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ON THE GROWTH OF ITERATED ENTIRE FUNCTIONS

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Abstract. We consider iteration of two entire functions of (p,q)-order and study some growth properties of iterated entire functions to generalise some earlier results.

1. Introduction

For any two transcendental entire functions $f(z)$ and $g(z)$, $\lim_{r\to\infty} \frac{M(r,f\circ g)}{M(r,f)} = \infty$ and Clunie [2] proved that the same is true for the ratio $\frac{T(r,f\circ g)}{T(r,f)}$. In [7] Singh proved some results dealing with the ratios of $\log T(r, f\circ g)$ and $T(r,f)$ under some restrictions on the orders of f and g. In this paper, we generalise the results of Singh [7] for iterated entire functions of (p,q)-orders. Following Sato [6], we write $\log^{[0]} x = x$, $\exp^{[0]} x = x$ and for positive integer m, $\log^{[m]} x = \log(\log^{[m-1]} x)$, $\exp^{[m]} x = \exp(\exp^{[m-1]} x)$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Then the (p,q)-order and lower (p,q)-order of $f(z)$ are denoted by $\rho_{(p,q)}(f)$ and $\lambda_{(p,q)}(f)$ respectively and defined by [1]

$\rho_{(p,q)}(f) = \lim_{r\to\infty} \sup \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}$

and $\lambda_{(p,q)}(f) = \lim_{r\to\infty} \inf \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}$, $p \geq q \geq 1$.

According to Lahiri and Banerjee [4] if$f(z)$ and $g(z)$ be entire functions then the

iteration of f with respect to g is defined as follows:

$f_1(z) = f(z)$
$f_2(z) = f(g(z)) = f(g_1(z))$
$f_3(z) = f(g(f(z))) = f(g_2(z))$
. . . . .　　　. . . . . . .　　　. . . . . . .
$f_n(z) = f(g(f(g(....(f(z)$ or $g(z)$ according as n is odd or even))))
and so are $g_n(z)$.
Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

The main purpose of this paper is to study growth properties of iterated entire functions to that of the generating functions under some restriction on (p,q)-orders and lower (p,q)-orders of f and g.

Throughout we assume f, g etc., are non-constant entire functions having finite (p,q)-orders.

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2. Lemmas

Following two lemmas will be needed throughout the proof of our theorems.

Lemma 1[5]. Let $f(z)$ and $g(z)$ be entire functions. If $M(r,g) > \frac{2+\epsilon}{\epsilon}|g(0)|$ for any $\epsilon > 0$, then $T(r, f\circ g) \le (1+\epsilon)T(M(r,g), f)$.

In particular, if $g(0) = 0$ then $T(r, f\circ g) \le T(M(r,g), f)$for all $r > 0$.

Lemma 2[3]. If $f(z)$ be regular in $|z| \le R$, then for $0 \le r < R$

$T(r,f) \le \log^{+} M(r,f) \le \frac{R+r}{R-r}T(R,f)$.

In particular if f be entire, then

$T(r,f) \le \log^{+} M(r,f) \le 3T(2r,f)$.

3. Main Results

First we shall show that if we put some restriction on (p,q)-orders of f and g then the limit superior of the ratio is bounded above by a finite quantity. The following two theorems admit the results.

Theorem 1. Let $f(z)$ and $g(z)$ be two entire functions with $f(0) = g(0) = 0$ and $\rho_{(p,q)}(g) < \lambda_{(p,q)}(f)$. Then for even n

$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[q-1]}T(2^{n-2}r,f)} \le \rho_{(p,q)}(f).$$

Proof. We have by Lemma 1 and Lemma 2

$$\log^{[p]}T(r,f_n) \le \log^{[p]}T(M(r,g_{n-1}),f)$$
$$< (\rho_{(p,q)}(f)+\epsilon)\log^{[q]}M(r,g_{n-1}), \quad \text{for all large values of } r$$

