinate independence of path integrals, Eur. Phys. J. C Part. Fields 19(4)(2001), 743-747.

[11] B. Fisher, A. Kananthai, G. Sritanatana and K. Nonlaopon, The composition of the distributions $x_-^{ms}\ln x_-$ and $x_+^{r-p/m}$, Integral Transforms Spec. Funct., 16(1)(2005), 13-20.
[12] B. Fisher and E. Ozcag, Some results on the neutrix composition of the delta function, Filomat, 26(6)(2012), 1247-1256.
[13] B. Fisher and K. Taş, On the composition of the distributions $x^{-1}\ln|x|$ and x_+^r, Integral Transforms Spec. Funct., 16(7)(2005), 533-543.
[14] S. Gasiorowics, Elementery Particle Physics, John Wiley and Sons, New York, 1966.
[15] I.M. Gel'fand and G.E. Shilov, Generalized Functions, Volume I, Academic Press, New York and London, 1st edition, 1964.
[16] D. S. Jones, Hadamard's Finite Part., Math. Methods Appl. Sci. 19(13)(1996), 1017-1052.
[17] E. L. Koh and C. K. Li, On distributions δ^k and $(\delta')^k$, Math. Nachr., 157(1992), 243-248.
[18] H. Kou and B. Fisher, On Composition of Distributions, Publ. Math. Debrecen, 40(3-4)(1992), 279-290.
[19] L. Lazarova, B. Jolevska-Tuneska, I. Akturk and E. Ozcag Note on the Distribution Composition $(x_+^\mu)^\lambda$. Bull. Malaysian Math. Soc., (2016). doi:10.1007/s40840-016-0342-2.
[20] C. K. Li and C. Li, On defining the distributions δ^k and $(\delta')^k$ by fractional derivatives, Appl. Math. Compt., 246(2014), 502-513.
[21] E. Ozcag, Defining the k-th powers of the Dirac-delta distribution for negative integers, Appl. Math. Letters, 14(2001), 419-423.
[22] E. Ozcag, U. Gulen and B. Fisher, On the distribution δ_+^k, Integral Transforms Spec. Funct., 9(2000), 57-64.
[23] E. Ozcag, L. Lazarova and B. Jolevska-Tuneska, Defining Compositions of $x_+^\mu, |x|^\mu, x^{-s}$ and $x^{-s}\ln|x|$ as Neutrix Limit of Regular Sequences, Commun. Math. Stat., 4(1)(2016), 63-80.
[24] G. Temple, The Theory of generalized Functions, Proc. Roy. Soc. ser. A 28(1955), 175-190.

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