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BOUNDING THE DIFFERENCE AND RATIO BETWEEN THE WEIGHTED ARITHMETIC AND GEOMETRIC MEANS

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Abstract. In the paper, making use of two integral representations for the difference and ratio of the weighted arithmetic and geometric means and employing the weighted arithmetic-geometric-harmonic mean inequality, the author bounds the difference and ratio between the weighted arithmetic and geometric means in the form of double inequalities.

1. Main results

In [3, Theorem 2.3, Eq. (2.16)], the difference $A(a,b;\lambda)-G(a,b;\lambda)$ between the weighted arithmetic mean $A(a,b;\lambda)=\lambda a+(1-\lambda)b$ and the weighted geometric mean $G(a,b;\lambda)=a^{\lambda}b^{1-\lambda}$ was expressed as an integral representation

$$A(a,b;\lambda)-G(a,b;\lambda)=\frac{\sin(\lambda\pi)}{\pi}\int_a^b\frac{G(t-a,b-t;\lambda)}{t}\,\mathrm{d}\,t \tag{1.1}$$

for $b>a>0$ and $\lambda\in(0,1)$. In [4, Remark 4.1], the ratio $\frac{A(a,b;\lambda)}{G(a,b;\lambda)}$ between the weighted arithmetic mean $A(a,b;\lambda)$ and the weighted geometric mean $G(a,b;\lambda)$ was expressed as an integral representation

$$\frac{A(a,b;\lambda)}{G(a,b;\lambda)}=1+\frac{\sin(\lambda\pi)}{\pi}\int_a^b\frac{G(t-a,b-t;1-\lambda)}{t^2}\,\mathrm{d}\,t \tag{1.2}$$

for $b>a>0$ and $\lambda\in(0,1)$.

In this paper, making use of the integral representations (1.1) and (1.2) and employing the weighted arithmetic-geometric-harmonic mean inequality

$$A(a,b;\lambda)>G(a,b;\lambda)>H(a,b;\lambda) \tag{1.3}$$

for $b>a>0$ and $\lambda\in(0,1)$, where $H(a,b;\lambda)=\frac{1}{\frac{\lambda}{a}+\frac{1-\lambda}{b}}$, for $b>a>0$ and $\lambda\in(0,1)$ is called the weighted harmonic mean, we will bound the difference $A(a,b;\lambda)-G(a,b;\lambda)$ and the ratio $\frac{A(a,b;\lambda)}{G(a,b;\lambda)}$ of the weighted arithmetic mean $A(a,b;\lambda)$ and the geometric mean $G(a,b;\lambda)$ in the form of double inequalities.

Our main results can be stated as the following theorems.

Theorem 1.1. *For $b>a>0$ and $\lambda\in(0,1)$, the difference between the weighted arithmetic and geometric means can be bounded by*

$$\begin{aligned}\frac{\sin(\lambda\pi)}{\pi}\biggl((2\lambda-1)(b-a)+[(1-\lambda)b-\lambda a]\ln\frac{b}{a}\biggr)&>[\lambda a+(1-\lambda)b]-a^{\lambda}b^{1-\lambda}\\ &>\begin{cases}\dfrac{\sin(\lambda\pi)}{\pi}\biggl[\dfrac{\lambda(1-\lambda)(b-a)^2}{(2\lambda-1)^2[\lambda b-(1-\lambda)a]}\ln\biggl(\dfrac{1}{\lambda}-1\biggr)-\dfrac{ab(\ln b-\ln a)}{\lambda b-(1-\lambda)a}+\dfrac{b-a}{2\lambda-1}\biggr], & \lambda\ne\dfrac12;\\ \dfrac{1}{\pi}\dfrac{b^2-2ab(\ln b-\ln a)-a^2}{b-a}, & \lambda=\dfrac12.\end{cases}\end{aligned} \tag{1.4}$$