and $\epsilon > 0$

$$\le (\rho_{(p,q)}(f)+\epsilon)\log^{[q-1]}\{3T(2r,g_{n-1})\}$$

$$= (\rho_{(p,q)}(f)+\epsilon)\log^{[q-1]}T(2r,g_{n-1}) + O(1).$$

So, $\log^{[p+(p+1-q)]}T(r,f_n) < \log^{[p]}T(2r,g_{n-1}) + O(1)$

$$< (\rho_{(p,q)}(g)+\epsilon)\log^{[q-1]}T(2^2r,f_{n-2}) + O(1).$$

Proceeding similarly after $(n-2)$ steps we get

$$\log^{[p+(n-2)(p+1-q)]}T(r,f_n) < \log^{[p]}T(2^{n-2}r,f(g)) + O(1)$$

$$\le \log^{[p]}T(M(2^{n-2}r,g),f) + O(1)$$
$$< (\rho_{(p,q)}(f)+\epsilon)\log^{[q]}M(2^{n-2}r,g) + O(1)$$
$$< (\rho_{(p,q)}(f)+\epsilon)\{\exp^{[p-q]}(\log^{[q-1]}(2^{n-2}r))^{\rho_{(p,q)}(g)+\epsilon}\} + O(1) \tag{3.1}$$

for all large values of r

$$< (\rho_{(p,q)}(f)+\epsilon)\{\exp^{[p-q]}(\log^{[q-1]}(2^{n-2}r))^{\lambda_{(p,q)}(f)-\epsilon}\} + O(1)$$

by choosing $\epsilon > 0$ so small that $\rho_{(p,q)}(g) + \epsilon < \lambda_{(p,q)}(f) - \epsilon$.

On the other hand,

$T(r,f) > \exp^{[p-1]}(\log^{[q-1]}r)^{\lambda_{(p,q)}(f)-\epsilon}$, for all $r \ge r_0$

or, $\log^{[q-1]}T(r,f) > \exp^{[p-q]}(\log^{[q-1]}r)^{\lambda_{(p,q)}(f)-\epsilon}$, for all $r \ge r_0$.

Therefore, from above

$$\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{[\log^{[q-1]}T(2^{n-2}r,f)} < \frac{(\rho_{(p,q)}(f)+\epsilon)\{\exp^{[p-q]}(\log^{[q-1]}(2^{n-2}r))^{\lambda_{(p,q)}(f)-\epsilon}\}+O(1)}{\exp^{[p-q]}(\log^{[q-1]}(2^{n-2}r))^{\lambda_{(p,q)}(f)-\epsilon}},$$

for all $r \ge r_0$.

Hence,

$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[q-1]}T(2^{n-2}r,f)}\leq\rho_{(p,q)}(f)+\epsilon.$$

The theorem now follows since ϵ (>0) is arbitrary.

Note 1. From the hypothesis it is clear that f must be transcendental.

Theorem 2. Let f and g be two entire functions with $f(0)=g(0)=0$ and $\rho_{(p,q)}(f)<\lambda_{(p,q)}(g)$. Then for odd n

$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[q-1]}T(2^{n-2}r,g)}\leq\rho_{(p,q)}(g).$$

The proof of the theorem is on the same line as that of Theorem 1.

If $\rho_{(p,q)}(g)>\rho_{(p,q)}(f)$ holds in Theorem 1 we shall show that the limit superior will tend to infinity. Now we prove the following two theorems.

Theorem 3. Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p,q)-orders with $\rho_{(p,q)}(g)>\rho_{(p,q)}(f)$. Then for even n

$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[q-1]}T(\frac{r}{4^{n-1}},f)}=\infty.$$

Proof. We have,

$$\begin{aligned}T(r,f_n)&=T(r,f(g_{n-1}))\\&\geq\tfrac{1}{3}\log M(\tfrac{1}{8}M(\tfrac{r}{4},g_{n-1})+o(1),f)\qquad\{\text{ see [7], page 100 }\}\\&\geq\tfrac{1}{3}\log M(\tfrac{1}{9}M(\tfrac{r}{4},g_{n-1}),f)\\&\geq\tfrac{1}{3}T(\tfrac{1}{9}M(\tfrac{r}{4},g_{n-1}),f)\\&>\tfrac{1}{3}\exp^{[p-1]}\{\log^{[q-1]}\tfrac{1}{9}M(\tfrac{r}{4},g_{n-1})\}^{\lambda_{(p,q)}(f)-\epsilon},\quad\text{for all } r\geq r_0\\&=\tfrac{1}{3}\exp^{[p-1]}\{\log^{[q-1]}M(\tfrac{r}{4},g_{n-1})\}^{\lambda_{(p,q)}(f)-\epsilon}+O(1),\quad\text{for all } r\geq r_0.\end{aligned}$$