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Theorem 1.2. *For $b > a > 0$ and $\lambda \in (0,1)$, the ratio between the weighted arithmetic and geometric means can be bounded by*

$$1+\frac{\sin(\lambda\pi)}{\pi}\left[\lambda\frac{b^2-a^2}{ab}+(1-2\lambda)\ln\frac{b}{a}+\frac{a}{b}-1\right] > \frac{\lambda a+(1-\lambda)b}{a^\lambda b^{1-\lambda}}$$
$$> \begin{cases} 1+\dfrac{\sin(\lambda\pi)}{\pi}\left\{\dfrac{(1-\lambda)b^2-\lambda a^2}{[\lambda a-(1-\lambda)b]^2}\ln\dfrac{b}{a}+\dfrac{(b-a)[\lambda a-(1-\lambda)b]}{[\lambda a-(1-\lambda)b]^2}\right. \\ \qquad \left. +\dfrac{(1-\lambda)\lambda(b-a)^2}{(2\lambda-1)[\lambda a-(1-\lambda)b]^2}\ln\left(\dfrac{1}{\lambda}-1\right)\right\}, \quad \lambda\ne\dfrac{1}{2}; \\ 1+\dfrac{2}{\pi}\left[\dfrac{(a+b)}{b-a}\ln\dfrac{b}{a}-2\right], \quad \lambda=\dfrac{1}{2}. \end{cases} \tag{1.5}$$

2. Proofs of Theorems 1.1 and 1.2

Now we start out to prove our main results.

Proof of Theorem 1.1. By virtue of the inequality in the left-hand side of (1.3), we obtain

$$\int_a^b \frac{(t-a)^\lambda(b-t)^{1-\lambda}}{t}\,\mathrm{d}\,t < \int_a^b \frac{\lambda(t-a)+(1-\lambda)(b-t)}{t}\,\mathrm{d}\,t$$
$$= [(1-\lambda)b-\lambda a](\ln b-\ln a)+(2\lambda-1)(b-a).$$

Substituting this into (1.1) yields

$$[\lambda a+(1-\lambda)b]-a^\lambda b^{1-\lambda} < \frac{\sin(\lambda\pi)}{\pi}\left\{[(1-\lambda)b-\lambda a]\ln\frac{b}{a}+(2\lambda-1)(b-a)\right\}.$$

By virtue of the inequality in the right-hand side of (1.3), we obtain

$$\int_a^b \frac{(t-a)^\lambda(b-t)^{1-\lambda}}{t}\,\mathrm{d}\,t > \int_a^b \frac{1}{t}\frac{1}{\frac{\lambda}{t-a}+\frac{1-\lambda}{b-t}}\,\mathrm{d}\,t$$
$$= \frac{\lambda(1-\lambda)(b-a)^2}{(2\lambda-1)^2(\lambda b-(1-\lambda)a)}\ln\left(\frac{1}{\lambda}-1\right)-\frac{ab(\ln b-\ln a)}{\lambda b-(1-\lambda)a}+\frac{b-a}{2\lambda-1}.$$

Substituting this into (1.1) yields

$$[\lambda a+(1-\lambda)b]-a^\lambda b^{1-\lambda} > \frac{\sin(\lambda\pi)}{\pi}\left\{\frac{\lambda(1-\lambda)(b-a)^2}{(2\lambda-1)^2[\lambda b-(1-\lambda)a]}\ln\left(\frac{1}{\lambda}-1\right)\right.$$
$$\left.-\frac{ab(\ln b-\ln a)}{\lambda b-(1-\lambda)a}+\frac{b-a}{2\lambda-1}\right\} \to \frac{1}{\pi}\frac{b^2-2ab(\ln b-\ln a)-a^2}{b-a}$$

as $\lambda\to\frac{1}{2}$. The double inequality (1.4) is thus proved. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. By virtue of the inequality in the left-hand side of (1.3), we obtain

$$\int_a^b \frac{(t-a)^{1-\lambda}(b-t)^\lambda}{t^2}\,\mathrm{d}\,t < \int_a^b \frac{(1-\lambda)(t-a)+\lambda(b-t)}{t^2}\,\mathrm{d}\,t$$
$$= \lambda\frac{b^2-a^2}{ab}+(1-2\lambda)\ln\frac{b}{a}+\frac{a}{b}-1.$$

Substituting this into (1.2) yields

$$\frac{\lambda a+(1-\lambda)b}{a^\lambda b^{1-\lambda}}-1 < \frac{\sin(\lambda\pi)}{\pi}\left[\lambda\frac{b^2-a^2}{ab}+(1-2\lambda)\ln\frac{b}{a}+\frac{a}{b}-1\right].$$

By virtue of the inequality in the right-hand side of (1.3), we obtain

$$\int_a^b \frac{(t-a)^{1-\lambda}(b-t)^\lambda}{t^2}\,\mathrm{d}\,t > \int_a^b \frac{1}{t^2}\frac{1}{\frac{1-\lambda}{t-a}+\frac{\lambda}{b-t}}\,\mathrm{d}\,t$$
$$= \frac{(1-2\lambda)\big[\lambda a^2-(1-\lambda)b^2\big]\ln\frac{b}{a}+(b-a)\big\{(2\lambda-1)[\lambda a-(1-\lambda)b]+(1-\lambda)\lambda(b-a)\ln\big(\frac{1}{\lambda}-1\big)\big\}}{(2\lambda-1)[\lambda a-(1-\lambda)b]^2}$$
$$\to \frac{2(a+b)}{b-a}\ln\frac{b}{a}-4$$

as $\lambda\to\frac{1}{2}$. Substituting this into (1.2) yields

$$\frac{\lambda a+(1-\lambda)b}{a^\lambda b^{1-\lambda}}-1 > \frac{\sin(\lambda\pi)}{\pi}\bigg\{\frac{(1-\lambda)b^2-\lambda a^2}{[\lambda a-(1-\lambda)b]^2}\ln\frac{b}{a}+\frac{(b-a)[\lambda a-(1-\lambda)b]}{[\lambda a-(1-\lambda)b]^2}$$
$$+\frac{(1-\lambda)\lambda(b-a)^2}{(2\lambda-1)[\lambda a-(1-\lambda)b]^2}\ln\bigg(\frac{1}{\lambda}-1\bigg)\bigg\} \to \frac{2}{\pi}\bigg[\frac{(a+b)}{b-a}\ln\frac{b}{a}-2\bigg]$$

as $\lambda\to\frac{1}{2}$. The double inequality (1.5) is thus proved. The proof of Theorem 1.2 is complete. □

3. Remarks

Finally we list several remarks on our main results.

Remark 3.1. When $\lambda=\frac{1}{2}$ and $b>a>0$, the double inequality (1.4) can be written as

$$\frac{1}{\pi}\bigg(\frac{b-a}{2}\ln\frac{b}{a}\bigg) > \frac{a+b}{2}-\sqrt{ab} > \frac{2}{\pi}\bigg(\frac{a+b}{2}-ab\frac{\ln b-\ln a}{b-a}\bigg) > 0.$$

When $\lambda=\frac{1}{2}$ and $b>a>0$, the double inequality (1.5) can be written as

$$\frac{b-a}{\pi}\bigg(\frac{a+b}{2ab}-\frac{1}{b}\bigg) > \frac{a+b}{2\sqrt{ab}}-1 > \frac{2}{\pi}\bigg[\frac{(a+b)}{b-a}\ln\frac{b}{a}-2\bigg] > 0.$$

Remark 3.2. From the integral representations (1.1) and (1.2), we can easily see that all inequalities for bounding the (weighted) geometric mean can be used to construct inequalities for bounding the difference and ratio between the (weighted) arithmetic and geometric means.