Therefore,

$$\begin{aligned}\log^{[p]}T(r,f_n)&>\log\{\log^{[q-1]}M(\tfrac{r}{4},g_{n-1})\}^{\lambda_{(p,q)}(f)-\epsilon}+O(1),\\&=(\lambda_{(p,q)}(f)-\epsilon)\log^{[q]}M(\tfrac{r}{4},g_{n-1})+O(1).\end{aligned}\tag{3.2}$$

So, we have for all $r\geq r_0$

$$\begin{aligned}\log^{[p+(p+1-q)]}T(r,f_n)&>\log^{[p]}[\log M(\tfrac{r}{4},g_{n-1})]+O(1)\\&\geq\log^{[p]}T(\tfrac{r}{4},g_{n-1})+O(1)\\&>(\lambda_{(p,q)}(g)-\epsilon)\log^{[q]}M(\tfrac{r}{4^2},f_{n-2})+O(1),\text{ using (3.2)}\end{aligned}$$

$$\begin{aligned}\text{or, }\log^{[p+2(p+1-q)]}T(r,f_n)&>\log^{[p]}T(\tfrac{r}{4^2},f_{n-2})+O(1)\\&>(\lambda_{(p,q)}(f)-\epsilon)\log^{[q]}M(\tfrac{r}{4^3},g_{n-3})+O(1),\text{ using (3.2).}\end{aligned}$$

Proceeding similarly after some steps we get

$$\begin{aligned}\log^{[p+(n-2)(p+1-q)]}T(r,f_n)&>(\lambda_{(p,q)}(f)-\epsilon)\log^{[q]}M(\tfrac{r}{4^{n-1}},g)+O(1)\\&>(\lambda_{(p,q)}(f)-\epsilon)\exp^{[p-q]}(\log^{[q-1]}(\tfrac{r}{4^{n-1}}))^{\rho_{(p,q)}(g)-\epsilon}+O(1)\end{aligned}\tag{3.3}$$

for a sequence of values of $r\to\infty$.

On the other hand for all $r\geq r_0$ we have,

$$\begin{aligned}&T(r,f)<\exp^{[p-1]}(\log^{[q-1]}r)^{\rho_{(p,q)}(f)+\epsilon}\\&\text{or, }\log^{[q-1]}T(r,f)<\exp^{[p-q]}(\log^{[q-1]}r)^{\rho_{(p,q)}(f)+\epsilon}.\end{aligned}\tag{3.4}$$

So, from (3.3) and (3.4) we have for a sequence of values of $r\to\infty$,

$$\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[q-1]}T(\frac{r}{4^{n-1}},f)}>\frac{(\lambda_{(p,q)}(f)-\epsilon)\exp^{[p-q]}(\log^{[q-1]}(\frac{r}{4^{n-1}}))^{\rho_{(p,q)}(g)-\epsilon}}{\exp^{[p-q]}(\log^{[q-1]}\frac{r}{4^{n-1}})^{\rho_{(p,q)}(f)+\epsilon}}+o(1)$$

and so,

$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[q-1]}T(\frac{r}{4^{n-1}},f)}=\infty$$
since we can choose ϵ (>0) such that $\rho_{(p,q)}(g)-\epsilon>\rho_{(p,q)}(f)+\epsilon$.
This proves the theorem.

An immediate consequence of Theorem 3 for odd n is the following theorem.

Theorem 4. Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p,q)-orders with $\rho_{(p,q)}(g)<\rho_{(p,q)}(f)$. Then for odd n
$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[q-1]}T(\frac{r}{4^{n-1}},g)}=\infty.$$

Next if we consider the ratios $\frac{\log^{[p+(n-1)(p+1-q)]}T(r,f_n)}{\log^{[p]}T(2^{n-2}r,g)}$ or $\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[p]}T(\frac{r}{4^{n-1}},g)}$ we have obtained the following four theorems.