Remark 3.3. Let $0<a_k<a_{k+1}$ for $1\le k\le n-1$, $w_k>0$ for $1\le k\le n$, and $z\in\mathbb{C}\setminus[-a_n,-a_1]$. Theorem 3.1 in [7] states that the principal branch of the weighted geometric mean $\prod_{k=1}^n(z+a_k)^{w_k}$ has the integral representation

$$\prod_{k=1}^n(z+a_k)^{w_k}=\sum_{k=1}^n w_ka_k+z-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\Bigg[\Bigg(\sum_{j=1}^\ell w_j\Bigg)\pi\Bigg]\int_{a_\ell}^{a_{\ell+1}}\prod_{k=1}^n|a_k-t|^{w_k}\frac{\mathrm{d}\,t}{t+z}. \tag{3.1}$$

By the same arguments as in proofs of Theorems 1.1 and 1.2, we can derive from (3.1) lower and upper bounds for the difference $\sum_{k=1}^n w_ka_k-\prod_{k=1}^n a_k^{w_k}$ between the weighted arithmetic mean $\sum_{k=1}^n w_ka_k$ and the geometric mean $\prod_{k=1}^n a_k^{w_k}$.

Remark 3.4. In [4], it was obtained that, for $a_k<a_{k+1}$, $z\in\mathbb{C}\setminus[-a_n,-a_1]$, and $w_k>0$ with $\sum_{k=1}^n w_k=1$, the principal branch of the reciprocal of the weight geometric mean $\prod_{k=1}^n(z+a_k)^{w_k}$ can be represented by

$$\frac{1}{\prod_{k=1}^n(z+a_k)^{w_k}}=\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\Bigg(\pi\sum_{k=1}^\ell w_k\Bigg)\int_{a_\ell}^{a_{\ell+1}}\frac{1}{\prod_{k=1}^n|t-a_k|^{w_k}}\frac{1}{t+z}\,\mathrm{d}\,t. \tag{3.2}$$

The integral representation (3.2) generalizes corresponding results in [2,3] and [5, Lemma 2.4].

Remark 3.5. This paper is a companion of the articles [1–4,6–10].

References

[1] B.-N. Guo and F. Qi, On the degree of the weighted geometric mean as a complete Bernstein function, Afr. Mat. 26 (7) (2015), 1253–1262.
[2] F. Qi and B.-N. Guo, The reciprocal of the geometric mean of many positive numbers is a Stieltjes transform, J. Comput. Appl. Math. 311 (2017), 165–170.
[3] F. Qi and B.-N. Guo, The reciprocal of the weighted geometric mean is a Stieltjes function, Bol. Soc. Mat. Mex. (2016). doi:10.1007/s40590-016-0151-5.
[4] F. Qi and B.-N. Guo, The reciprocal of the weighted geometric mean of many positive numbers is a Stieltjes function, ResearchGate Working Paper (2016), doi:10.13140/RG.2.2.23822.36163.
[5] F. Qi, B.-N. Guo, V. Čerňanová, and X.-T. Shi, Explicit expressions, Cauchy products, integral representations, convexity, and inequalities of central Delannoy numbers, ResearchGate Working Paper (2016), doi:10.13140/RG.2.1.4889.6886.
[6] F. Qi, X.-J. Zhang, and W.-H. Li, An elementary proof of the weighted geometric mean being a Bernstein function, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 77 (1) (2015), 35–38.
[7] F. Qi, X.-J. Zhang, and W.-H. Li, An integral representation for the weighted geometric mean and its applications, Acta Math. Sin. (Engl. Ser.) 30 (1) (2014), 61–68.
[8] F. Qi, X.-J. Zhang, and W.-H. Li, Lévy-Khintchine representation of the geometric mean of many positive numbers and applications, Math. Inequal. Appl. 17 (2) (2014), 719–729.
[9] F. Qi, X.-J. Zhang, and W.-H. Li, Lévy-Khintchine representations of the weighted geometric mean and the logarithmic mean, Mediterr. J. Math. 11 (2) (2014), 315–327.
[10] F. Qi, X.-J. Zhang, and W.-H. Li, The harmonic and geometric means are Bernstein functions, Bol. Soc. Mat. Mex. (2016), doi:10.1007/s40590-016-0085-y.

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