Theorem 5. Let $f(z)$ and $g(z)$ be two transcendental entire functions with $f(0)=g(0)=0$ and let $\lambda_{(p,q)}(g)>0$. Then for even n
$$\limsup_{r\to\infty}\frac{\log^{[p+(n-1)(p+1-q)]}T(r,f_n)}{\log^{[p]}T(2^{n-2}r,g)}\le\frac{\rho_{(p,q)}(g)}{\lambda_{(p,q)}(g)}.$$

Proof. We get from (3.1), for all large values of r
$\log^{[p+(n-2)(p+1-q)]}T(r,f_n)<(\rho_{(p,q)}(f)+\epsilon)\{\exp^{[p-q]}(\log^{[q-1]}(2^{n-2}r))^{\rho_{(p,q)}(g)+\epsilon}\}+O(1)$
or, $\log^{[p+(n-1)(p+1-q)]}T(r,f_n)<(\rho_{(p,q)}(g)+\epsilon)\log^{[q]}(2^{n-2}r)+O(1)$.

On the other hand,
$\log^{[p]}T(r,g)>(\lambda_{(p,q)}(g)-\epsilon)\log^{[q]}r$, for all $r\ge r_0$.
Thus for all $r\ge r_0$
$$\frac{\log^{[p+(n-1)(p+1-q)]}T(r,f_n)}{\log^{[p]}T(2^{n-2}r,g)}<\frac{(\rho_{(p,q)}(g)+\epsilon)\log^{[q]}(2^{n-2}r)+O(1)}{(\lambda_{(p,q)}(g)-\epsilon)\log^{[q]}(2^{n-2}r)}.$$
Therefore,
$$\limsup_{r\to\infty}\frac{\log^{[p+(n-1)(p+1-q)]}T(r,f_n)}{\log^{[p]}T(2^{n-2}r,g)}\le\frac{\rho_{(p,q)}(g)}{\lambda_{(p,q)}(g)}.$$
Hence the theorem is proved.

Theorem 6. Let $f(z)$ and $g(z)$ be two transcendental entire functions with $f(0)=g(0)=0$ and let $\lambda_{(p,q)}(f)>0$. Then for odd n
$$\limsup_{r\to\infty}\frac{\log^{[p+(n-1)(p+1-q)]}T(r,f_n)}{\log^{[p]}T(2^{n-2}r,f)}\le\frac{\rho_{(p,q)}(f)}{\lambda_{(p,q)}(f)}.$$
The proof is omitted.

Theorem 7. Let $f(z)$ and $g(z)$ be two transcendental entire functions of positive lower (p,q)-orders with $\rho_{(p,q)}(g)>0$. Then for even n
$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[p]}T(\frac{r}{4^{n-1}},g)}=\infty.$$

Proof. From (3.3), we have for a sequence of values of $r\to\infty$
$\log^{[p+(n-2)(p+1-q)]}T(r,f_n)>(\lambda_{(p,q)}(f)-\epsilon)\exp^{[p-q]}(\log^{[q-1]}(\frac{r}{4^{n-1}}))^{\rho_{(p,q)}(g)-\epsilon}+O(1)$.
Also, $\log^{[p]}T(r,g)<(\rho_{(p,q)}(g)+\epsilon)\log^{[q]}r$, for all $r\ge r_0$.
Thus
$$\frac{\log^{[p+(n-2)(p+1-q)]}T(r,f_n)}{\log^{[p]}T(\frac{r}{4^{n-1}},g)}\ge\frac{(\lambda_{(p,q)}(f)-\epsilon)}{(\rho_{(p,q)}(g)+\epsilon)}\frac{\exp^{[p-q]}(\log^{[q-1]}(\frac{r}{4^{n-1}}))^{\rho_{(p,q)}(g)-\epsilon}}{\log^{[q]}\frac{r}{4^{n-1}}}$$

which tends to infinity as $r\to\infty$, through this sequence since $\rho_{(p,q)}(g)>0$.

Theorem 8. Let $f(z)$ and $g(z)$ be two transcendental entire functions of positive lower (p,q)-orders with $\rho_{(p,q)}(f)>0$. Then for odd n

$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_n)}{\log^{[p]} T(\frac{r}{4^{n-1}},f)} = \infty.$$

The proof is omitted, since it follows easily as in Theorem 7.

Note 2. If we put $n = 2$, $p = q = 1$ in the Theorem 1 and Theorem 5 we get the results of A.P. Singh [7].

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